Units, functions, etc

$\mu m \equiv m \cdot 10^{-6}$ $\quad nm \equiv m \cdot 10^{-9}$

$GHz \equiv 10^{9} \cdot Hz$ $\quad THz \equiv 10^{12} \cdot Hz$

$\mu g \equiv 10^{-6} \cdot gm$ $\quad ng \equiv 10^{-9} \cdot gm$ $\quad pg \equiv 10^{-12} \cdot gm$

$ppm \equiv 10^{-6}$ $\quad ppb \equiv 10^{-9}$

$hour \equiv 60 \cdot min$ $\quad ps \equiv 10^{-12} \cdot s$ $\quad ns \equiv 10^{-9} \cdot s$ $\quad ms \equiv s \cdot 10^{-3}$ $\quad \mu s \equiv s \cdot 10^{-6}$

$mrad \equiv rad \cdot 10^{-3}$

$mW \equiv W \cdot 10^{-3}$ $\quad mK \equiv K \cdot 10^{-3}$ $\quad eV \equiv erg \cdot 1.60219 \cdot 10^{-12}$

Velocity of light in vacuum $\quad c \equiv 299792458 \cdot m \cdot sec^{-1}$

Planck's constant $\quad h \equiv 6.6260755 \cdot 10^{-34} \cdot joule \cdot sec$

Boltzmann's constant $\quad k \equiv 1.380658 \cdot 10^{-23} \cdot joule \cdot K^{-1}$

In line division operators: $\quad /(a,b) := \frac{a}{b} \quad \div(a,b) := \frac{a}{b}$

$\pm$ operator: $\quad \pm(a,b) := \begin{pmatrix} a+b \\ a-b \end{pmatrix}$

A match function that doesn't choke if there is no match $\quad MATCH(x,V) := \text{"No Match"} \;\; \text{on error} \;\; match(x,V)$

Functions to convert range values to vectors. These functions may or may not work for range variables - it depends on the Mathcad version.

$$\Downarrow(X) := \left| \begin{array}{l} v \leftarrow 0 \\ i \leftarrow ORIGIN \\ \text{for } \; x \in X \\ \quad \left| \begin{array}{l} v_i \leftarrow x \\ i \leftarrow i+1 \end{array} \right. \\ \text{return } \; v_0 \;\; \text{if } \; rows(v) = 1 \\ \text{return } \; v \;\; \text{otherwise} \end{array} \right.$$

$\Uparrow(X) := reverse(\Downarrow(X))$

$\Rightarrow(X) := \Downarrow(X)^T$

$\Leftarrow(X) := reverse(\Downarrow(X))^T$

$$
\log\Downarrow(X) := \begin{array}{|l} i \leftarrow v \leftarrow 0 \\ \text{for } x \in X \\ \quad \begin{array}{|l} v_i \leftarrow x \\ i \leftarrow i + 1 \\ \text{break if } i = 2 \end{array} \\ start \leftarrow \log(v_0) \\ step \leftarrow \log(v_1) - start \\ \text{for } x \in X \\ \quad end \leftarrow x \\ \text{for } i \in 0 .. \operatorname{floor}\left(\dfrac{\log(end) - start}{step}\right) \\ \quad V_i \leftarrow 10^{start + i \cdot step} \\ V \end{array}
$$

$$\log\Uparrow(X) := \operatorname{reverse}(\log\Downarrow(X))$$

$$\log\Rightarrow(X) := \log\Downarrow(X)^T$$

$$\log\Leftarrow(X) := \operatorname{reverse}(\log\Downarrow(X))^T$$

Units, functions, etc

Discrete Fourier Transform Sums

Take an example function: $F(x) := \operatorname{erf}(x)$

$$
F(x) := \begin{array}{|l} \text{" function from Valery Aug. 09, 2007 "} \\ \dfrac{1}{2} \cdot 2^{\frac{1}{2}} \cdot x^{\frac{1}{2}} \quad \text{if } 0 < x \\ \text{otherwise} \\ \quad \begin{array}{|l} (x + 1.5)^2 \quad \text{if } x < -1.5 \\ \min\left(1, \left|\sin(2\pi x)\, x^2\right|\right) \quad \text{otherwise} \end{array} \end{array}
$$

Define the number of points: $N := 2000$

Start and ending values for the time axis: $t_{start} := -2$ $\quad t_{end} := 2$

The time values: $i := 0 .. N - 1$ $\quad times_i := t_{start} + i \cdot \dfrac{(t_{end} - t_{start})}{(N - 1)}$

$$\Delta t := \frac{times_{N-1} - times_0}{N - 1}$$

The Y values: $Y := \overrightarrow{F(times)}$

We can now calculate the Fourier coefficients. The obvious way to do this is using one of the built in functions, for example CFFT

$$Y_f := \operatorname{CFFT}(Y)$$

Note that the second half of Y_f is the reverse of the complex conjugate of the first half, with the exception of either 1 or 2 points. If the number of points is odd, then point 0 is unique. If the number of points is even then points 0 and N/2 are unique.

Separate the coefficients: $A := \mathrm{Re}(Y_f)$ $\quad B := \mathrm{Im}(Y_f)$

Recover just the unique points:

$$A := \mathrm{submatrix}\left(A, 0, \mathrm{ceil}\left(\frac{\mathrm{last}(A)}{2}\right), 0, 0\right) \qquad B := \mathrm{submatrix}\left(B, 0, \mathrm{ceil}\left(\frac{\mathrm{last}(B)}{2}\right), 0, 0\right)$$

We can generate the same coefficients using the classic sums for the DFT. The normalization coefficients and sign convention have been chosen to match CFFT

$$n := 0 .. \mathrm{floor}\left(\frac{N}{2}\right)$$

$$a_n := \frac{1}{N} \cdot \sum_{k=0}^{N-1} \left(Y_k \cdot \cos\left(\frac{2 \cdot \pi \cdot n \cdot k}{N}\right)\right) \qquad b_n := \frac{-1}{N} \cdot \sum_{k=0}^{N-1} \left(Y_k \cdot \sin\left(\frac{2 \cdot \pi \cdot n \cdot k}{N}\right)\right)$$

We can verify that the coefficents determined using the two methods are the same to within roundoff error:

$$\sum \overrightarrow{|A - a|} = 6.72583185 \times 10^{-13} \qquad \sum \overrightarrow{|B - b|} = 7.204675803 \times 10^{-13}$$

To perform the inverse DFT to rebuild Y we could of course just use ICFFT (if the original data were to be obtained from the a and b coefficients we would of course have to first recreate the complex vector Y_f, including the second "half"). We can also do the inverse FT via discreet sums though. If we do that then unique points must be treated differently, because we only want to add in half their values relative to the other points. The forms of the sums are therefore slightly different depending on whether the number of points is even or odd.

Inverse Fourier Sum for N odd

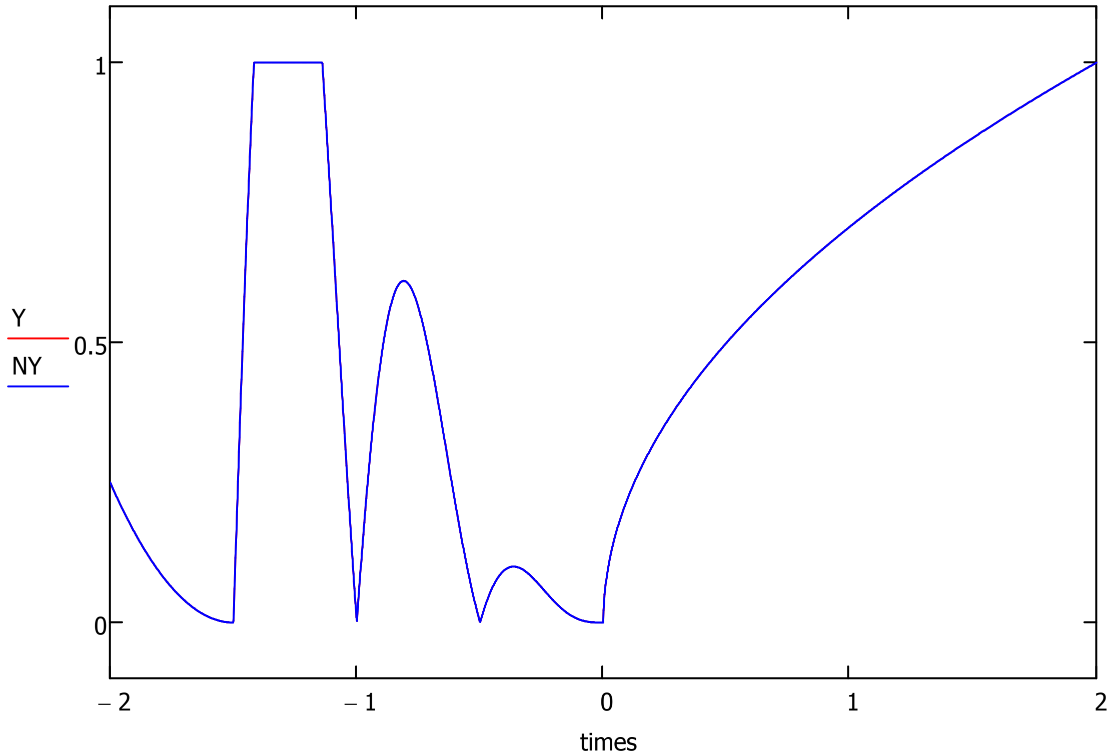
$$\mathrm{NewY\_odd}(t) := A_0 + 2 \cdot \sum_{k=1}^{\mathrm{floor}(N \div 2)} \left[A_k \cdot \cos\left[\frac{2 \cdot k \cdot \pi \cdot (t - \mathrm{times}_0)}{N \cdot \Delta t}\right] - B_k \cdot \sin\left[\frac{2 \cdot k \cdot \pi \cdot (t - \mathrm{times}_0)}{N \cdot \Delta t}\right]\right]$$

Inverse Fourier Sum for N even

$$\mathrm{NewY\_even}(t) := A_0 + 2 \cdot \sum_{k=1}^{(N \div 2)-1} \left[A_k \cdot \cos\left[\frac{2 \cdot k \cdot \pi \cdot (t - \mathrm{times}_0)}{N \cdot \Delta t}\right] - B_k \cdot \sin\left[\frac{2 \cdot k \cdot \pi \cdot (t - \mathrm{times}_0)}{N \cdot \Delta t}\right]\right] + \left[A_{N \div 2} \cdot \cos\left[\frac{2 \cdot N \cdot \pi \cdot (t - \mathrm{times}_0)}{2 \cdot N \cdot \Delta t}\right] - B_{N}\right.$$

$$\mathrm{NewY}(t) := \mathrm{if}\left(\mathrm{round}\left(\frac{N}{2}\right) = \frac{N}{2}, \mathrm{NewY\_even}(t), \mathrm{NewY\_odd}(t)\right)$$

$$NY := \overrightarrow{\mathrm{NewY}(\mathrm{times})} \qquad \sum \overrightarrow{|NY - Y|} = 2.8411 \times 10^{-12}$$



Y
NY
1
0.5
0
− 2
− 1
0
1
2
times

$$N \div 2 \cdot \sin\left[\frac{2 \cdot N \cdot \pi \cdot \left(t - times_0\right)}{2 \cdot N \cdot \Delta t}\right]\Bigg]$$