NUMERICAL INTEGRATION OF THE MOTIONS OF THREE NEWTONIAN PARTICLES IN THREE DIMENSIONS

Roger L. Mansfield - Mathcad Virtual Event, April 14, 2011

Mathcad Prime 1.0 Version

This Mathcad worksheet is a 3D version of a worksheet posted to the PlanetPTC website by Valery F. Ochkov (OchkovVF@rmpei.ru -- see http://communities.ptc.com/videos/1356).

In his 3-Planets worksheet, Dr. Ochkov uses Mathcad's Odesolve function to solve three second-order ODEs, one 2-vector ODE for each of the three particles. Each 2-vector ODE has component equations in x and y. By design, the motion takes place entirely in the xy plane.

This worksheet uses reduction of order to produce six first-order ODEs, each 3-vector ODE having component equations in x, y, and z. The reduction yields a system of eighteen scalar, first-order ODEs, which we solve using Mathcad's Radau function.

Input Data

$G := 1$ Gravity constant

$m1 := 20$ Mass of Newtonian particle 1 (Black)

$m2 := 2$ Mass of Newtonian particle 2 (Red)

$m3 := 0.1$ Mass of Newtonian particle 3 (Blue)

$x1b := -3$	$y1b := 3$	$z1b := 0$	Position and velocity of particle 1
$vx1b := 1$	$vy1b := 0$	$vz1b := 0$	
$x2b := -3$	$y2b := -0.2$	$z2b := 0$	Position and velocity of particle 2
$vx2b := 1$	$vy2b := 0$	$vz2b := 0$	
$x3b := -3.1$	$y3b := -0.1$	$z3b := 0$	Position and velocity of particle 3
$vx3b := 2$	$vy3b := 0$	$vz3b := 0$	

$node := 1000$ Number of output integration points to be requested from Radau numerical integrator.

$FRAME := 999$ FRAME variable declaration (must delete this assignment before animating).

$te := 1.1$ Integration span endpoint.

$t1 := \frac{te}{100} \cdot (FRAME + 1)$ Time variable for animation.

Set up correspondence between Valery's Odesolve data and my Radau state vector.

$X0_1 := x1b$ $X0_2 := y1b$ $X0_3 := z1b$

$X0_4 := vx1b$ $X0_5 := vy1b$ $X0_6 := vz1b$

$X0_7 := x2b$ $X0_8 := y2b$ $X0_9 := z2b$

$X0_{10} := vx2b$ $X0_{11} := vy2b$ $X0_{12} := vz2b$

$X0_{13} := x3b$ $X0_{14} := y3b$ $X0_{15} := z3b$

$X0_{16} := vx3b$ $X0_{17} := vy3b$ $X0_{18} := vz3b$

Specify derivative function for Radau, Xdot(X).

<u>Note</u> Derivative function Xdot(X) implements a system of eighteen first-order ODEs, six equations each for Newtonian particle 1, 2, and 3, in turn. The six equations for each particle are equations for x, y, z, xdot, ydot, and zdot, in turn. (I am using a subscript "dot" rather than an over-dot to denote the first derivative with respect to time.)

$$X_{dot}(X) := \left\| \begin{array}{l} r21 \leftarrow \sqrt{\left(X_7 - X_1\right)^2 + \left(X_8 - X_2\right)^2 + \left(X_9 - X_3\right)^2} \\ r31 \leftarrow \sqrt{\left(X_{13} - X_1\right)^2 + \left(X_{14} - X_2\right)^2 + \left(X_{15} - X_3\right)^2} \\ r32 \leftarrow \sqrt{\left(X_{13} - X_7\right)^2 + \left(X_{14} - X_8\right)^2 + \left(X_{15} - X_9\right)^2} \\ \left[\begin{array}{c} X_4 \\ X_5 \\ X_6 \end{array} \right. \end{array} \right.$$

$$\left[\begin{array}{c}
G \cdot m2 \cdot \dfrac{\left(X_7 - X_1\right)}{r21^3} + G \cdot m3 \cdot \dfrac{\left(X_{13} - X_1\right)}{r31^3} \\
G \cdot m2 \cdot \dfrac{\left(X_8 - X_2\right)}{r21^3} + G \cdot m3 \cdot \dfrac{\left(X_{14} - X_2\right)}{r31^3} \\
G \cdot m2 \cdot \dfrac{\left(X_9 - X_3\right)}{r21^3} + G \cdot m3 \cdot \dfrac{\left(X_{15} - X_3\right)}{r31^3} \\
X_{10} \\
X_{11} \\
X_{12} \\
G \cdot m1 \cdot \dfrac{\left(X_1 - X_7\right)}{r21^3} + G \cdot m3 \cdot \dfrac{\left(X_{13} - X_7\right)}{r32^3} \\
G \cdot m1 \cdot \dfrac{\left(X_2 - X_8\right)}{r21^3} + G \cdot m3 \cdot \dfrac{\left(X_{14} - X_8\right)}{r32^3} \\
G \cdot m1 \cdot \dfrac{\left(X_3 - X_9\right)}{r21^3} + G \cdot m3 \cdot \dfrac{\left(X_{15} - X_9\right)}{r32^3} \\
X_{16} \\
X_{17} \\
X_{18} \\
G \cdot m1 \cdot \dfrac{\left(X_1 - X_{13}\right)}{r31^3} + G \cdot m2 \cdot \dfrac{\left(X_7 - X_{13}\right)}{r32^3} \\
G \cdot m1 \cdot \dfrac{\left(X_2 - X_{14}\right)}{r31^3} + G \cdot m2 \cdot \dfrac{\left(X_8 - X_{14}\right)}{r32^3} \\
G \cdot m1 \cdot \dfrac{\left(X_3 - X_{15}\right)}{r31^3} + G \cdot m2 \cdot \dfrac{\left(X_9 - X_{15}\right)}{r32^3}
\end{array}\right]$$

$D(t, X0) := X_{dot}(X0)$

Tell the Radau integrator what the derivative function is.

$Q := \mathrm{Radau}(X0, 0, t1, node, D)$

Invoke Radau integrator.

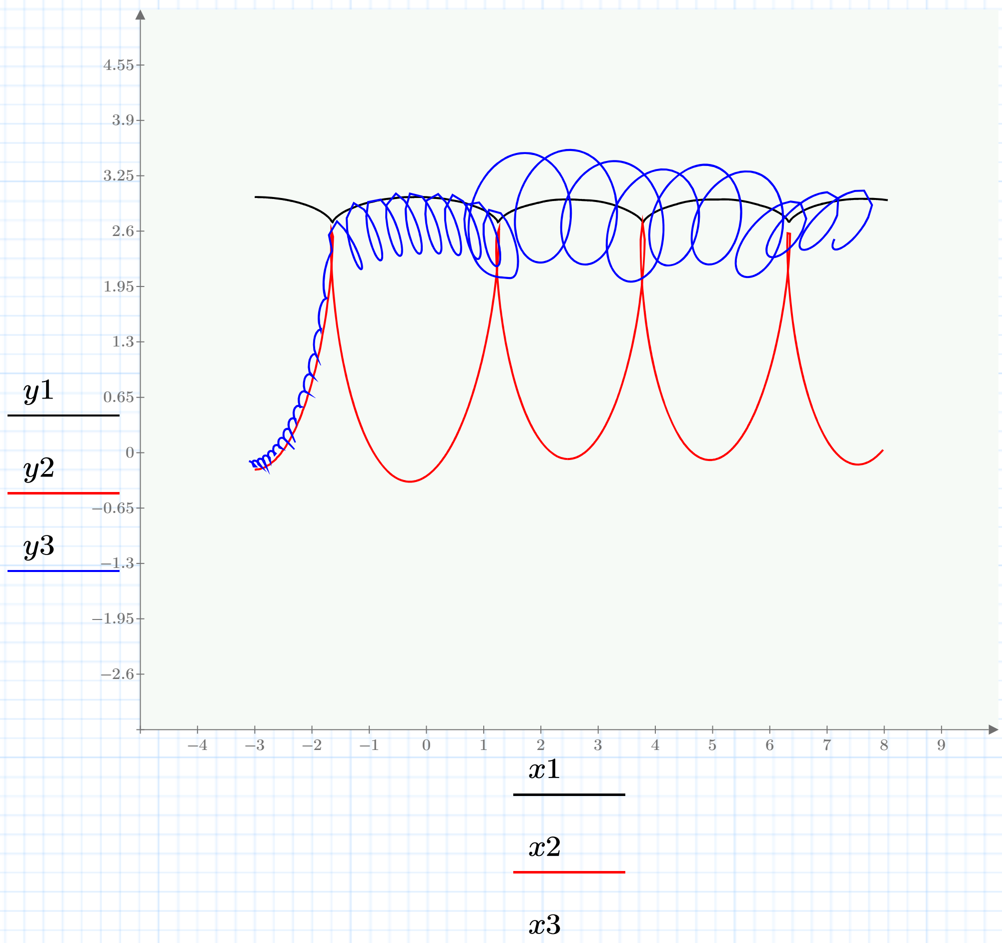
$x1 := Q^{\langle 2 \rangle}$ $y1 := Q^{\langle 3 \rangle}$ $z1 := Q^{\langle 4 \rangle}$

$x2 := Q^{\langle 8 \rangle}$ $y2 := Q^{\langle 9 \rangle}$ $z2 := Q^{\langle 10 \rangle}$

$x3 := Q^{\langle 14 \rangle}$ $y3 := Q^{\langle 15 \rangle}$ $z3 := Q^{\langle 16 \rangle}$

This worksheet was set up to integrate the motions of three bodies in three dimensions. But since Mathad Prime 1.0 cannot do 3D plots, we set the z positions and velocities of the three bodies to zero. In Mathcad Prime 2.0 and later we expect to be able to do 3D plots, then we will be able to explore the motions in 3D.

Two-dimensional (2D) Plot



Final Comments

1. This worksheet was converted from Mathcad 15 to Mathcad Prime 1.0 using the Mathcad Prime worksheet converter. Features in the Mathcad 15 worksheet that could not be converted to Mathcad Prime 1.0 include:

a. Subscripts in text blocks, e.g. xdot, ydot, zdot, and Xdot with "dot" as literal subscripts.

b. Clickable, rotatable 3D plot.

2. Mathcad's Radau integrator is intended for stiff systems of first-order, ordinary differential equations (ODEs). Its RADAU5 integrator is based upon Hairer and Wanner (1996), as documented in

Hairer, Ernst and Wanner, Gerhard, *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems,* Springer Series in Computational Mathematics (Springer, March 2010).

3. This worksheet supports my claim that Mathcad Prime 1.0 includes most of the functions from of Mathcad 15 and earlier versions, to include the more robust numerical integrators (Mathcad 14) and the functionality of Design of Experiments (Mathcad 15).