TwoPDETestb.xmcd. September 11, 2009.. This worksheet modifies the PDE Solve Quicksheet solve the linearized partial differential equations describing undrained, adiabatic shear of a veloci strengthening fault zone. The equations are written in terms of the gradients of pore pressure and temperature.

Adds results for FD solution to previous worksheets.

In these equations, Φ is the gradient of pore pressure and Ψ is gradient of temperature in non-dimensional form and perturbations from the uniform solution. cth and chy are thermal and hydraulic diffusivities, z is a parameter that gives the magnitude of the velocity strengthening (she rate strengthening) and Hc is a measure of magnitude of the shear heating (all in nondimensiona form). Representative values are given below.

$cth := 10^{-2}$ $\qquad Hc := 0.3$ $\qquad chy := 1\cdot 10^{-4}$ $\qquad z := 40$

Linearized solution for temperature and pressure (and their gradients) is unstable (grows exponentially) for wavenumbers less than a critical value given by (Critical value of the velocity is slightly different because of the effect of the time dependence of the homogeneous solution.

$$Ncrit := \frac{1}{\pi}\cdot\sqrt{\frac{z\cdot Hc}{(cth + chy)}} \qquad Ncrit = 10.972$$

$N1 := floor(Ncrit)$ $\qquad floor(Ncrit) = 10$

$N2 := ceil(Ncrit)$ $\qquad ceil(Ncrit) = 11$

$\Phi 0 := 1$ $\qquad \Psi 0 := 0$

$P0(y) := \Phi 0\cdot\sin(\pi\cdot Nnum\cdot y)$ $\qquad T0(y) := \Psi 0\cdot\sin(\pi\cdot Nnum\cdot y)$

Given

$$\Psi_t(y,t) = cth\cdot\Psi_{yy}(y,t) + Hc\cdot z\cdot\Phi(y,t)$$

$$\Phi_t(y,t) = \left(chy\cdot\Phi_{yy}(y,t) + cth\cdot\Psi_{yy}(y,t)\right) + Hc\cdot z\cdot\Phi(y,t)$$

Initial conditions (for single Fourier mode):

$\Phi(y,0) = P0(y)$

$\Psi(y,0) = T0(y)$

Boundary conditions (since solution is anti-symmetric about y = 0, bc's given for half the domain

$\Psi(L,t) = 0$ $\qquad \Phi(L,t) = 0$

$\Psi(0,t) = 0$ $\qquad \Phi(0,t) = 0$

Solve over the ranges 0 to L in y and 0 to T in t.

$$\begin{pmatrix} \Phi num \\ \Psi num \end{pmatrix} := \text{Pdesolve}\left[\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, y, \begin{pmatrix} 0 \\ L \end{pmatrix}, t, \begin{pmatrix} 0 \\ T \end{pmatrix}, \text{spacepts}, \text{timepts}\right]$$

Exact (i.e., analytical) Solution

Exact solution (verified and tested in LinearSolFulla.xmcd. Solutions are of the form exp(pt) sin(N*πY) (because of the boundary conditions). Then p must satisfy a quadratic equation with tl following coefficients::

$$Bc(n) := \frac{1}{2}\cdot\left[(\pi\cdot n)^2(cth + chy) - z\cdot Hc\right] \qquad Cc(n) := (\pi\cdot n)^4\cdot(cth\cdot chy)$$

$$Discrim(n) := \sqrt{(Bc(n))^2 - Cc(n)}$$

Two solutions are:

$$pplus := -Bc(Nnum) + Discrim(Nnum) \qquad pminus := -Bc(Nnum) - Discrim(Nnum)$$

$$pplus = -1.177 + 0.796i \qquad pminus = -1.177 - 0.796i$$

$$\Phi plus := \frac{pplus + cth\cdot(Nnum\cdot\pi)^2}{pplus - pminus}\cdot\left[\Phi 0 - \frac{\Psi 0}{Hc\cdot z}\cdot\left[pminus + cth\cdot(Nnum\cdot\pi)^2\right]\right]$$

$$\Phi minus := \Phi 0 - \Phi plus$$

$$\Psi plus := \frac{Hc\cdot z\cdot\Phi plus}{\left[pplus + cth\cdot(Nnum\cdot\pi)^2\right]} \qquad \Psi minus := \frac{Hc\cdot z\cdot\Phi minus}{\left[pminus + cth\cdot(Nnum\cdot\pi)^2\right]}$$

$$\Psi time(s) := \Psi plus\cdot\exp(pplus\cdot s) + \Psi minus\cdot\exp(pminus\cdot s)$$

$$\Phi time(s) := \Phi plus\cdot\exp(pplus\cdot s) + \Phi minus\cdot\exp(pminus\cdot s)$$

$$\Psi anal(y,s) := \Psi time(s)\cdot\sin(Nnum\cdot\pi\cdot y) \qquad \Phi anal(y,s) := \Phi time(s)\cdot\sin(Nnum\cdot\pi\cdot y)$$

Explicit Finite Difference Scheme

$$nypts := 100 \qquad ky := 0..nypts \qquad yL := 1.0 \qquad deltay := \frac{yL}{nypts}$$

$dtmin1 := 0.5 \cdot \frac{deltay^2}{chy}$ $\qquad$ $dtmin2 := 0.5 \cdot \frac{deltay^2}{cth}$

Time step must be smaller than dtmin1 and dtmin2 for convergence.

$$
\mathrm{Sol2}(ntpts, deltat) :=
\begin{array}{|l}
Rth \leftarrow cth \cdot \frac{deltat}{deltay^2} \\
Rhy \leftarrow chy \cdot \frac{deltat}{deltay^2} \\
\text{error("Rth is greater than 0.5")} \ \text{ if } Rth \ge 0.5 \\
\text{error("Rhy is greater than 0.5")} \ \text{ if } Rhy \ge 0.5 \\
\text{for } ky \in 0..nypts \\
\quad \begin{array}{|l}
P_{ky,0} \leftarrow P0(ky \cdot deltay) \\
T_{ky,0} \leftarrow T0(ky \cdot deltay)
\end{array} \\
\text{for } kt \in 0..ntpts \\
\quad \begin{array}{|l}
P_{0,kt} \leftarrow 0 \\
T_{0,kt} \leftarrow 0 \\
P_{nypts,kt} \leftarrow 0 \\
T_{nypts,kt} \leftarrow 0 \\
\text{for } ky \in 1..nypts-1 \\
\quad \begin{array}{|l}
T_{ky,kt+1} \leftarrow T_{ky,kt} + deltat \cdot Hc \cdot z \cdot P_{ky,kt} + Rth \cdot \left(T_{ky-1,kt} - 2 \cdot T_{ky,kt}\right. \\
P_{ky,kt+1} \leftarrow P_{ky,kt} + T_{ky,kt+1} - T_{ky,kt} + Rhy \cdot \left(P_{ky-1,kt} - 2 \cdot P_{ky,kt}\right.
\end{array}
\end{array} \\
\begin{pmatrix} P \\ T \end{pmatrix}
\end{array}
$$

Compare Analytical and Numerical Solutions

$L \equiv 1$ $\qquad$ $T \equiv 3$ $\qquad$ $spacepts \equiv 100$ $\qquad$ $timepts \equiv 100$ $\qquad$ $Nnum \equiv 12$

$y := 0, 0.01 .. L$ $\qquad$ $t := 0, 0.01 .. T$

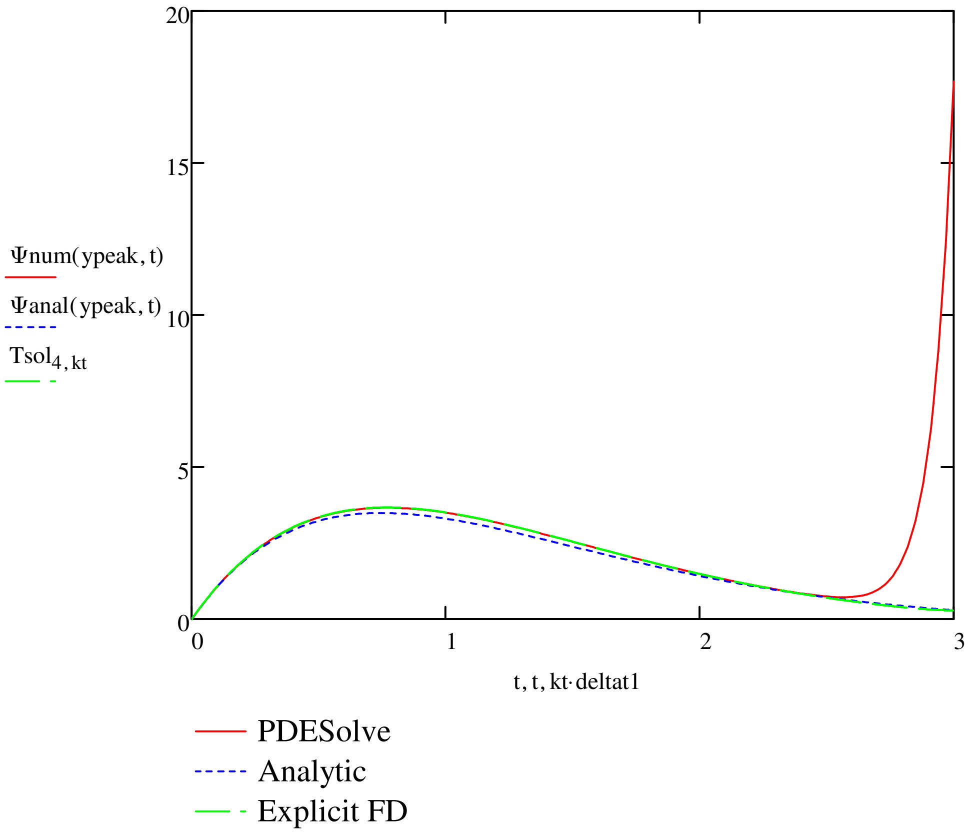
$deltat1 := 0.00125$

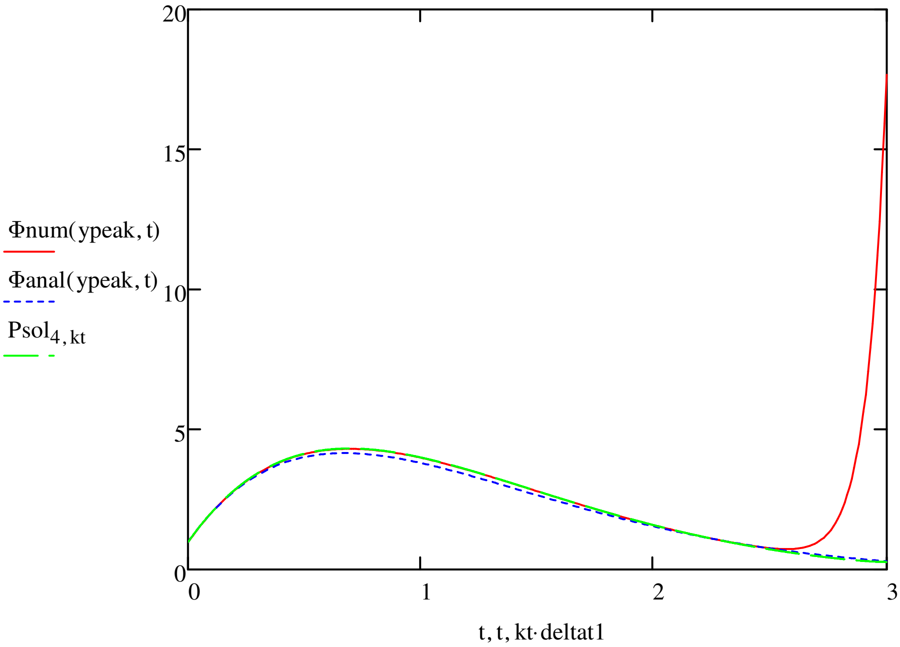
$ntpts := 2400$ $\quad kt := 0..ntpts$ $\quad Tfinal := deltat1 \cdot ntpts$

$SOL := Sol2(ntpts, deltat1)$ $\quad Psol := SOL_0$ $\quad Tsol := SOL_1$

Compare time history at $ypeak := 0.04$. This is near the first peak in the sine wave, i.e., at

$$\frac{1}{2 \cdot Nnum} = 0.042$$





PDESolve
Analytic
Explicit FD

Note agreement of analytical and finite difference solutions, but divergence of the PDESolve solution as the time approaches 3. Increasing the number of time points in PDESolve does not seem to mitigate the problem (I have tried the same number of points as in the FD scheme).