Signals and Systems Using Mathcad (Tutorial) by Derose and Veronis.
Chapter 2 Time Domain Analysis
Entered by: Karl S Bogha Dhaliwal
Part 2 of 2

Example 20 Convolution Step Function

$u(t) := \Phi(t)$ step funtion

$x(t) := u(t)$

$h(t) := e^{-t} \cdot u(t)$

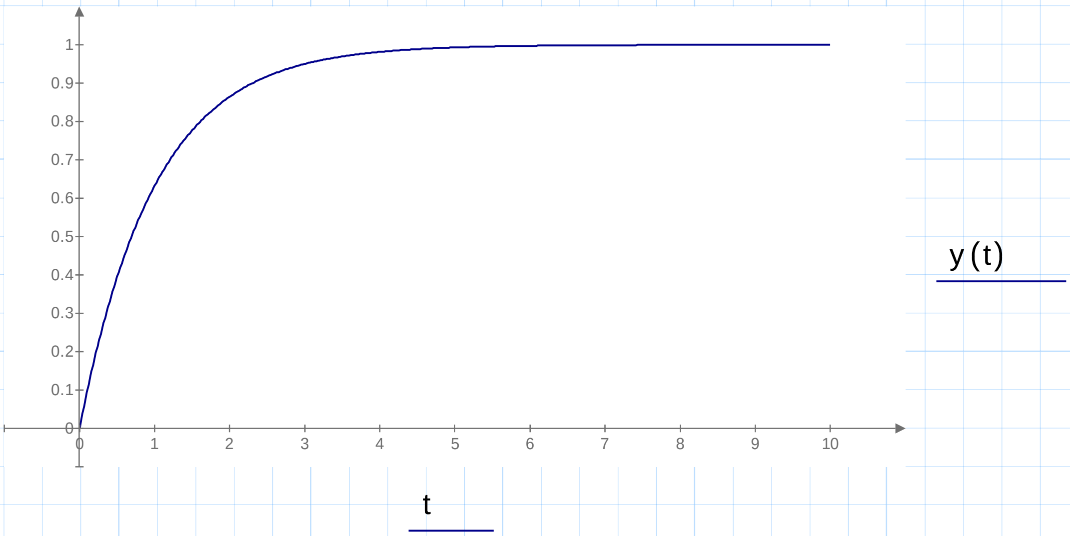
$$y(t) := \int_0^t x(\tau) \cdot h(t-\tau)\, d\tau$$

$y(t) \rightarrow 1 - e^{-t}$ result computed by Mathcad

Going back to page 75-77 the function u(t) is included so the answer is:

y(t) = (1-e^-t)u(t) answer

This result can also be plotted:



Using eq 2.17 as shown below:

$$y(t) := \int_0^t h(\tau) \cdot u(t-\tau)\, d\tau$$ equation 2.17

$y(t) \rightarrow 1 - e^{-t}$

so y(t) = 1 - (e^-t)u(t) answer

Lets describe the process of Convolution:

1. idea is to multiply two signals x(t) and h(t) as described in previous example
2. fold one of the signals ie h(t) (not easily understood, look at symmetrical signal)
3. then shift h(t) for different values of t - ie something like (tau)...(tau - 2)...(tau - 6)..
4. then multiply to x(t)
5. calculate-compute to get the total area ie the result

Not easy to visualise, see the sketches below.
Refer to engineering mathematics textbook.

A convolution is an integral that expresses the amount of overlap of one function g as it is shifted over another function f. It therefore "blends" one function with another. The convolution is sometimes also known by its German name, *faltung* ("folding").

$$[f * g](t) \equiv \int_0^t f(\tau)\, g(t-\tau)\, d\tau,$$

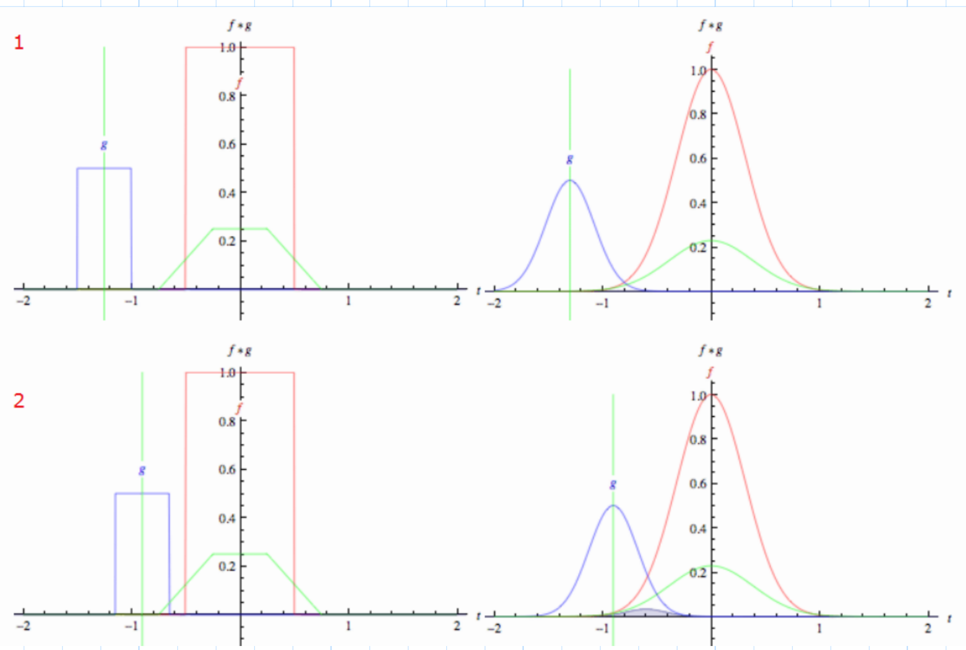
where the symbol $[f * g](t)$ denotes convolution of f and g.

Convolution is more often taken over an infinite range,

$$f * g \equiv \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} g(\tau)\, f(t-\tau)\, d\tau$$

The frames below graphically illustrate the convolution of two boxcar functions (left) and two Gaussians (right). In the plots, the green curve shows the convolution of the blue and red curves as a function of t, the position indicated by the vertical green line. The gray region indicates the product $g(\tau) f(t-\tau)$ as a function of t, so its area as a function of t is precisely the convolution. One feature to emphasize and which is not conveyed by these illustrations (since they both exclusively involve symmetric functions) is that the function g must be mirrored before lagging it across f and integrating.

The grey shaded region is the area of concern.



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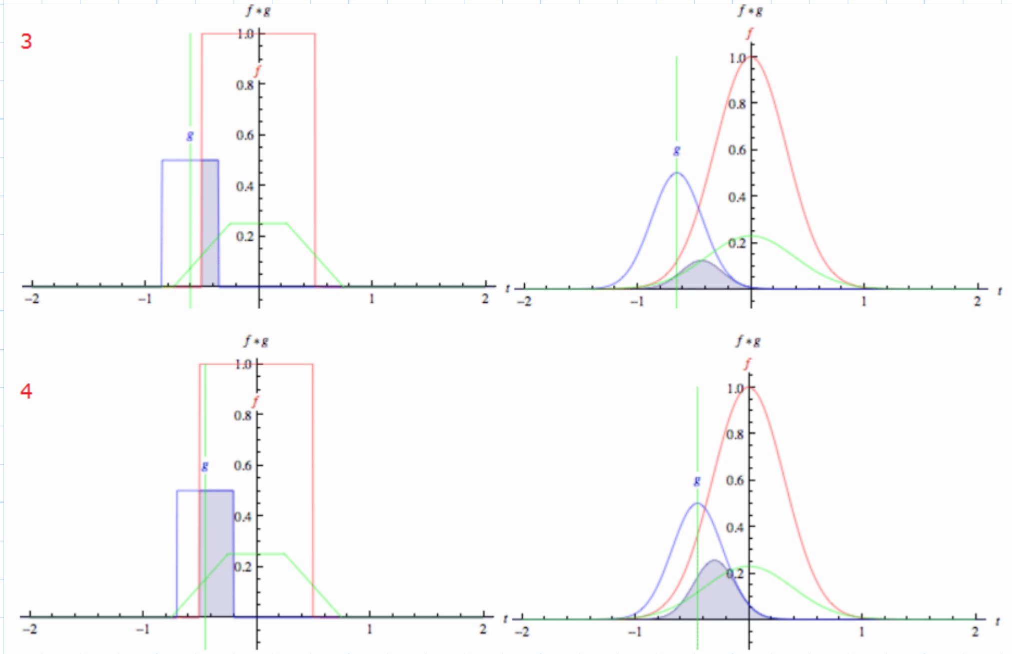

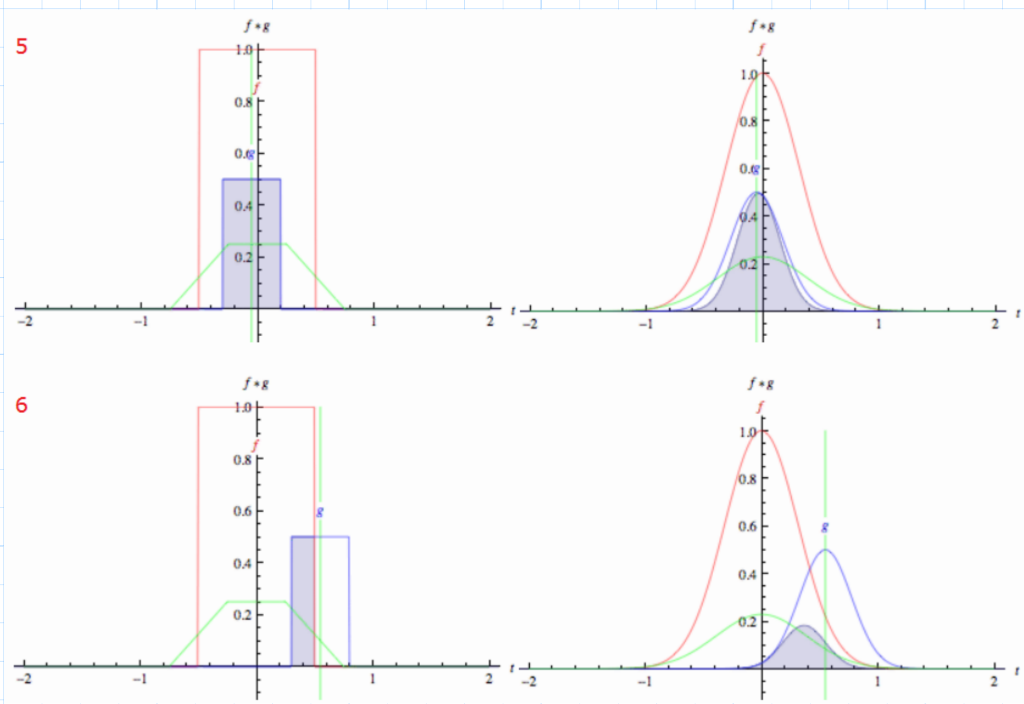

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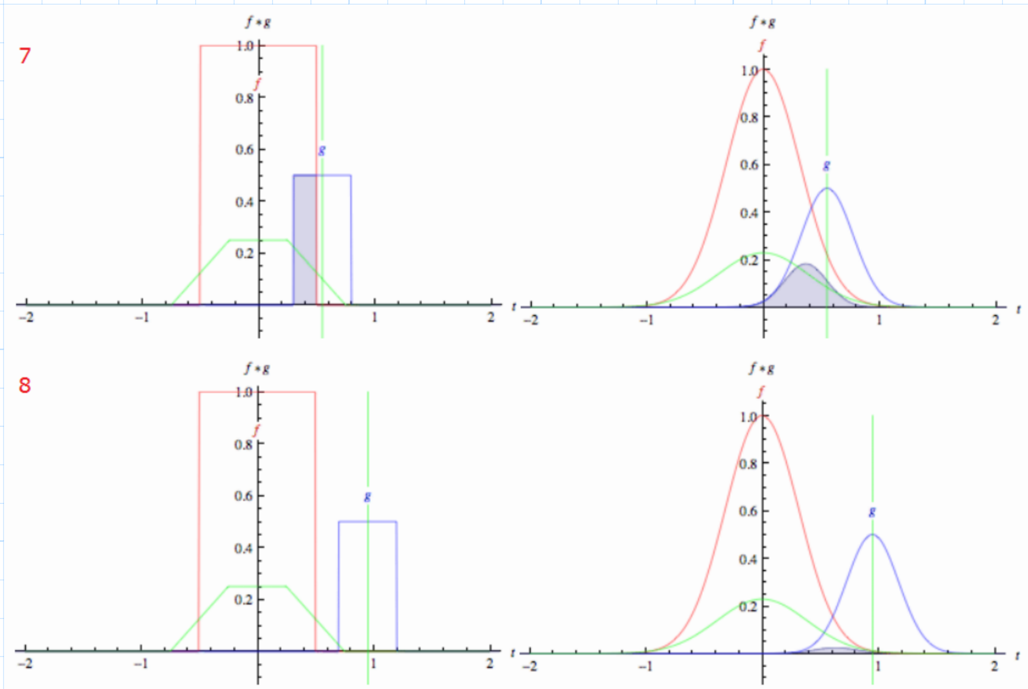
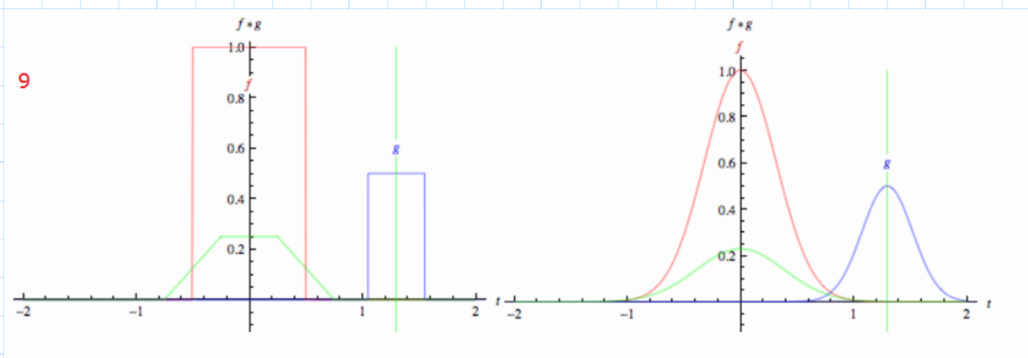




Example 21 Convolution

clear (t) clears the variable t

$u(t) := \Phi(t)$

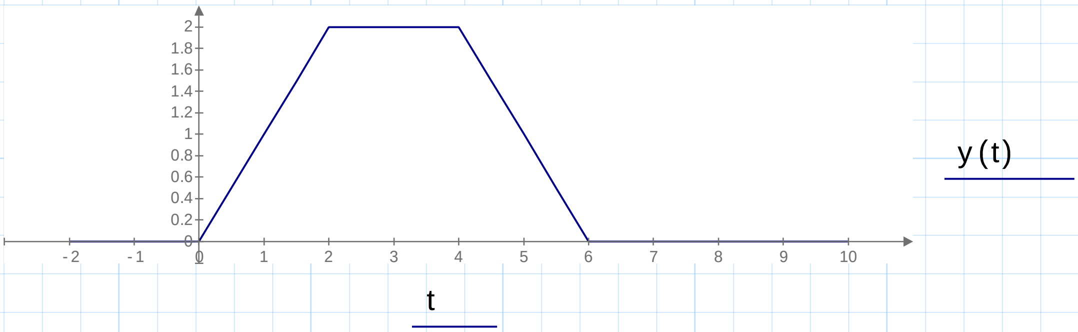
$x(t) := u(t) - u(t-4)$

$h(t) := u(t) - u(t-2)$

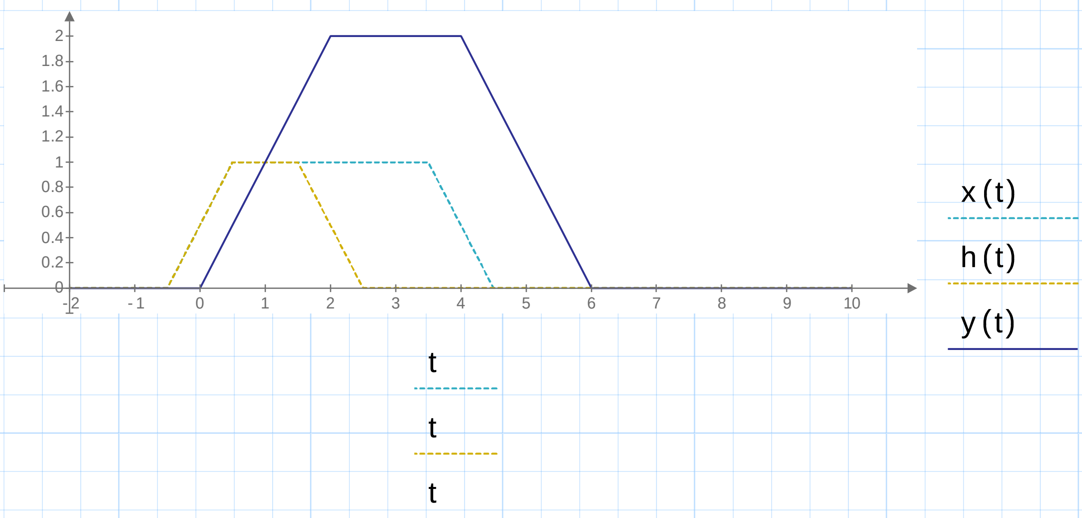
$y(t) := \int_{t-2}^{t} x(\tau) \cdot h(t-\tau)\, d\tau$

For the graphical result:

$t := -2, -1.5 .. 10$



Plots below shows the interaction of x(t) and h(t) to show y(t).



clear(t)

clear (t) - clears the symbolic value of x, otherwise t would be working as it was set in previous example.

If Mathcad's symbolic result was to be obtained, the following expression is what we will get due to the fact that u(t) is not a mathematical function. Add u(t) at the end.

$$y(t) \rightarrow \int_{t-2}^{t} (\Phi(\tau) - \Phi(\tau - 4)) \cdot (\Phi(t - \tau) - \Phi(t - \tau - 2)) \, d\tau$$

Example 2.22 Convolution

$h(t) := t \cdot e^{-t} \cdot u(t)$
$x(t) := u(t)$

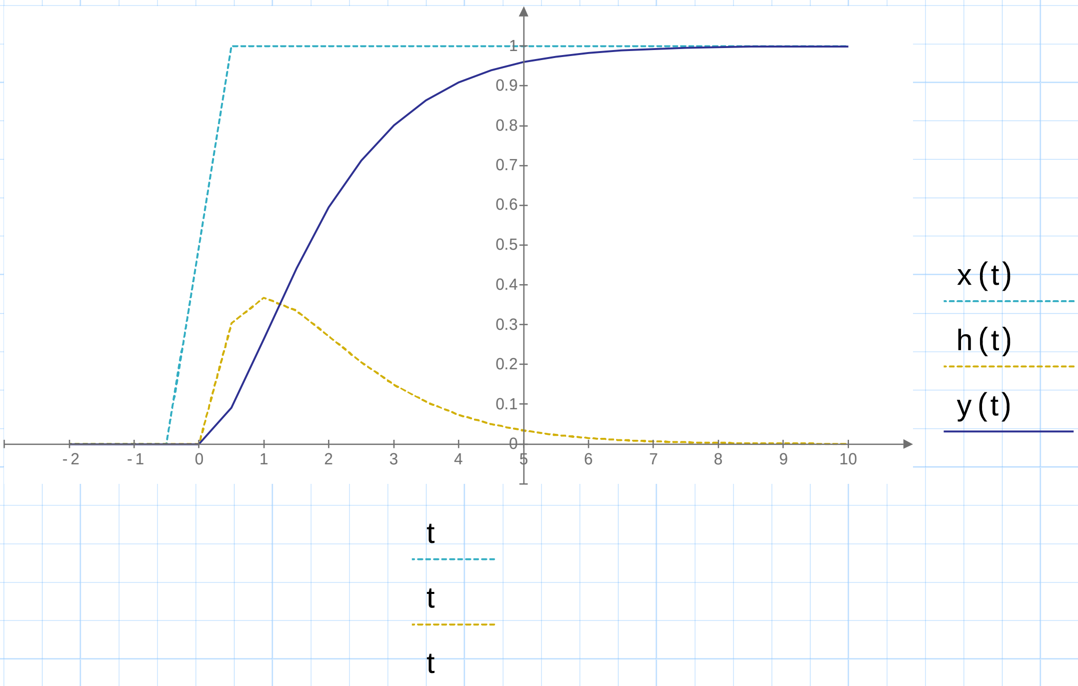
$$y(t) := \int_0^t x(\tau) \cdot h(t-\tau)\, d\tau$$

clear (t)
$y(t) \rightarrow 1 - t \cdot e^{-t} - e^{-t}$

The real answer would be y(t) = [1-t(e^-t) - (e^-t)]u(t)

For the approximate graphical result since u(t) is not included/defined:

$t := -2, -1.5 .. 10$



Example 2.23 Convolution Animation Not Available in Prime 1, 2 and 3.
Its purpose was demonstrated with sketches and multiple plots above.

Example 2.24 Convolution with the delta-impulse function.

Notes:
When a signal convolutes with a delta (impulse) function the resultant signal is shifted towards the direction of the shifted delta-impulse function.

Differenting a step fucntion returns an impulse function
Intergrating a impulse function returns a step function

Example 24-1 - First method
clear(t)
$t := -5, -4.99 .. 5$

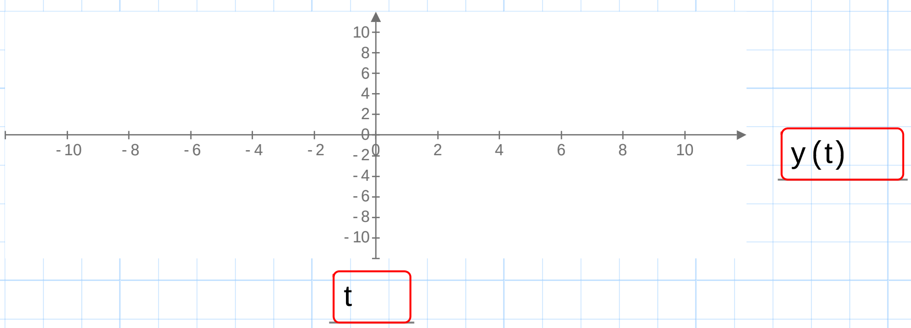
$u(t) := \Phi(t)$ define unit step function

$x(t) := u(t) - u(t-4)$ define input signal

$h(t) := \delta(t)$ show the delta function but not define it

$$y(t) := \int_{t-5}^{t+5} x(\tau) \cdot \Delta(t-\tau)\, d\tau$$

$y(t)$ Calculation is not giving a solution - try to fix it.



Plotting failed. Replace complex values and NaNs by real numbers.

Lets replace h(t) = delta(t-2)
clear(t)
$h(t) := \delta(t-2)$
$t := -7, -6.99 .. 7$

redifine y(t) below and replace t by (t-2) in del(t-tau)

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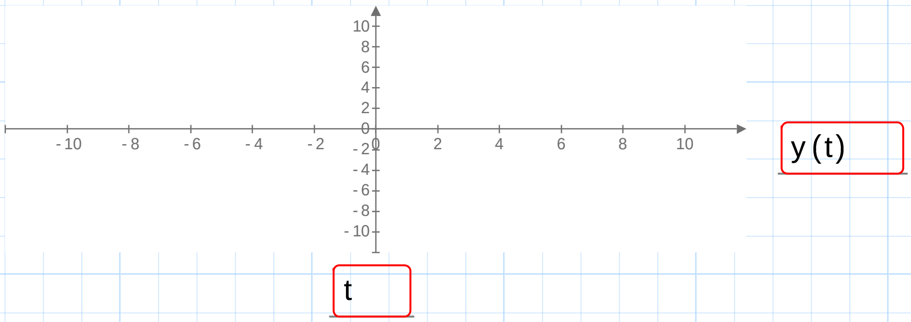
$$y(t) := \int_{t-5}^{t+5} x(\tau)\cdot\Delta(t-2-\tau)\,d\tau$$

$y(t)$ Calculation is not giving a solution - try to fix it.



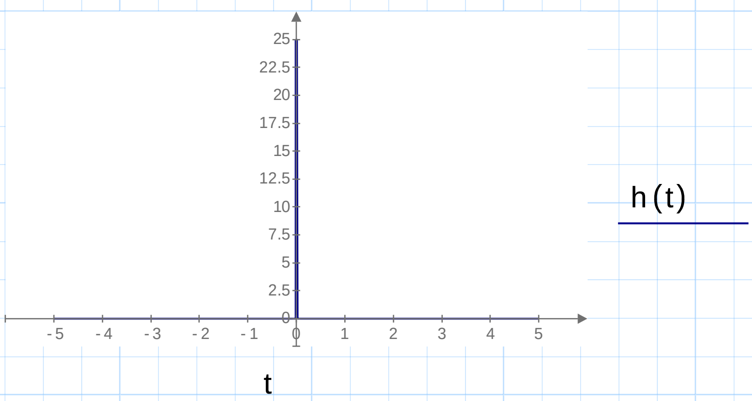
Plotting failed. Replace complex values and NaNs by real numbers.

Second method, since we intergrate the delta function we get 0, (due to the delta function has no area - no range), we must not place the delta function directly in the intergral. We must define a function with an area small enough to give us one.

clear(t)
$t := -5, -4.99 .. 5$
$n := 50$

$$h(t) := \textbf{if}\left(\left(\frac{-1}{n}\right) < t < \left(\frac{1}{n}\right), \left(\frac{n}{2}\right), 0\right)$$

defining h(t) small enough to get an area of unity see plot of function of impulse below.

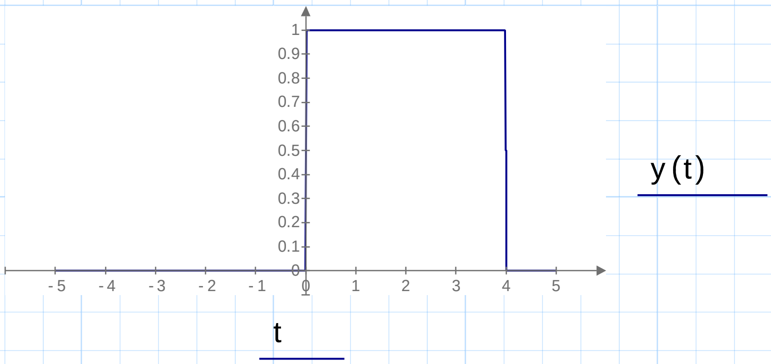


$u(t) := \Phi(t)$ defining the unit step function

$x(t) := u(t) - u(t-4)$ defining the input signal

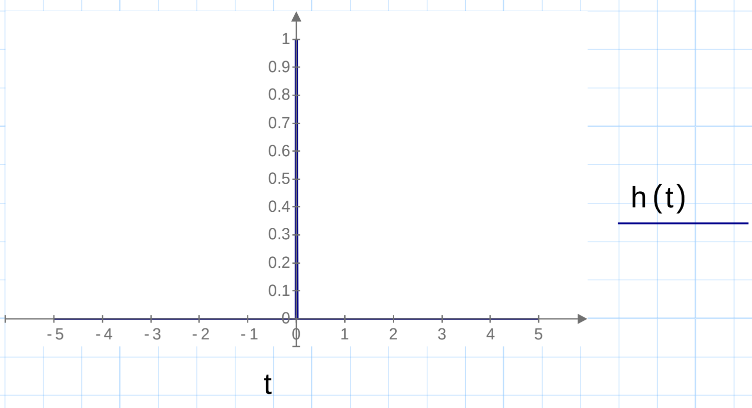
$$y(t) := \int_{t-5}^{t+5} x(\tau) \cdot h(t-\tau)\, d\tau$$

$y(t) \rightarrow ?$ unable to evaluate expression but it does plot the result



The impulse function can be made to unity by dividing by 25 (n/2) - scaling

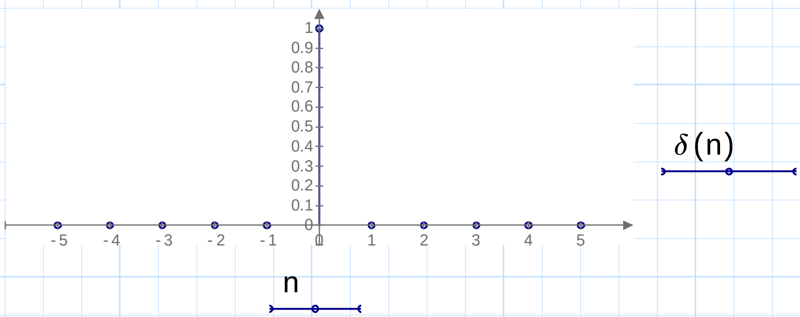
$$h(t) := \frac{h(t)}{\left(\frac{n}{2}\right)}$$



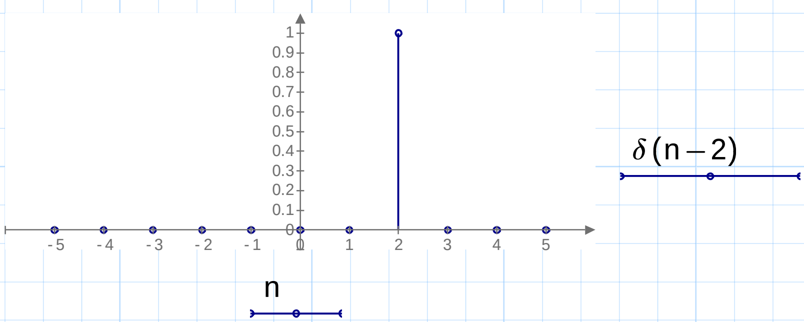
The above example no 24 in the second method, successful one, showed that when we convolute a signal with delta function, the result of the convolution is identical to the original signal shifted towards the direction of the delta function.
It moved to the right by 4. (t-4=0, so t=4) positive 4 to the right.

Example 2.26

$n := -5..5$
$\delta(n) := \textbf{if}(n = 0, 1, 0)$



Delayed by n=2, so when n = 2, n-2=0, this is when delta = 1.



Example 2.27 Discrete Time Systems

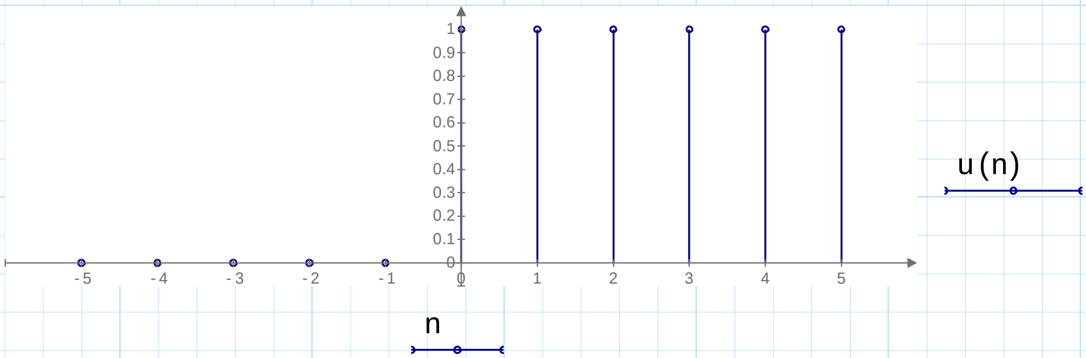
clear(n)

$n := -5..5$

$\delta(n) := \textbf{if}(n = 0, 1, 0)$

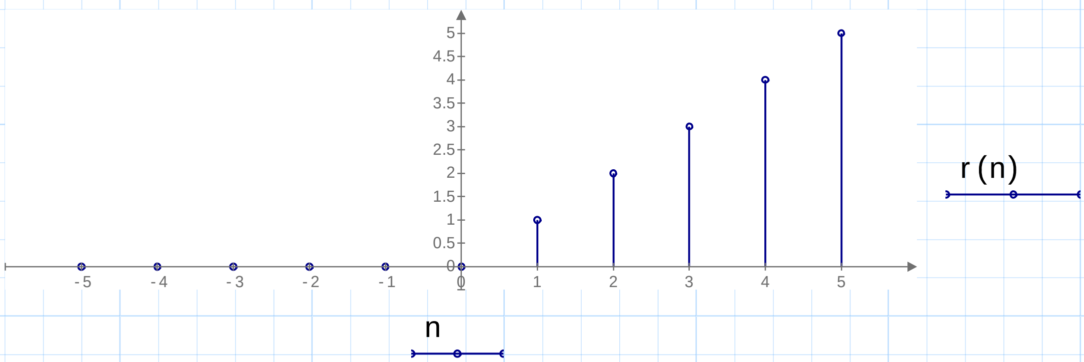
$u(n) := \delta(n)$

$$u(n) := \sum_{k=0}^{5} \delta(n-k)$$



Unit Ramp Function:

$r(n) := n \cdot u(n)$



Example 2.28 Mathcad/Prime Convolution function - convolve(,)

clear (n)

Origin := 0

$n := 0..5$

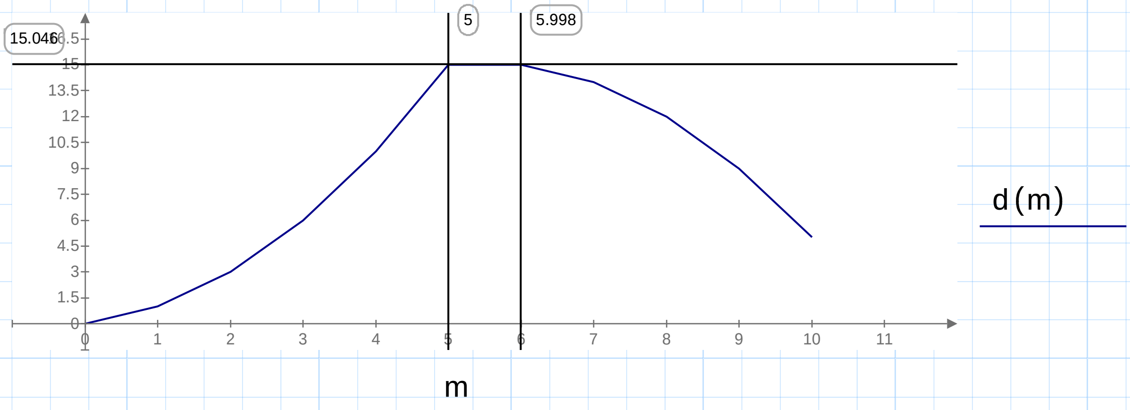
$x_n := 1$ $y_n := n$

$d := \text{convolve}(x, y)$ convolution(a,b)

$d^T = [1.026 \cdot 10^{-15} \;\; 1 \;\; 3 \;\; 6 \;\; 10 \;\; 15 \;\; 15 \;\; 14 \;\; 12 \;\; 9 \;\; 5]$ transpose matrix: ctrl + shift + T

$M := \text{length}(x) + \text{length}(y - 1)$

$m := 0..M - 1$



Example 2.29 Discrete time convolution

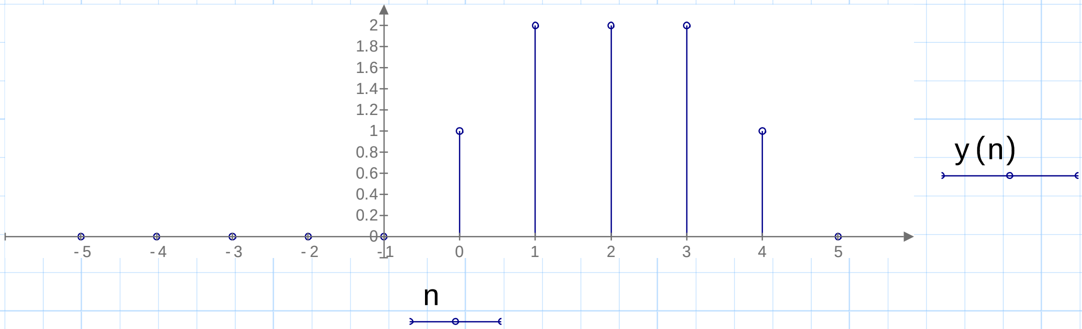
clear (n)

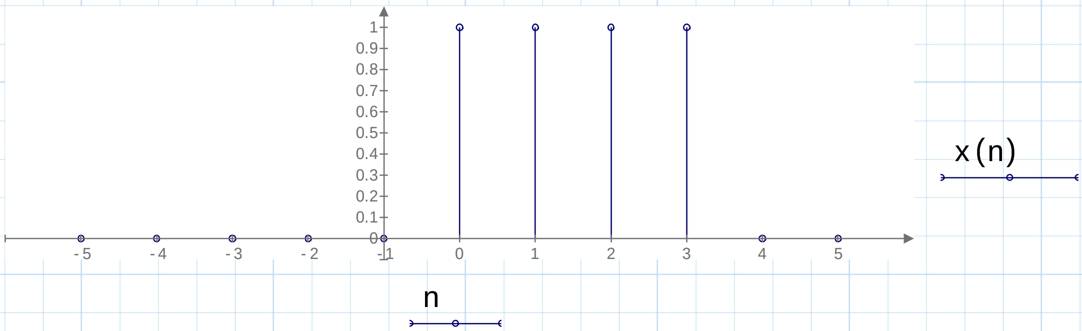
$n := -5..5$

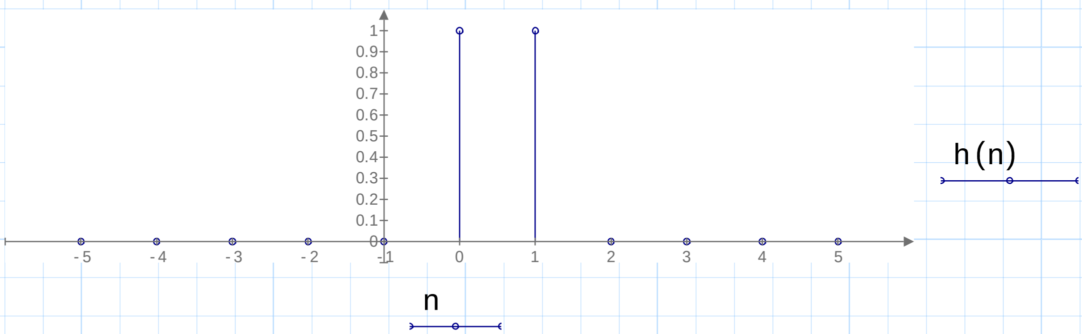
$x(n) := u(n) - u(n - 4)$ input signal

$h(n) := u(n) - u(n - 2)$ impulse function

$y(n) := \sum_{k=n-2}^{n} x(k) \cdot h(n - k)$ output or the convolution sum


y(n)
n

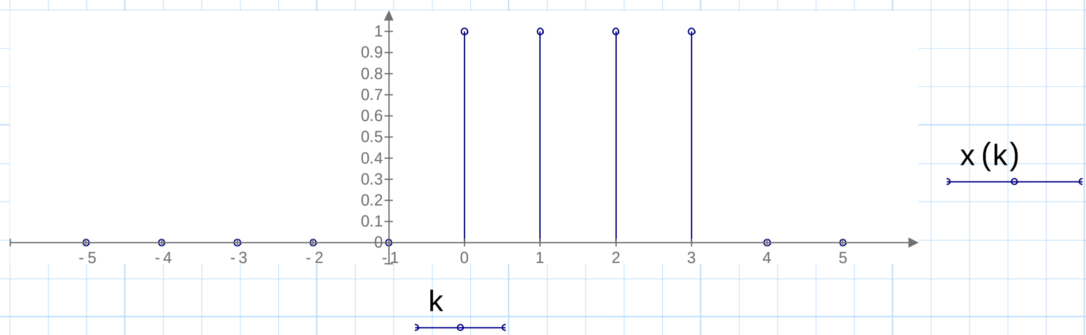

x(n)
n


h(n)
n

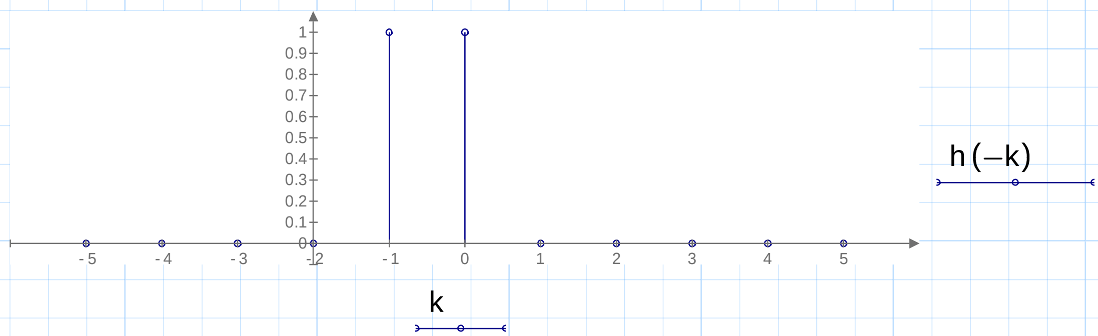
Xk is the signal we will multiply each specific time (in y(n)), it is worthwhile to show it here.
The figure below shows x(n) at n=k

$k := n$

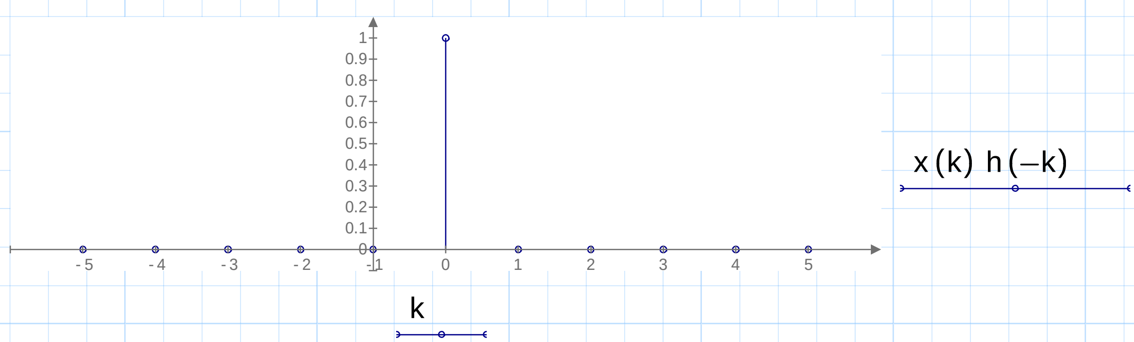
$x(n) := u(n) - u(n-4)$ x(n) at n = k



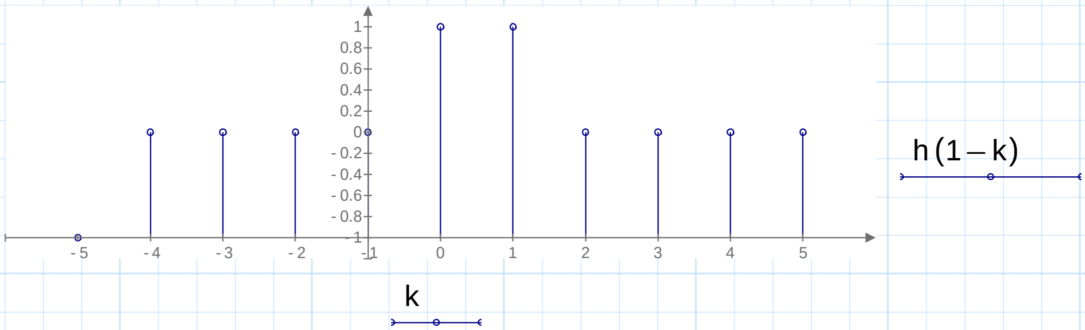
h(n-k) at n = 0, so h(-k)



at n = 0, x(k)h(-k) = 1
product of the input and the shifted impulse for 0 unit

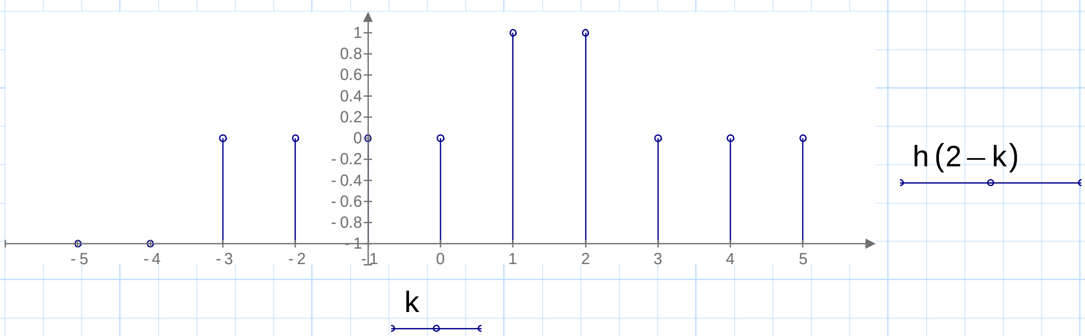


h(n-k) at n = 1



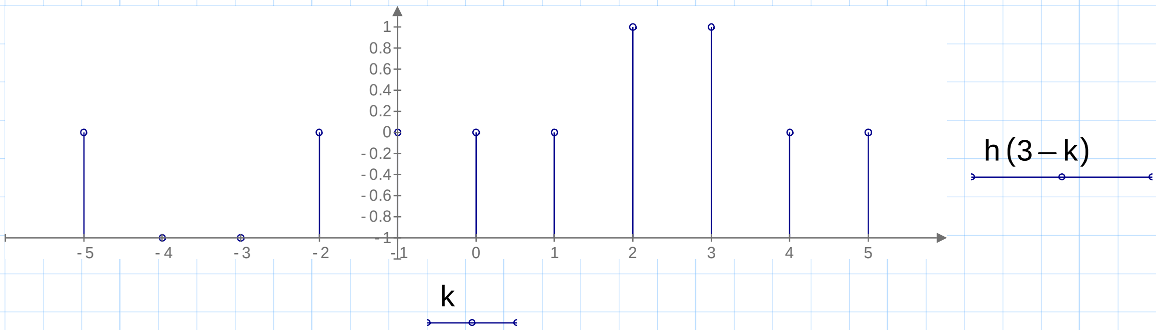
at n=1, x(k)h(1-k)=2
product of the input and shifted impulse for 1 unit

h(n-k) at n = 2

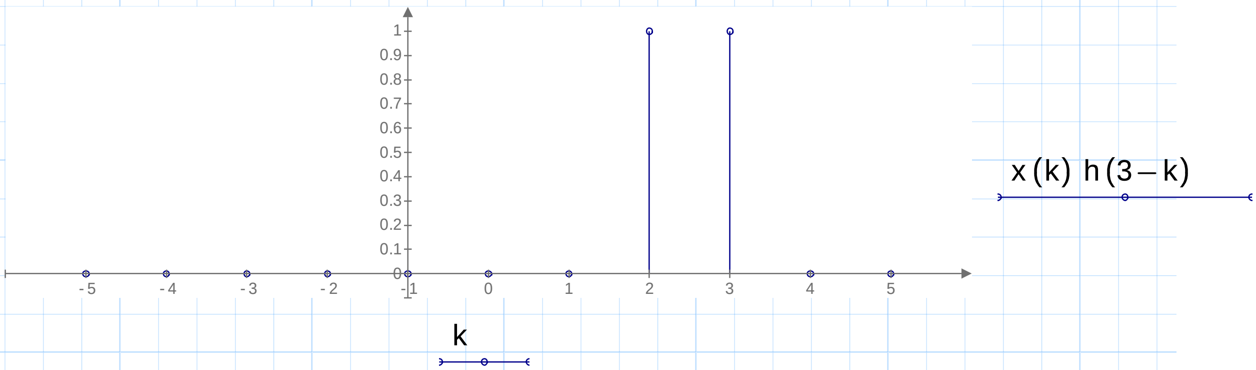


at n = 2, x(k)h(2-k)=2
product of the input and the shifted impulse for 2 units

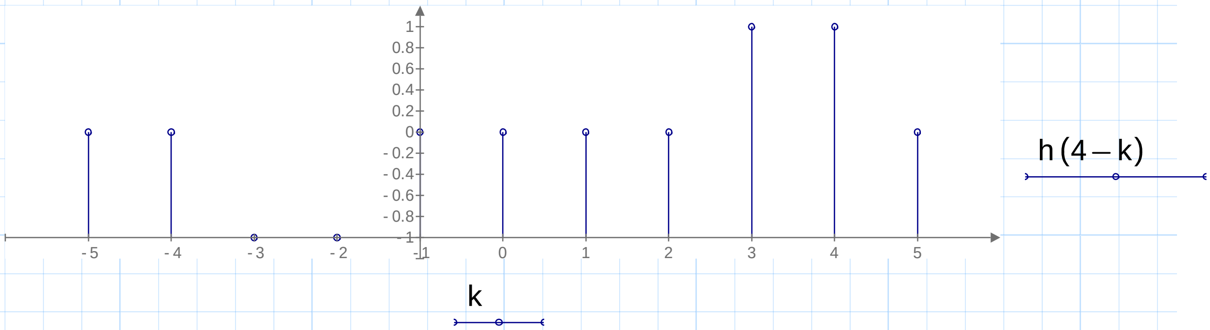
h(n-k) at n = 3



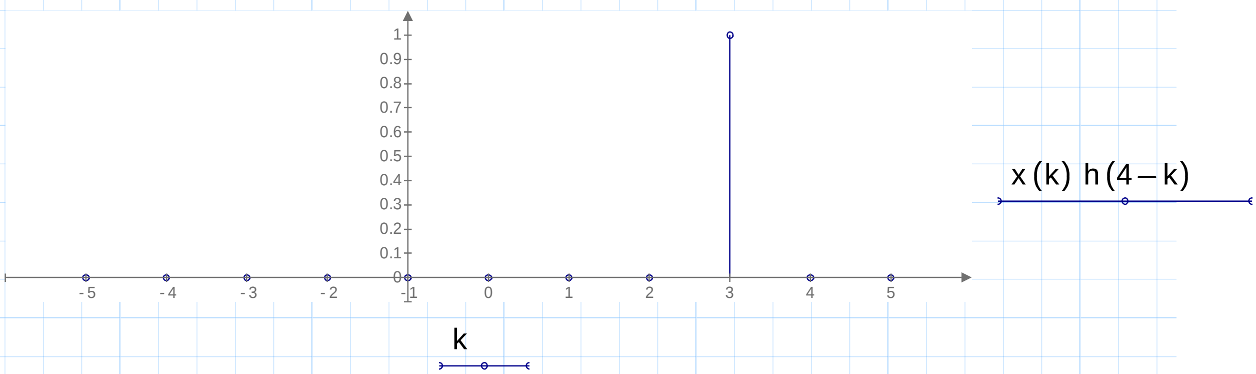
at n = 3, x(k)h(3-k)=2
product of the input and the shifted impulse for 3 units



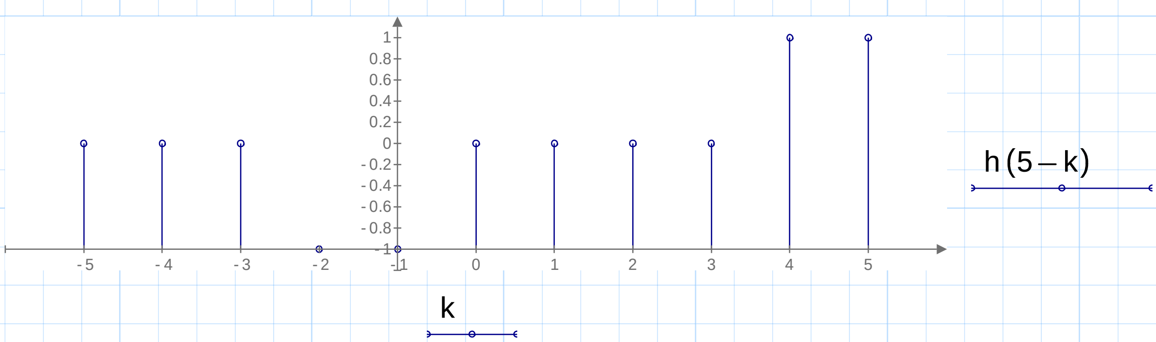
h(n-k) at n = 4



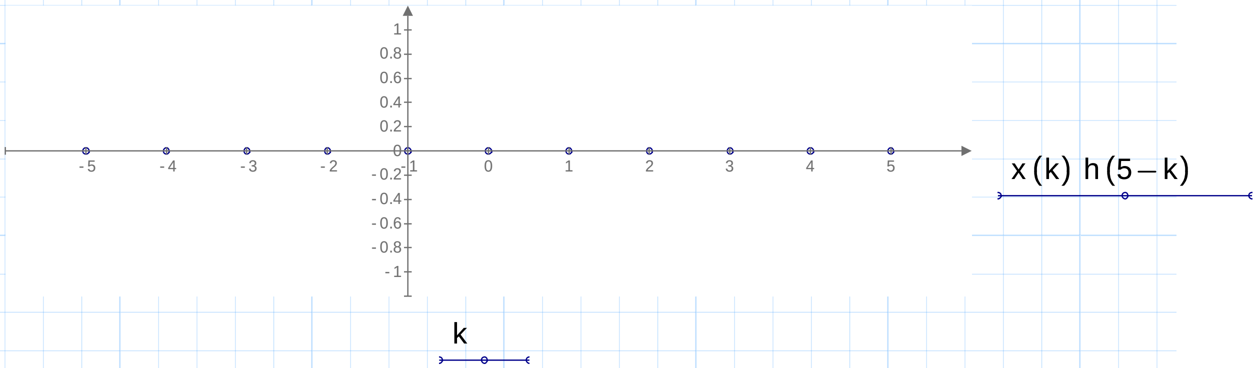
at n = 4, x(k)h(4-k)=1
product of the input and the shifted impulse for 4 units



h(n-k) at n = 5



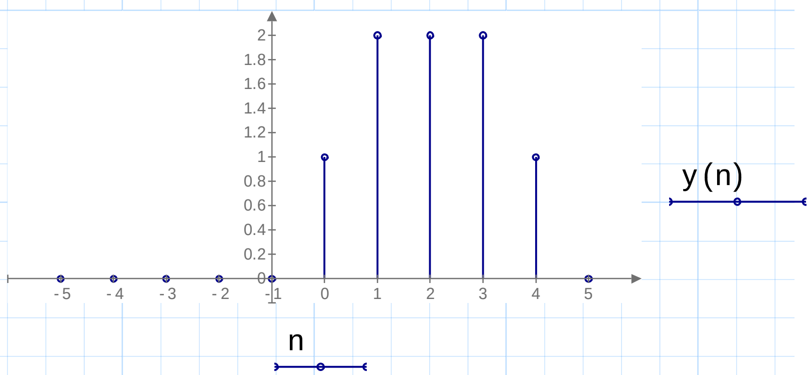
at n = 5, x(k)h(5-k)=0
product of the input and the shifted impulse for 5 units



By adding the value of x(k)h(n-k) from n=0 to n=5, we can plot the result of the convolution y(k), by tabulating the vlaue.

As shown below, we place the values directly in a table to plot them.

n	y
0	1
1	2
2	2
3	2
4	1
5	0



The result of the convolution shown in plot.

Example 2.30 Mathcad/Prime symbolic result for digital convolution

clear $(n) := 0$

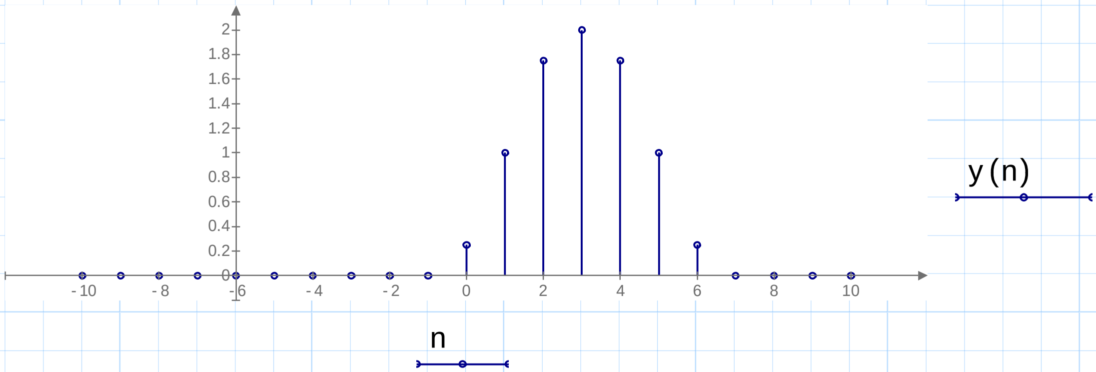
$n := -10 .. 10$

$u(n) := \Phi(n)$

$x(n) := u(n) - u(n-4)$

$h(n) := u(n) - u(n-2)$

$y(n) := \sum_{k=n-2}^{n} x(k) \cdot h(n-k)$ Convolution using summation method



Plot above did not match results of example 29

clear (n)

$n := 0..5$ Origin := 0

$u(n) := \Phi(n)$

$x_n := u(n) - u(n-4)$ input signal

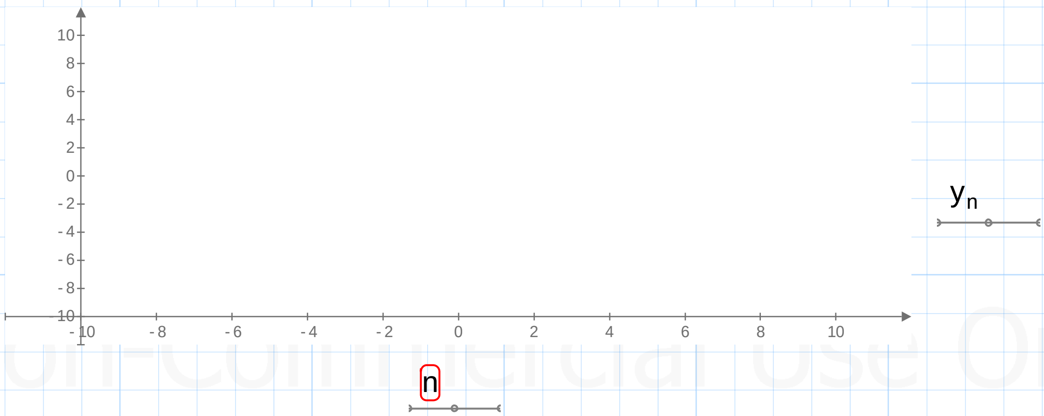
$h_n := u(n) - u(n-2)$ impulse function

$y_n := \text{convolve}(x, h)$

$y_{n\_transp} := y_n^T$

$y_{n\_transp} = [0\ 1\ 2\ 2\ 2\ 1\ 0\ 0\ 0\ 0\ 0]$

The magnitude of the result match but plotting the function is not achieved. The transposed values revealed more values and were in complex format. So there maybe imitations to using the builtin convolve function.



Example 31 Convolution with Impulse function

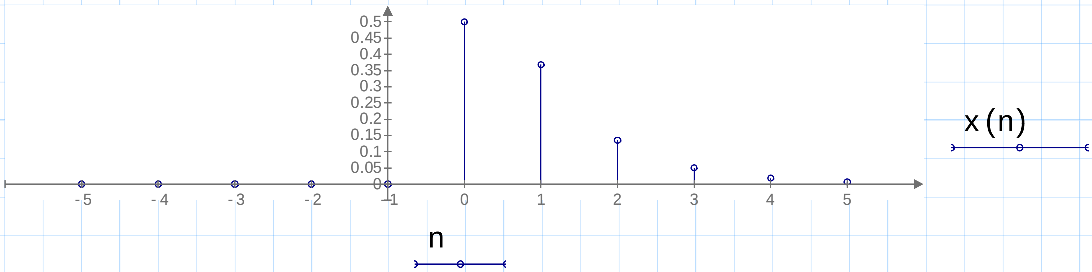
clear(n)

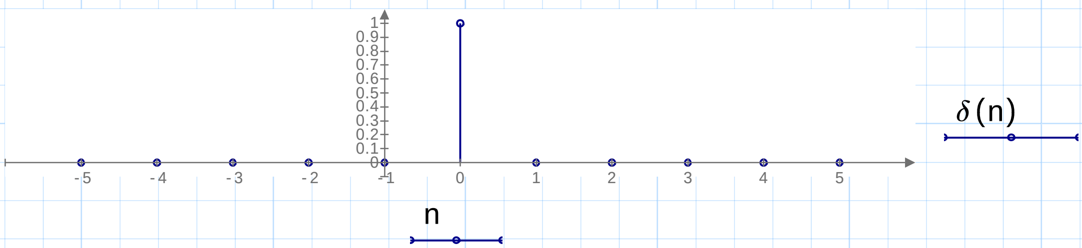
$u(n) := \Phi(n)$

$x(n) := \left(e^{-n}\right) \cdot u(n)$ unit step function

$n := -5, -4 .. 5$ set the range

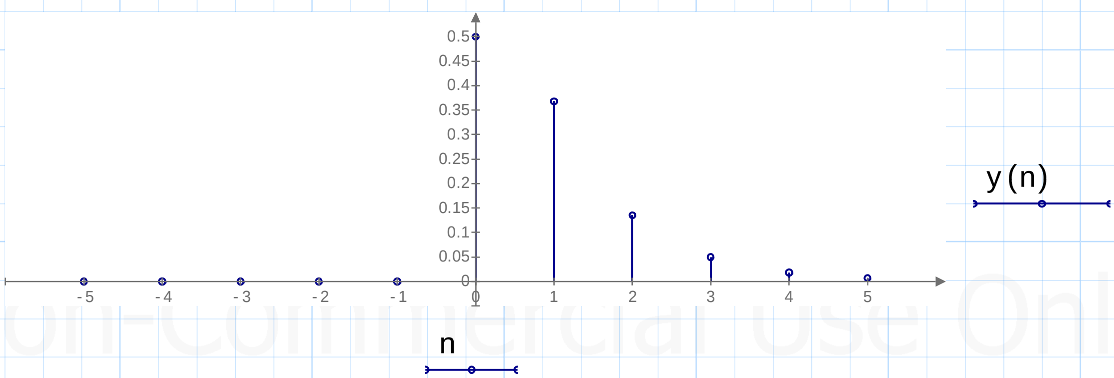
$\delta(n) := \textbf{if}(n = 0, 1, 0)$ defining the impulse function - which is the response





Now using the summation method to define the convolution:

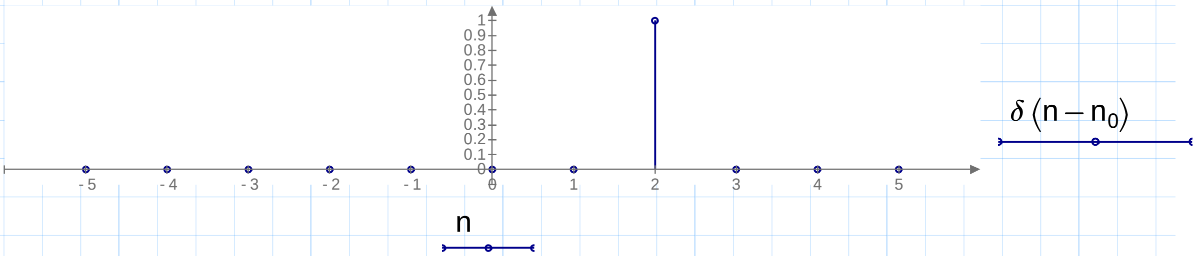
$$y(n) := \sum_{k=0}^{n} x(k) \cdot \delta(n-k)$$

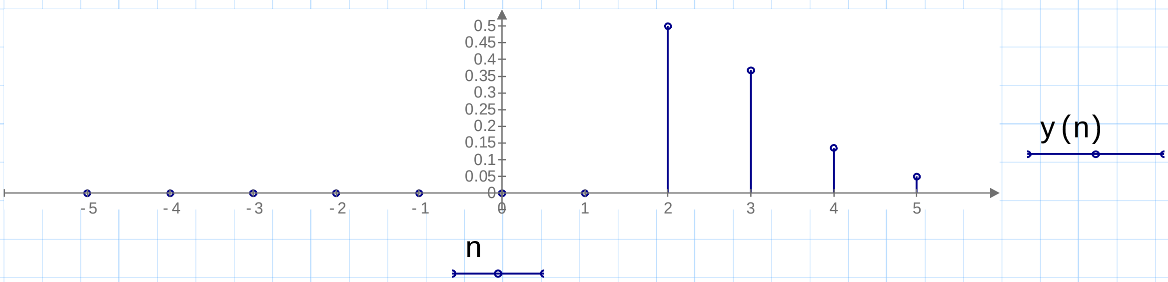


Lets shift the impulse response by a factor of n0, where n0 can take any value:

$n_0 := 2$

$$y(n) := \sum_{k=0}^{n} x(k) \cdot \delta(n - k - n_0)$$





The shifted impulse and the output result (above 2 plots).

Note:

We see clearly that a signal xn convolutes with hn, where hn is equal to delta_n returning xn.

The same if hn = delta(n-n0) results in xn-x0.

Section 2.4.10 Systems described by difference quations.
Pages 104-106.
Feedback signals content for discrete signals.
Figures 2.85-2.87 explains the ideas-theory.

Example 2.32 Mathcad implementation of difference equations

$Origin := -1$

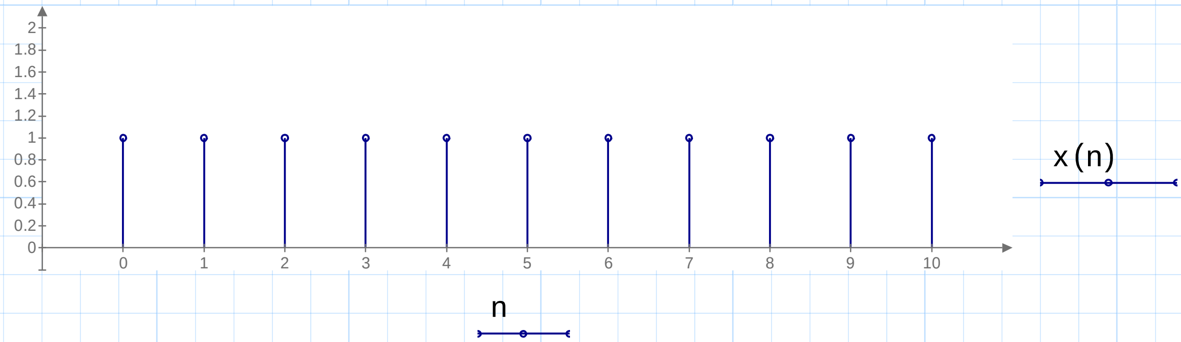
If in Prime for this example, the results dont look accurate set the Origin to 0 or -1 in the Calculation tab. One setting of origin is for the whole work sheet.

clear (n)

$n := 0, 1 .. 10$

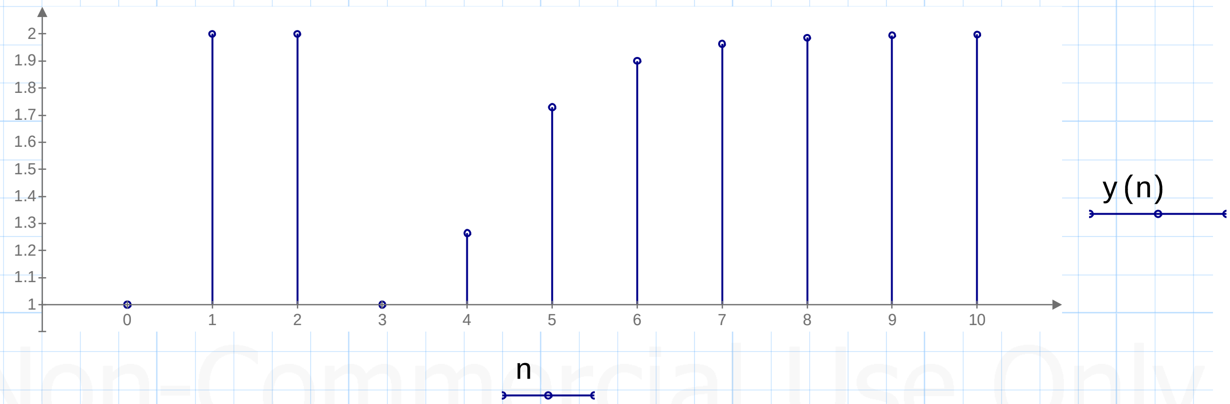
$u(n) := \mathbf{if}(n \geq 0, 1, 0)$

$x(n) := u(n)$



Check: If we set the initial condition when n= -1 equal zero --> This is done automatically when we set n=-1 for the starting calculation, otherwise we set n = 0,1..10. Above n was set as n = -1,0..10.
- this can be the case when the system had not started at time -1s

$y(n) := x(n) + x(n-1) - 2 \cdot y(n-1)$ Defining the difference equation of the system

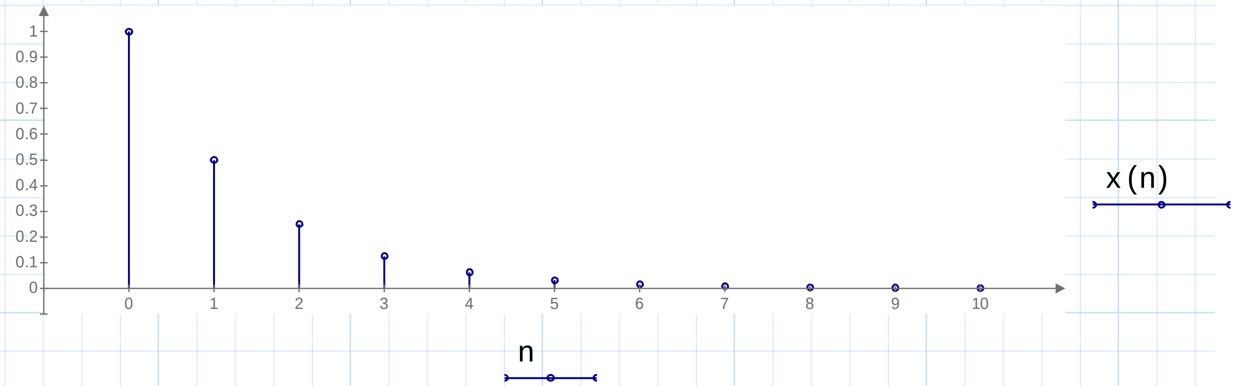


Example 2.33

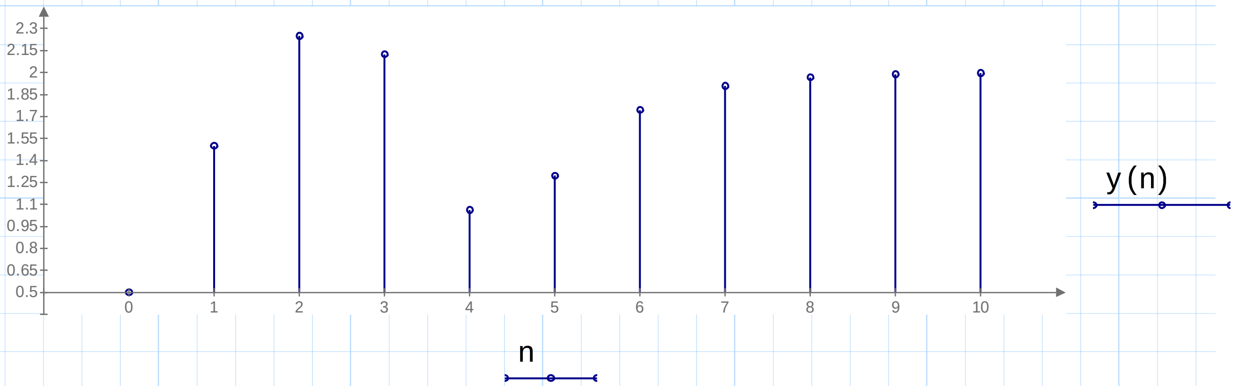
Continuing using the functions of example 2.32

$$x(n) := \left(\frac{1}{2}\right)^{n} \cdot u(n)$$

$$y(n) := \left(\frac{1}{2}\right) \cdot x(n) + \left(\frac{1}{4}\right) \cdot x(n-1) + y(n-1)$$



The input of the system



The response of system