Flexible Sled

$ORIGIN := 1$

Basic Data

$XZ :=$

	1	2
1	500	100
2	498.997	...

$XZ := \frac{XZ \cdot ft}{m}$

These distances are supplied in feet, but it's easier to deal with Odesolve later if everything is unitless. Because this worksheet has SI units as its default, I've divided by metres to get the correct dimensionless distances.

Extract the seperate x and z values.

$X := XZ_{1,1} - XZ^{\langle 1 \rangle}$

Reset the x distances so they are measured from zero.

$Z := XZ^{\langle 2 \rangle}$

Calculate distances along surface at data points

$i := 2 .. \mathrm{last}(X)$

$S_1 := 0 \qquad S_i := \sqrt{\left(X_i - X_{i-1}\right)^2 + \left(Z_i - Z_{i-1}\right)^2} + S_{i-1}$

Calculate angles to the horizontal of the surface between succesive data points (assuming a linear change between them).

$j := 1 .. \mathrm{last}(X) - 1$

$\Theta_j := \mathrm{atan}\left(\frac{Z_{j+1} - Z_j}{X_{j+1} - X_j}\right) \qquad \Theta_{\mathrm{last}(X)} := 0$

Gravitational acceleration

$g := 9.807$

Mass of vehicle

$mass := \frac{600lb}{kg}$ $\quad mass = 272.155$ Again we remove units

Friction coefficient

$\mu := 0.08$

Initial velocity

$v0 := \frac{20\frac{ft}{s}}{\frac{m}{s}}$ $\quad v0 = 6.096$ with units removed

Basic Data

Useful Functions

Function to obtain z value at any x, by linear interpolation

$z1(x) := linterp(X, Z, x)$

Function to obtain x value at any s, by linear interpolation

$x(s) := linterp(S, X, s)$

Function to obtain θ at any x by linear interpolation

$\theta(x) := linterp(X, \Theta, x)$

Gradient function g1

$$g1(x) := \tan(\theta(x))$$

Function g2 to obtain rate of change of gradient g1 wrt x (i.e. dg1(x)/dx)

$$g2(x) := \left(\tan(\theta(x))^2 + 1\right)\cdot\frac{d}{dx}\theta(x)$$

Radius of curvature of surface function: R
(see http://en.wikipedia.org/wiki/Radius_of_curvature_(mathematics) for the basic expression).
There is a sign associated with the radius in order to get the right sign for the centripetal force on the mass - see the Normal force function below.

$$R(x) := \left| \begin{array}{l} dg1dx \leftarrow g2(x) \\ \text{return } \infty \text{ if } |dg1dx| < 10^{-6} \\ \dfrac{\left(1 + g1(x)^2\right)^{\frac{3}{2}}}{dg1dx} \end{array} \right.$$

Normal force function N. When radius is positive the centripetal force increases the normal force; when negative it reduces it.

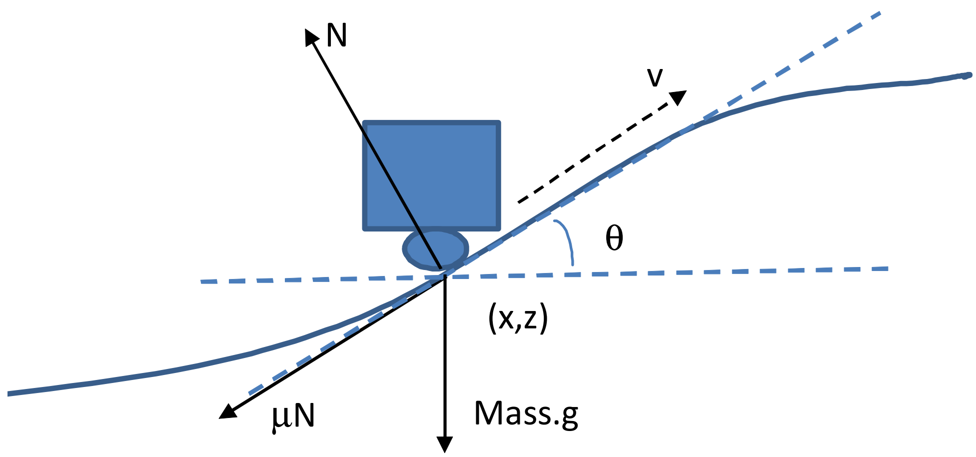
$$N(x, v) := \max\left[\text{mass}\cdot\left(g\cdot\cos(\theta(x)) + \frac{v^2}{R(x)}\right), 0\right]$$

Useful Functions

Single point model

Let's start by considering a single point model. That is, the whole mass of the sled is considered to be concentrated in a point. This is not going to be a very accurate model for the flexible extended sled, but it will provide a first step to a more complicated representation.

The picture below shows the forces (weight, normal and friction) at an arbitrary point (x,z).



Solve ODEs

Initially assume a long end time (seconds)

$t_{end} := 10$

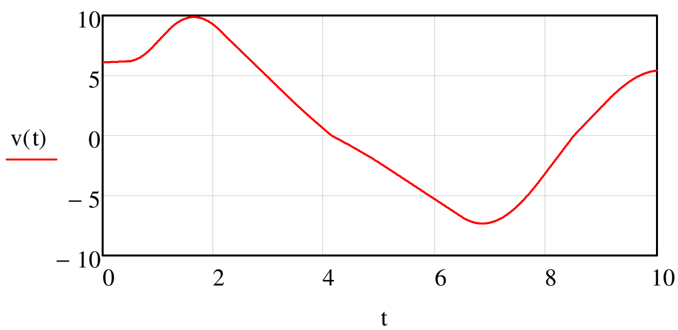
Given

$$\frac{d}{dt}s(t) = v(t) \qquad\qquad s(0) = 0$$

$$\frac{d}{dt}v(t) = -g\cdot\sin(\theta(x(s(t)))) - \frac{\mu}{mass}\cdot N(x(s(t)), v(t))\cdot \mathrm{sign}(v(t)) \qquad\qquad v(0) = v0$$

$$\begin{pmatrix} s \\ v \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} s \\ v \end{pmatrix}, t, t_{end}\right]$$

Plot velocity against time and look for first time at which velocity goes to zero.



Find end time of forward movement (which will occur just after 4 seconds)

$\tau := 5$

Given $\quad v(\tau) = 0 \quad \tau := \text{Find}(\tau)$

$\tau = 4.128$

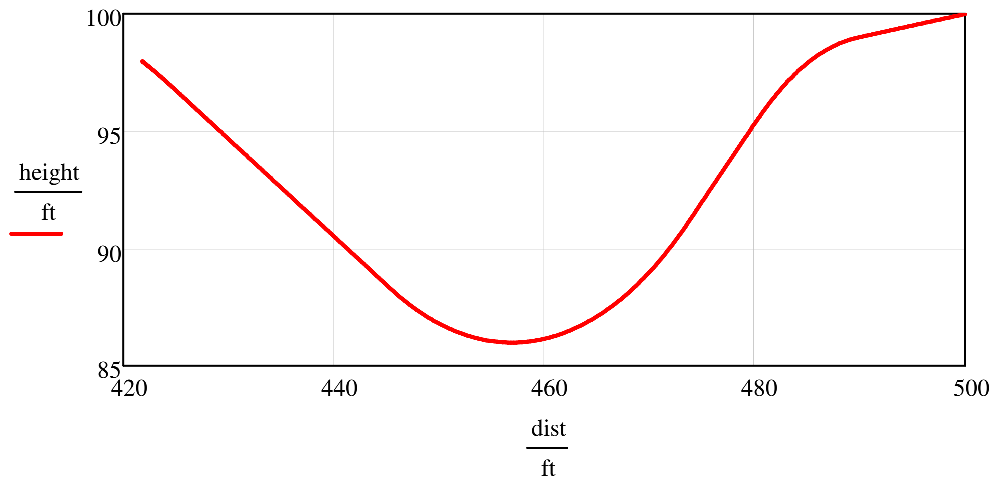
Find values of x and z at time τ

$x(s(\tau)) = 23.833$ metres $\quad z1(x(s(\tau))) = 29.864$ metres

Plot surface profile from time 0 to time τ

$npts := 500 \quad k := 1 .. npts \quad t_k := \tau \cdot \dfrac{k-1}{npts-1}$

$height_k := z1\left(x\left(s\left(t_k\right)\right)\right) \cdot m \quad dist_k := 500ft - x\left(s\left(t_k\right)\right) \cdot m$



Validity checks

Need to check that normal force is always positive as model is invalid otherwise.

tt is the time at which the normal force is closest to 0.

$tt := 0.5$ initial guess

Given

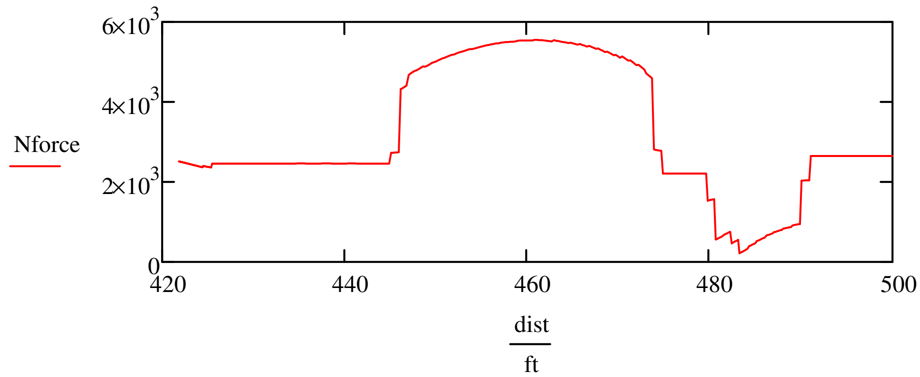
$N(x(s(tt)), v(tt)) = 0$

$tt := \mathrm{Minerr}(tt) = 0.83$

$N(x(s(tt)), v(tt)) = 213.867$ smallest value of normal force

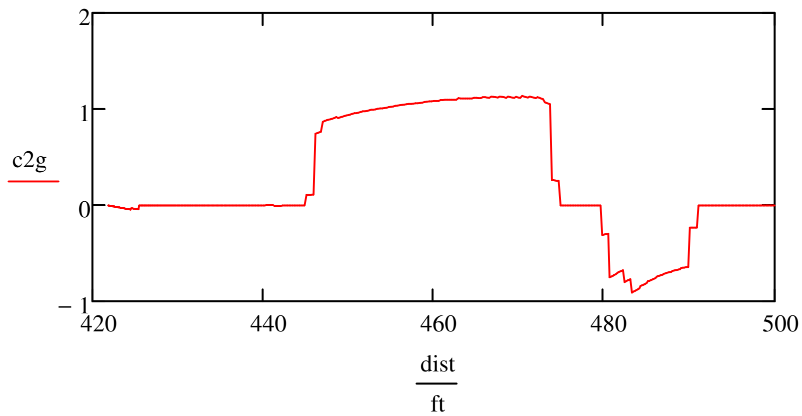
Plot normal force as a function of horizontal distance

$Nforce_k := N\left(x\left(s\left(t_k\right)\right), v\left(t_k\right)\right)$



Look at centripetal component relative to gravitional component in normal force

$$c2g_k := \frac{v(t_k)^2}{R(x(s(t_k)))\cdot g\cdot \cos(\theta(x(s(t_k))))}$$



We can see that the centripetal component is sometimes of comparable size to the gravitational component.

Let's check that the sled doesn't immediately fly off the track at the start. We'll do a conservative calculation assuming there is no resistance and that the sled initially travels horizontally at its starting speed and is in free-fall vertically. We'll compare its trajectory with the track profile. If its trajectory lies above the track profile for a while then we won't be able to use the above model for that period of time. So:

Use xx for horizontal distance, zz for height, vx and vy for horizontal and vertical velocities respectively. (Of course, there is an analytical solution for this free-fall motion, but I'll use mathcad's odesolve here).

Given

$$\frac{d}{dt}xx(t) = vx(t) \qquad xx(0) = 0$$

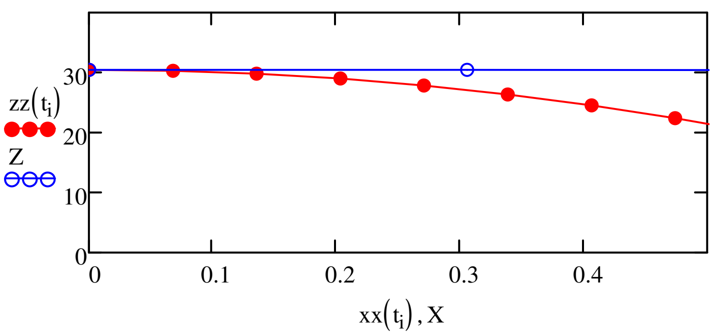
$$\frac{d}{dt}zz(t) = vz(t) \qquad zz(0) = Z_1$$

$$\frac{d}{dt}vx(t) = 0 \qquad vx(0) = v0$$

$$\frac{d}{dt}vz(t) = -mass \cdot g \qquad vz(0) = 0$$

$$\begin{pmatrix} xx \\ zz \\ vx \\ vz \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} xx \\ zz \\ vx \\ vz \end{pmatrix}, t, 0.1\right]$$

$i := 1 .. 10 \qquad t_i := \frac{i-1}{10-1} \cdot 0.1$



We can see that in free fall the sled would immediately go below the track if the track weren't there; so it's reasonable to assume the sled follows the track profile at the start.

Validity checks