Tables from Linear Systems and Signals 2nd ed by B.P. Lathi.

TABLE 4.1 A Short Table of (Unilateral) Laplace Transforms

No.	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$
9a	$e^{-at}\cos bt\, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at}\sin bt\, u(t)$	$\frac{b}{(s+a)^2+b^2}$
10a	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta - br\sin(\theta))}{s^2+2as+(a^2+b^2)}$
10b	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb}+\frac{0.5re^{-j\theta}}{s+a+jb}$
10c	$re^{-at}\cos(bt+\theta)\, u(t)$ $r=\sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$ $\theta=\tan^{-1}\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right)$ $b=\sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$
10d	$e^{-at}\left[A\cos bt+\frac{B-Aa}{b}\sin bt\right]u(t)$ $b=\sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$

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TABLE 4.2 The Laplace Transform Properties

Operation	$x(t)$	$X(s)$
Addition	$x_1(t) + x_2(t)$	$X_1(s) + X_2(s)$
Scalar multiplication	$kx(t)$	$kX(s)$
Time differentiation	$\dfrac{dx}{dt}$	$sX(s) - x(0^-)$
	$\dfrac{d^2x}{dt^2}$	$s^2X(s) - sx(0^-) - \dot{x}(0^-)$
	$\dfrac{d^3x}{dt^3}$	$s^3X(s) - s^2x(0^-) - s\dot{x}(0^-) - \ddot{x}(0^-)$
	$\dfrac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$
Time integration	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(t)\,dt$
Time shifting	$x(t-t_0)u(t-t_0)$	$X(s)e^{-st_0}$ $\quad t_0 \geq 0$
Frequency shifting	$x(t)e^{s_0t}$	$X(s-s_0)$
Frequency differentiation	$-tx(t)$	$\dfrac{dX(s)}{ds}$
Frequency integration	$\dfrac{x(t)}{t}$	$\int_{s}^{\infty} X(z)\,dz$
Scaling	$x(at),\ a \geq 0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi j}X_1(s) * X_2(s)$
Initial value	$x(0^+)$	$\lim_{s\to\infty} sX(s)$ $\quad (n > m)$
Final value	$x(\infty)$	$\lim_{s\to 0} sX(s)$ $\quad$ [poles of $sX(s)$ in LHP]

Notes from textbook:

3.11 The Laplace Transform

We have seen earlier that the output of a Linear Time Invariant (LTI) system can be expressed as the convolution of the input with the impulse response as show on the block diagram of Figure 3.29. This assumption is made provided that the convolution is expressed always in the time domain. The Laplace transform, on the other hand, converts the impulse response of the system from the time domain to the frequency domain.

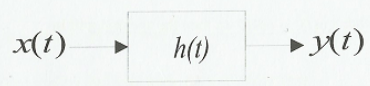


Figure 3.29: *A typical LTI system*

The Laplace Transform of a system can be calculated using the following formula; if $h(t)$ is the impulse response of a system, the Laplace Transform of $h(t)$ can be expressed from this formula.

$$H(s)=\int_0^{\infty} h(t)e^{-st}dt \qquad \text{(Equ.3.9)}$$

For a given *H(s)*, the Inverse Laplace Transform can be evaluated also by partial fraction expansion.

Example 3.9

$j := \sqrt{-1}$ $\qquad \omega := 2\cdot\pi$ $\qquad i := j$

$s := j\cdot\omega$

$a := 2$ 'a' is a constant
it is the a=1 in the solution's denominator (s+a) or (s+2)

$u(t) := \text{if}(|t| \ge 0, 1, 0)$ unit step function

$h(t) := e^{-a\cdot t}\cdot u(t)$ d(t) delta function is a Prime function

$$H(s) := \int_0^{\infty} h(t)\cdot e^{-s\cdot t}\,dt \rightarrow \frac{1}{s+2}$$ Laplace solution

$$\int_0^{\infty} h(t)\cdot e^{-s\cdot t}\,dt \rightarrow \frac{1}{2+2i\cdot\pi}$$ s here is substituted by 2 i pi, in the integral solution

$H(s) = 0.046 - 0.144513j$ When s = j 2 pi

Example 3.10 - Inverse Laplace Transform

$H(s) := \frac{2}{s+2}$

Now to go back to the time domain solution for the inverse

The Lapalce transofrm looks like the solution in example 3.9, by inspection it looks like the H(s) is equal to

H(s) = 2 (1/s+2) => 2 (1/s + a)

So the time domain inverse Laplace is

2 e^-2t u(t)

$H(s) := \frac{2}{s+2} \xrightarrow{\text{invlaplace}} 2 \cdot e^{-2 \cdot t}$

Correct Answer.
Use the Evaluation Operator and fill in the label invlaplace

Example 3.11 - Poles and Zeros

clear(j)   clear(i)

Poles and zeros for system stability see textbook on systems

A system's transfer function is given below H(s) plot the poles and zeros

$H(s) := \frac{(s+2)}{s \cdot (s+1) \cdot (s-1)}$

Define the poles and zeros as a vector:

$p := 0..2$ number of poles is 3, since there are three 's' in the function
$z := 0..0$ 0 to 0 since there is only one '0'

From the function H(s) the poles and zeros are:
Poles: s(s + 1) (s - 1)
s = 0
s + 1 = 0
s - 1 = 0
s0=0, s1=-1, and s2= 1 by setting each term = 0 in the denominator

Zeros: (s+2)
s0 = -2 by setting the numerator term(s) equal zero

The pole and zero vectors are populated:

$ORIGIN := 0$

$j := \sqrt{-1}$

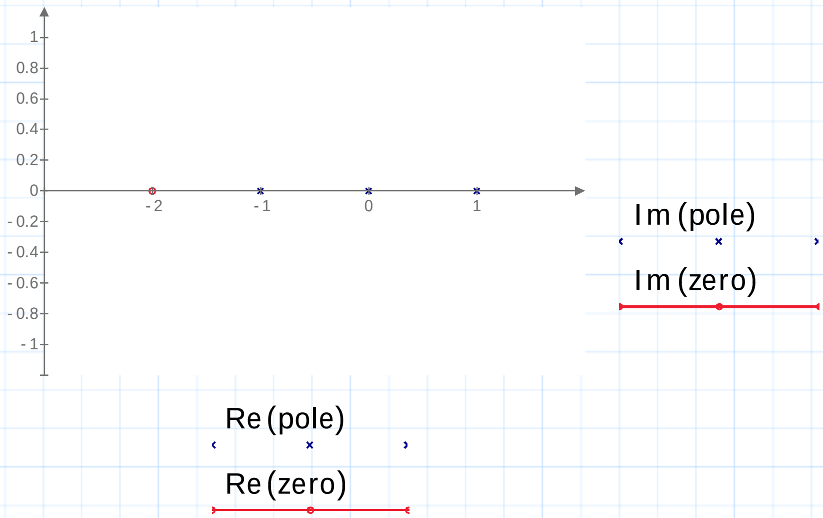
$pole := \begin{bmatrix} 0 + j \cdot 0 \\ -1 + j \cdot 0 \\ 1 + j \cdot 0 \end{bmatrix}$ $\quad pole = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ $\quad pole_1 = -1$

$zero := [-2 + j \cdot 0]$ $\quad zero = [-2]$ $\quad zero_0 = -2$

Now plot taking real and imaginary parts into consideration
Since s = j w , we have real and imaginary parts, so the poles and zeros are set similarly

Remember Prime/Mathcad has partial fractions function

The y axis is the imaginary axis in the plot, x axis the real part.



Example 3.12 - Poles and Zeros

$$H(s) := \frac{s^2 + \left(\frac{1}{2}\right)}{s^3 + s^2 + s + \frac{1}{2}}$$

Its a tedious task to get the roots of the equations above for the numerator and denominator.

So we use the 'polyroot' function in Prime/Mathcad

For polyroot a vector has to be created filling in the values for each power of and constant.

The top most element is the constant then starting with the variable, then variable squared,.....etc.

$$Z := \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

element 0 = 1/2 - constant
1 = 0 - s
2 = 1 - s^2

Zeros := polyroots(Z)

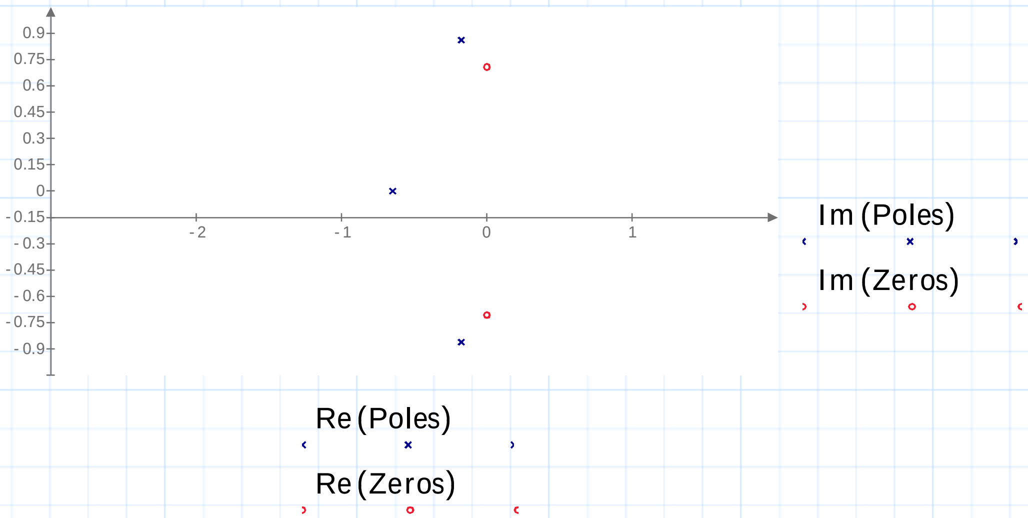
$$Zeros = \begin{bmatrix} -0.71j \\ 0.71j \end{bmatrix}$$

$$P := \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

element 0 = 1/2 - constant
1 = 1 - s
2 = 1 - s^2
3 = 1 - s^3

Poles := polyroots(P)

$$Poles = \begin{bmatrix} -0.65 \\ -0.18-0.86j \\ -0.18+0.86j \end{bmatrix}$$



Frequency Response in The Lapalce Transforms.

Example 3.13

$$H(s) := \frac{1}{s^3 + 2 \cdot s^2 + 2 \cdot s + 1}$$ System's transfer function

Plot the frequency response of the system and show the 3-dB:

$j := \sqrt{-1}$

$s := j \cdot \omega$

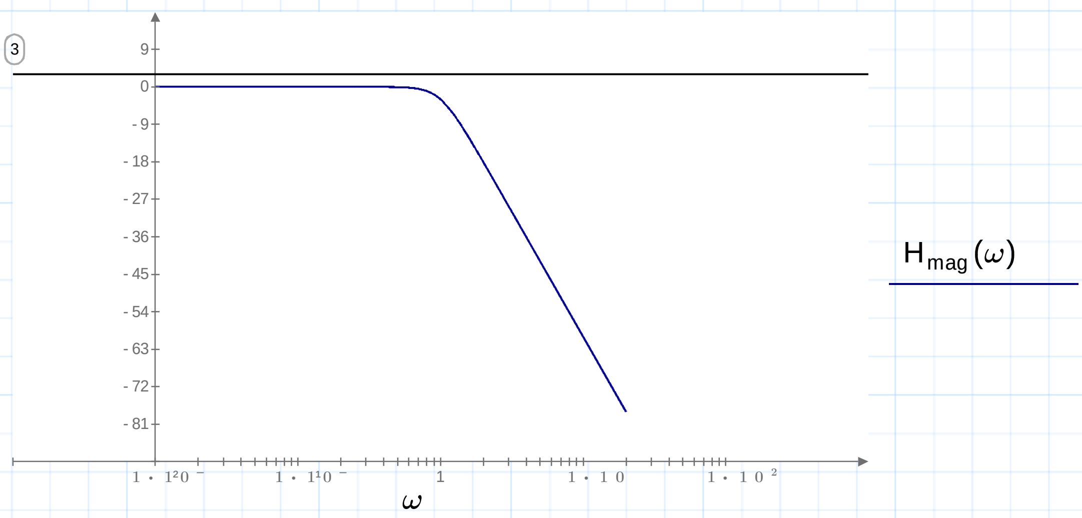
$\omega := 0.01, 0.02 .. 20$ this line has to be placed after s = jw

Next setup the transfer function H(s) to H(jw):

$$H(\omega) := \frac{1}{(j \cdot \omega)^3 + 2 \cdot (j \cdot \omega)^2 + 2 \cdot (j \cdot \omega) + 1}$$

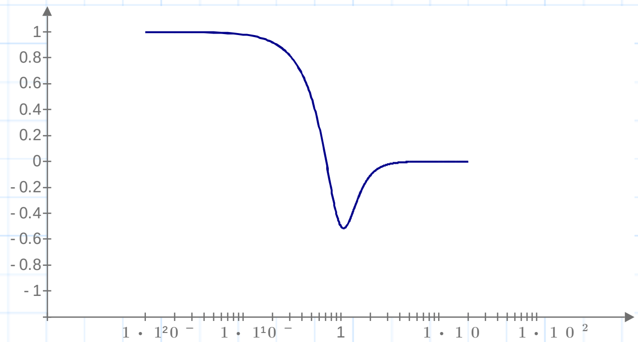
To plot the transfer function use the formula 20 log | H(s) | - note it is magnitude of |H(s)|:
Set plot in Logrithmic scale on x-axis w.

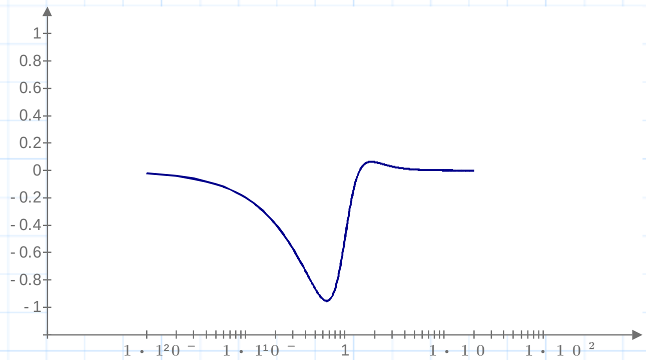
$$H_{mag}(\omega) := 20 \cdot \log\left(\left|\frac{1}{(j \cdot \omega)^3 + 2 \cdot (j \cdot \omega)^2 + 2 \cdot (j \cdot \omega) + 1}\right|\right)$$

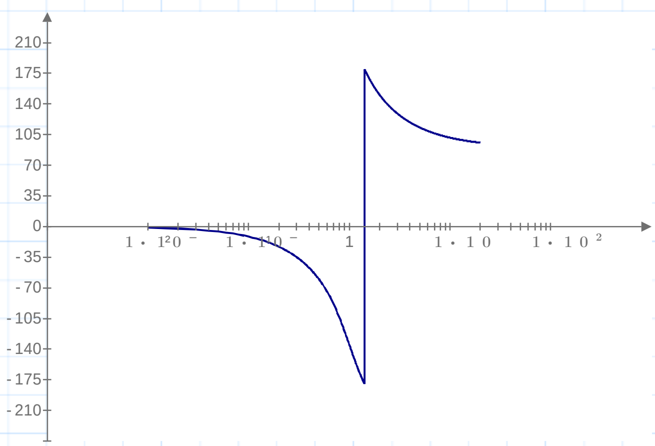


Plot above the horizontal marker is set at 3dB.

Next break the plot into the real part plot, and the phase in degrees plot.
Do NOT take the magnitude for the real, imaginary, and phase part.







Phase in degrees - multiply by 180/pi

Examples from Linear Systems and Signals 2nd Edition. Oxford. By B.P. Lathi.

Example 4.1

For a signal x(t) = e^-at u(t), find the Laplace transform X(s) and its region of convergence.

clear (s)   clear (X)

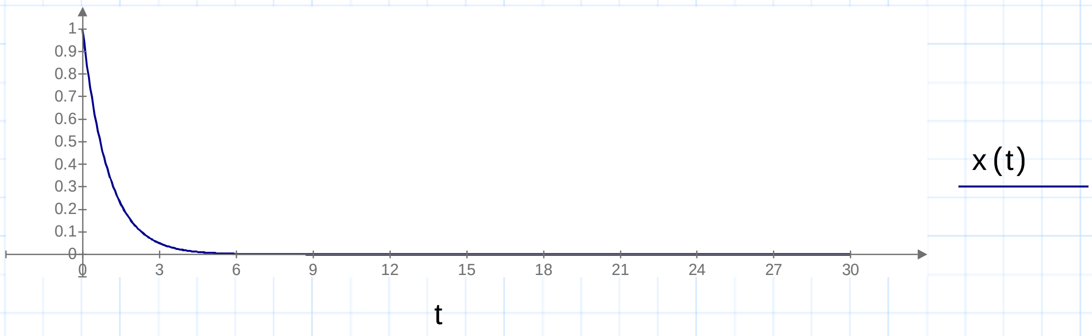
$j := \sqrt{-1}$ $\omega := 2 \cdot \pi$ $i := j$

$s := j \cdot \omega$

$a := 1$ 'a' is a constant
it is the a=1 in the solution's denominator (s+a) or (s+2)

$u(t) := if(|t| \geq 0, 1, 0)$ unit step function

$x(t) := e^{-a \cdot t} \cdot u(t)$ d(t) delta function is a Prime function

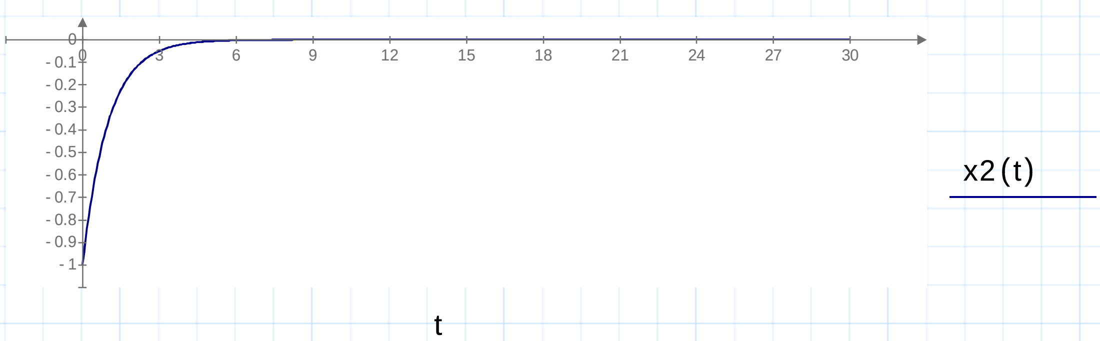


$$X(s) := \int_0^{\infty} x(t) \cdot e^{-s \cdot t} \, dt \rightarrow \frac{1}{s+1}$$ Laplace solution

There is a pole at s = -1, no zeros.
So the region of convergence is defined on the x-axis from -1 to infinty

Next a negative sign in x(t) and u(-t)

$x2(t) := -e^{-a \cdot t} \cdot u(-t)$ d(t) delta function is a Prime function



$$X2(s) := \int_0^{\infty} x2(t) \cdot e^{-s \cdot t} \, dt \rightarrow -\frac{1}{s+1}$$ Laplace solution

There is a pole at s = 1, no zeros.
-(s+1) = -s -1 ; s = 1
So the region of convergence is defined on the x-axis from 1 to -infinty

The valus of 'a' is a +ve of -ve integer value

Note: Page 385 - The Laplace transforms for the signal e^(-at) u(t) and -e^(at) u(t) are identical except for their region of convergence. Therefore for a given X(s) there may be more than one inverse transform, depending on the ROC region of occurence. In other words, unless the region of convergence is specified there is no one to one correspondence between X(s) and x(t). This fact increases complexity in using Laplace transform.

Example 4.2
Determine the Laplace transform of the following:
a). d(t) delta function
b). u(t) unit step function
c). cos w0 t u(t)

a)
$j := \sqrt{-1}$ $\omega := 2 \cdot \pi$ $i := j$ $s := j \cdot \omega$ $n := \omega$ $m := n$
clear (x)   clear (X)
$x(t) := \delta(m, n)$ return 1 when m=n

$$X(s) := \int_0^{\infty} x(t) \cdot e^{-s \cdot t}\, dt$$

The transform tables show the Laplace transform of delta (t) = 1 for all s.

b).
clear (x)   clear (X)
$u(t) := \text{if}(|t| \geq 0, 1, 0)$ unit step function

$x(t) := u(t)$

$$X(s) := \int_0^{\infty} x(t) \cdot e^{-s \cdot t}\, dt \rightarrow \frac{1}{s} - \frac{\lim_{t \to \infty} e^{-(s \cdot t)}}{s}$$

From tables the transform is 1/s for Re s > 0

c).
clear (x)   clear (X)   clear (x2)
$j := \sqrt{-1}$ $\omega_0 := 2 \cdot \pi$ $i := j$ $s := j \cdot \omega_0$
$u(t) := \text{if}(|t| \geq 0, 1, 0)$

$e\_cos(t) := \frac{1}{2} \cdot \left(e^{j \cdot \omega_0 \cdot t} + e^{-j \cdot \omega_0 \cdot t}\right)$ cosine term in exponential form

$x1(t) := \frac{1}{2} \cdot \left(e^{j \cdot \omega_0 \cdot t}\right) \cdot u(t)$ $x2(t) := \frac{1}{2} \cdot \left(e^{-j \cdot \omega_0 \cdot t}\right) \cdot u(t)$

From the tables no 8a:

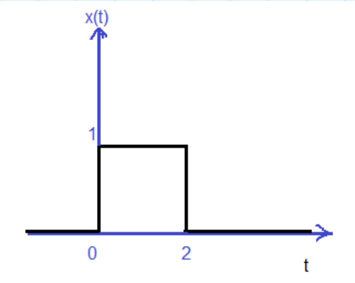
x1(t) = 1/2 (s/ (s^2 w0^2)
x2(t) = 1/2 (s/ (s^2 + w0^2)

$x(t) := x1(t) + x2(t)$

$$X(s) := \int_0^{\infty} x(t) \cdot e^{-s \cdot t} \, dt$$

X(s) = s / (s^2 + w0^2)   Answer - when real part of s >0. See tables.

Exercise 4.2



a)
$j := \sqrt{-1}$ $\omega := 2 \cdot \pi$ $i := j$ $s := j \cdot \omega$
clear (x)   clear (X)
$tt := 0, 1 .. 20$
$u(t) := \text{if}(0 \le t \le 2, 1, 0)$ return 1 when t is between 0 and 2

$x(t) := p(t)$

$$X(s) := \int_0^{\infty} x(t) \cdot e^{-s \cdot t} \, dt$$ cannot intergrate a conditional variable

Use the unit step function from 0 to 2
Start at 0 and subtract the rest starting at 2
t= 0 to 2 = 1, elsewhere 0
At t=0:
$x1(t) := u(0) = 1$

$$X1(s) := \int_0^{\infty} x1(t) \cdot e^{-s \cdot t} \, dt \rightarrow \frac{1}{s} - \frac{\lim_{t \to \infty} e^{-(s \cdot t)}}{s}$$

X1(s) = 1/s

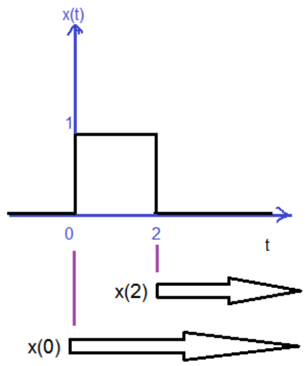
At t=2:

$x2(t) := u(2) = 1$

$X2(s) := \int_0^{\infty} x2(t) \cdot e^{-s \cdot 2} \, dt \rightarrow e^{-2 \cdot s} \cdot \infty$ substitue t with 2 --> e^-s2

X2(s) is shifted to 2, its = 1/s at 0 not at 2 with the value of at 2 to infinity of e^-2s

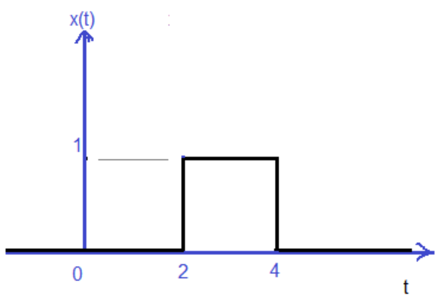
X2(s) = 1/s (e^-2s)



x(t --> 0-2) = x0 - x2
X(s) = X(0) - X(2)

X(s) = 1/s - 1/s(e^-2s)
X(s) = 1/s(1 - e^-2s) Answer - check with your results.
for all s

b).



This signal is shiftet by t=2
Apply shift property from tables.

Shift t = 2 --> e-2s
X(s) = 1/s(1 - e^-2s) x Shift
X(s) = 1/s(1 - e^-2s)e^-2s Answer - check with your results.
for all s

Exercise 4.3

Find the inverse Laplace transforms of the following:

$j := \sqrt{-1}$ $i := j$ $\omega := 2\cdot\pi$ $s := j\cdot\omega$

a).

$$H1(s) := \frac{7\cdot s-6}{s^2-s-6} \xrightarrow{\text{invlaplace}} 7\cdot e^{\frac{t}{2}}\cdot\left(\cosh\left(\frac{5\cdot t}{2}\right)-\frac{\sinh\left(\frac{5\cdot t}{2}\right)}{7}\right)$$

Prime/Mathcad solution above - not pleasant!
Apply PARTIAL FRACTIONS in Prime/Mathcad using function 'parfrac'.
To break the function into simpler parts

clear(s)   Make sure s had been cleared cannot have s = jw for the parfrac evaluation

$$\frac{7\cdot s-6}{s^2-s-6} \xrightarrow{\text{parfrac}} \frac{4}{s+2}+\frac{3}{s-3}$$

Now apply Laplace transform to each term

$ss := j\cdot\omega$

$H1a(s) := \frac{4}{s+2}$ $H1a(s) \xrightarrow{\text{invlaplace}} \frac{4\cdot\Delta(t)}{s+2}$

From Laplace transform table - no 5
4[1/(s+2)] = 4e^-2t

$H1b(s) := \frac{3}{s+2}$ $H1b(s) \xrightarrow{\text{invlaplace}} \frac{3\cdot\Delta(t)}{s+2}$

From Laplace transform table - no 5
3[1/(s+2)] = 3e^-2t

$H1(s) := H1a(s) + H1b(s)$

$H1(s) := \frac{4}{s+2}+\frac{3}{s+2}$

With a unit step function u(t) as part of the input signal the transfer function H1(s) in exponential form now:
x(t) = (4e^-2t + 3e^3t)u(t)   Answer - Inverse Laplace

b).

$$H2(s) := \frac{2 \cdot s^2 + 5}{s^2 + 3 \cdot s + 2} \xrightarrow{\text{invlaplace}} 2 \cdot \Delta(t) + 7 \cdot e^{-t} - 13 \cdot e^{-2 \cdot t}$$ - Prime solution

Apply PARTIAL FRACTIONS in Prime/Mathcad using function 'parfrac'.
To break the function into simpler parts

clear (s) Make sure s had been cleared it cannot be s = jw for the parfrac evaluation

$$\frac{2 \cdot s^2 + 5}{s^2 + 3 \cdot s + 2} \xrightarrow{\text{parfrac}} \frac{7}{s+1} - \frac{13}{s+2} + 2$$

Apply Laplace transform table no 5

7/(s+1) = 7 x (e^-at) a = 1 so 7 x (e^-t)

13/(s+2) = 13 x (e^-at) a = 2 so 13 x (e^-2t)

Apply Laplace transform table no 1

1 = d(t)
2 = 2d(t)

x(t) = [2d(t) + 7 (e^-at) - 13 (e^-2t)] u(t) with u(t) Answer Inverse Laplace

c).

$$H3(s) := \frac{6 \cdot (s + 34)}{s \cdot (s^2 + 10 \cdot s + 34)}$$

Apply PARTIAL FRACTIONS in Prime/Mathcad using function 'parfrac'.
To break the function into simpler parts

clear (s) Make sure s had been cleared it cannot be s = jw for the parfrac evaluation

$$\frac{6 \cdot (s + 34)}{s \cdot (s^2 + 10 \cdot s + 34)} \xrightarrow{\text{parfrac}} \frac{6}{s} - \frac{6 \cdot s + 54}{s^2 + 10 \cdot s + 34}$$

We proceed fresh using quadratic factors method:
Multiply both sides of equation by the denominator term

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BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

$$\frac{6(s + 34)}{s(s^2 + 10s + 34)} = \frac{k1}{s} + \frac{As + B}{(s^2 + 10s + 34)}$$

k1 = 6 from the partial fraction prior by Prime

6(s + 34) = 6(s^2 +10s + 34) + s(As + B)

Equating coefficients of s^2 and s on both sides

6s^2 + As^2
-6s^2 = As^2
A = -6

6s = 60s + Bs
Bs = s(-60+6) = -54s
B = -54

Now in the simpler form:

$$H3(s) := \frac{6}{s} + \frac{(-6 \cdot s - 54)}{s^2 + 10 \cdot s + 34}$$

Using transform table no 2 and 10c:

2	$u(t)$	$\frac{1}{s}$
10c	$re^{-at}\cos(bt + \theta)\,u(t)$	$\frac{As + B}{s^2 + 2as + c}$

For 10c the parameters are:
A = -6, B = -54
a: 10 = 2a
a = 5
b = sqrt(c - a^2) = sqrt(34-25) = sqrt(9)
b = 3

$$r = \sqrt{\frac{A^2c + B^2 - 2ABa}{c - a^2}}$$

$$\theta = \tan^{-1}\left(\frac{Aa - B}{A\sqrt{c - a^2}}\right)$$

$$b = \sqrt{c - a^2}$$

$A := -6 \quad B := -54 \quad a := 5 \quad b := 3 \quad c := 34$

$$r := \sqrt{\left(\frac{A^2 \cdot c + B^2 - 2 \cdot A \cdot B \cdot a}{c - a^2}\right)} = 10$$

$r = 10$

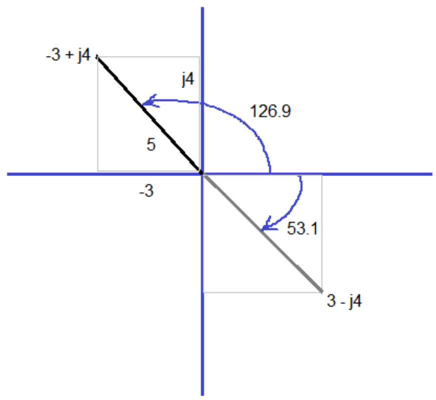
$A \cdot a - B = 24 \qquad A \cdot \sqrt{c - a^2} = -18$

24/-18= 4/-3 $\quad \theta_1 := \mathrm{atan}\left(\frac{4}{-3}\right) \quad \theta_1 = -53.130102\ \mathrm{deg}$

Or plugging straight into the formula:

$$\theta := \operatorname{atan}\left(\frac{A \cdot a - B}{A \cdot \sqrt{c - a^2}}\right) = -53.1\ \text{deg}$$

Where is the angle -53.81 located?
Right now at the lower right quadrant at 3 - 4j but the angle is pointing to the other direction so we rotate it anticlockwise 180 degrees to the vector -3 + j4 its conjugate.



$$\theta_{pos} := 180\ \text{deg} + \theta = 126.9\ \text{deg}$$

Now forming the laplace inverse equation:

x(t) = [6 + 10e^-5t cos(3t +126.9 deg)] u(t) Answer - Inverse Laplace

The solution Prime provided Not the exact same.

$$H3(s) := \frac{6 \cdot (s + 34)}{s \cdot \left(s^2 + 10 \cdot s + 34\right)} \xrightarrow{\text{invlaplace}} 6 - 8 \cdot \sin(3 \cdot t) \cdot e^{-5 \cdot t} - 6 \cdot \cos(3 \cdot t) \cdot e^{-5 \cdot t}$$

$$H3(s) := \frac{6}{s} + \frac{(-6 \cdot s - 54)}{s^2 + 10 \cdot s + 34} \xrightarrow{\text{invlaplace}} 6 - 8 \cdot \sin(3 \cdot t) \cdot e^{-5 \cdot t} - 6 \cdot \cos(3 \cdot t) \cdot e^{-5 \cdot t}$$

d).

$$H4(s) := \frac{8 \cdot s + 10}{(s + 1) \cdot (s + 2)^3}$$

Apply PARTIAL FRACTIONS in Prime/Mathcad using function 'parfrac'.
To break the function into simpler parts

$$H4(s) := \frac{8\cdot s+10}{(s+1)\cdot(s+2)^3} \xrightarrow{\text{parfrac}} \frac{2}{s+1} - \frac{2}{s+2} - \frac{2}{(s+2)^2} + \frac{6}{(s+2)^3}$$

Clean results above ready to use

5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$

Use nos 5, 6, and 7 above to finish the solution, check to the correct answer generated by Prime/Mathcad.

$$H4(s) := \frac{2}{s+1} - \frac{2}{s+2} - \frac{2}{(s+2)^2} + \frac{6}{(s+2)^3}$$

$$H4(s) \xrightarrow{\text{invlaplace}} 2\cdot e^{-t} - 2\cdot e^{-2\cdot t} - 2\cdot t\cdot e^{-2\cdot t} + 3\cdot t^2\cdot e^{-2\cdot t}$$ Answer Inverse Laplace

Exercise E4.2

i).

$j := \sqrt{-1}$ $\omega := 2\cdot\pi$ $i := j$ $s := j\cdot\omega$

$x(t) := 10\cdot e^{-3\cdot t}\cdot\cos(4\cdot t + 53.13\ deg)$ Find the Laplace Transform

10a	$re^{-at}\cos(bt+\theta)u(t)$	$\frac{(r\cos\theta)s + (ar\cos\theta - br\sin(\theta))}{s^2+2as+(a^2+b^2)}$

Use transform no 10a from the table:

$r := 10$ $a := 3$ $b := 4$ $\theta := 53.13\ deg$

$$X(\omega) := \frac{r\cdot\cos(\theta)\cdot s + a\cdot r\cdot\cos(\theta) - b\cdot r\cdot\sin(\theta)}{s^2 + 2\cdot a\cdot s + (a^2+b^2)}$$

$$X(\omega) := \frac{10\cdot\cos(\theta)\cdot s + 30\cdot\cos(\theta) - 40\cdot\sin(\theta)}{s^2 + 6\cdot s + 25}$$

$10\cdot\cos(\theta) = 6$ $30\cdot\cos(\theta) = 18$ $40\cdot\sin(\theta) = 32$

$$X(\omega) := \frac{6\cdot s + 18 - 32}{s^2 + 6\cdot s + 25}$$

$$X(\omega) := \frac{6\cdot s - 14}{s^2 + 6\cdot s + 25}$$ Answer

ii).
Find the inverse Laplace transform of the following:

a). (s + 17) / (s^2 + 4s -5)

clear (s)

$$X(\omega) := \frac{s + 17}{s^2 + 4\cdot s - 5}$$

$$\frac{s + 17}{s^2 + 4\cdot s - 5} \xrightarrow{parfrac} \frac{3}{s-1} - \frac{2}{s+5}$$

Apply no 5 in the list:

5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$

x(t): 3e^(st) - 2e(-5st)

x(t): [3e^(st) - 2e(-5st)] u(t)   Answer - Inverse Laplace

Using Prime/Mathcad:

$$\frac{3}{s-1} - \frac{2}{s+5} \xrightarrow{invlaplace} 3\cdot e^{t} - 2\cdot e^{-5\cdot t}$$ Verifies Answer

b).

$j := \sqrt{-1}$ $\omega := 2\cdot\pi$ $i := j$ $s := j\cdot\omega$
clear (s)

$$X(\omega) := \frac{3\cdot s - 5}{(s+1)\cdot(s^2 + 2\cdot s + 5)}$$

$$\frac{3\cdot s - 5}{(s+1)\cdot(s^2 + 2\cdot s + 5)} \xrightarrow{parfrac} \frac{2\cdot s + 5}{s^2 + 2\cdot s + 5} - \frac{2}{s+1}$$

Apply Laplace transform number 10c to first term, and number 5 to second term.

$$\frac{2 \cdot s + 5}{s^2 + 2 \cdot s + 5}$$

10c	$re^{-at}\cos(bt+\theta)\,u(t)$	$\dfrac{As+B}{s^2+2as+c}$

$a := 1 \qquad c := 5$

$A := 2 \qquad B := 5$

$$r := \sqrt{\left(\frac{A^2 \cdot c + B^2 - 2 \cdot A \cdot B \cdot a}{c - a^2}\right)} = 2.5$$

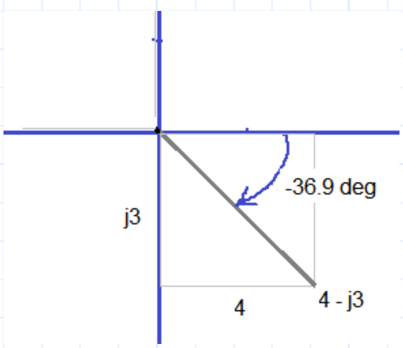
$$b := \sqrt{\left(c - a^2\right)} = 2$$

$$\theta := \operatorname{atan}\left(\frac{A \cdot a - B}{A \cdot \sqrt{c - a^2}}\right) = -36.9 \text{ deg}$$

next verify correct direction of the angle

$A \cdot a - B = -3 \qquad A \cdot \sqrt{c - a^2} = 4$

$$\theta := \operatorname{atan}\left(\frac{-3}{4}\right) = -36.9 \text{ deg}$$



Correct.

x(t) = [2.5e^(-t) cos(2t-36.9deg) -2e^(-t)] u(t)   Answer - Inverse Laplace

Verify quadratice term with Prime/Mathcad:

$$\frac{2 \cdot s + 5}{s^2 + 2 \cdot s + 5} \xrightarrow{\text{invlaplace}} 2 \cdot e^{-t} \cdot \left(\cos(2 \cdot t) + \frac{3 \cdot \sin(2 \cdot t)}{4}\right)$$

This instance for me the table solution is more suitable!

c).

$j := \sqrt{-1}$ $\omega := 2 \cdot \pi$ $i := j$ $s := j \cdot \omega$

clear (s)

$$X(\omega) := \frac{16 \cdot s + 43}{(s-2) \cdot (s+3)^2}$$

Expecting some combinations of from the table.

$$\frac{16 \cdot s + 43}{(s-2) \cdot (s+3)^2} \xrightarrow{parfrac} \frac{3}{s-2} - \frac{3}{s+3} + \frac{1}{(s+3)^2}$$

Prime partial fraction resulted with clear fractions, the time domain inverse transform expection is encouraging.

$\frac{3}{s-2}$ 3 e^(2t)

$\frac{3}{s+2}$ 3e^(-2t)

$\frac{1}{(s+3)^2}$ te^(-3t) from no 6 in table

x(t): [3 e^(2t) + 3e^(-2t) + te^(-3t)] u(t) Answer - Inverse Laplace Transform

Next notes on properties of Laplace transforms
from Signals and Systems 2nd ed by B.P. Lathi.

There are specific topics such as Bode Plots, Filters, Solutions of Differential and Integro-Differential Equations in the textbook in detail. These are specific to a course's content like Circuit Networks, Filters, Differential Equations, Controls,....., which you can continue on your own in context to those course's content.

The main objective:
1. to get started with Laplace for engineering problem solving
2. to get over the main hurdle in Laplace Transforms mathematics and using Prime/Mathcad.

4.2 Some Properties of the Laplace Transform

Properties of the Laplace transform are useful not only in the derivation of the Laplace trans[...] of functions but also in the solutions of linear integro-differential equations. A glance at Eqs. (4.[...] and (4.1) shows that there is a certain measure of symmetry in going from $x(t)$ to $X(s)$, and vi[...] versa. This symmetry or duality is also carried over to the properties of the Laplace transfor[...] This fact will be evident in the following development.

We are already familiar with two properties; linearity [Eq. (4.4)] and the uniqueness prop[...] of the Laplace transform discussed earlier.

4.2-1 Time Shifting

The time-shifting property states that if

$$x(t) \Longleftrightarrow X(s)$$

then for $t_0 \geq 0$

$$x(t - t_0) \Longleftrightarrow X(s)e^{-st_0} \qquad (4.19a)$$

Observe that $x(t)$ starts at $t = 0$, and, therefore, $x(t - t_0)$ starts at $t = t_0$. This fact is implicit, but is not explicitly indicated in Eq. (4.19a). This often leads to inadvertent errors. To avoid such a pitfall, we should restate the property as follows. If

$$x(t)u(t) \Longleftrightarrow X(s)$$

then

$$x(t - t_0)u(t - t_0) \Longleftrightarrow X(s)e^{-st_0} \qquad t_0 \geq 0 \qquad (4.19b)$$

Proof.

$$\mathcal{L}[x(t - t_0)u(t - t_0)] = \int_0^{\infty} x(t - t_0)u(t - t_0)e^{-st}\,dt$$

Setting $t - t_0 = \tau$, we obtain

$t - t_0 = \tau$

$t = \tau + t_0$

$$\mathcal{L}[x(t - t_0)u(t - t_0)] = \int_{-t_0}^{\infty} x(\tau)u(\tau)e^{-s(\tau + t_0)}\,d\tau$$

Because $u(\tau) = 0$ for $\tau < 0$ and $u(\tau) = 1$ for $\tau \geq 0$, the limits of integration can be taken from 0 to ∞. Thus

$$\mathcal{L}[x(t - t_0)u(t - t_0)] = \int_0^{\infty} x(\tau)e^{-s(\tau + t_0)}\,d\tau$$

$$= e^{-st_0}\int_0^{\infty} x(\tau)e^{-s\tau}\,d\tau$$

$$= X(s)e^{-st_0}$$

Note that $x(t - t_0)u(t - t_0)$ is the signal $x(t)u(t)$ delayed by t_0 seconds. The time-shifting property states that *delaying a signal by t_0 seconds amounts to multiplying its transform e^{-st_0}*.

e^-st0

This property of the unilateral Laplace transform holds only for positive t_0 because if t_0 were negative, the signal $x(t-t_0)u(t-t_0)$ may not be causal.

We can readily verify this property in Exercise E4.1. If the signal in Fig. 4.2a is $x(t)u(t)$, then the signal in Fig. 4.2b is $x(t-2)u(t-2)$. The Laplace transform for the pulse in Fig. 4.2a is $(1/s)(1-e^{-2s})$. Therefore, the Laplace transform for the pulse in Fig. 4.2b is $(1/s)(1-e^{-2s})e^{-2s}$.

The time-shifting property proves very convenient in finding the Laplace transform of functions with different descriptions over different intervals, as the following example demonstrates.

EXAMPLE 4.4

Find the Laplace transform of $x(t)$ depicted in Fig. 4.4a.

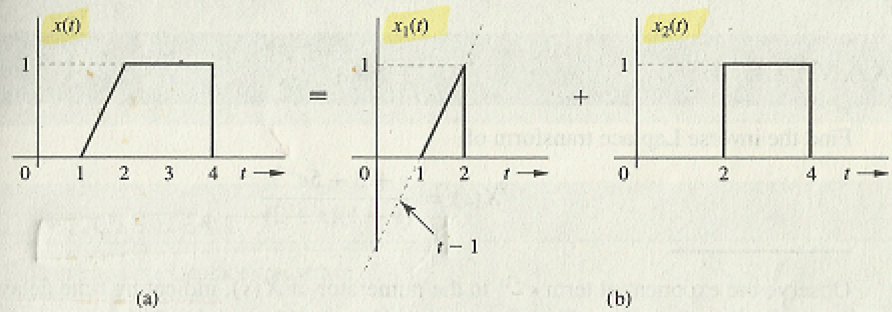


Figure 4.4 Finding a mathematical description of a function $x(t)$.

Describing mathematically a function such as the one in Fig. 4.4a is discussed in Section 1.4. The function $x(t)$ in Fig. 4.4a can be described as a sum of two components shown in Fig. 4.4b. The equation for the first component is $t-1$ over $1 \le t \le 2$, so that this component can be described by $(t-1)[u(t-1)-u(t-2)]$. The second component can be described by $u(t-2)-u(t-4)$. Therefore

$$\begin{aligned} x(t) &= (t-1)[u(t-1)-u(t-2)] + [u(t-2)-u(t-4)] \\ &= (t-1)u(t-1) - (t-1)u(t-2) + u(t-2) - u(t-4) \end{aligned} \quad (4.20a)$$

The first term on the right-hand side is the signal $tu(t)$ delayed by 1 second. Also, the third and fourth terms are the signal $u(t)$ delayed by 2 and 4 seconds, respectively. The second term, however, cannot be interpreted as a delayed version of any entry in Table 4.1. For this reason, we rearrange it as

$$(t-1)u(t-2) = (t-2+1)u(t-2) = (t-2)u(t-2) + u(t-2)$$

We have now expressed the second term in the desired form as $tu(t)$ delayed by 2 seconds plus $u(t)$ delayed by 2 seconds. With this result, Eq. (4.20a) can be expressed as

$$x(t) = (t-1)u(t-1) - (t-2)u(t-2) - u(t-4) \quad (4.20b)$$

1/s^2

Application of the time-shifting property to $tu(t) \Longleftrightarrow 1/s^2$ yields

$$(t-1)u(t-1) \Longleftrightarrow \frac{1}{s^2}e^{-s} \quad \text{and} \quad (t-2)u(t-2) \Longleftrightarrow \frac{1}{s^2}e^{-2s}$$

Also

$$u(t) \Longleftrightarrow \frac{1}{s} \quad \text{and} \quad u(t-4) \Longleftrightarrow \frac{1}{s}e^{-4s}$$

Therefore

$$X(s) = \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s}$$

EXAMPLE 4.5

Find the inverse Laplace transform of

$$X(s) = \frac{s+3+5e^{-2s}}{(s+1)(s+2)}$$

Observe the exponential term e^{-2s} in the numerator of $X(s)$, indicating time delay. In such a case we should separate $X(s)$ into terms with and without delay factor, as

$$X(s) = \underbrace{\frac{s+3}{(s+1)(s+2)}}_{X_1(s)} + \underbrace{\frac{5e^{-2s}}{(s+1)(s+2)}}_{X_2(s)e^{-2s}}$$

where

$$X_1(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$X_2(s) = \frac{5}{(s+1)(s+2)} = \frac{5}{s+1} - \frac{5}{s+2}$$

Therefore

$$x_1(t) = (2e^{-t} - e^{-2t})u(t)$$

$$x_2(t) = 5(e^{-t} - e^{-2t})u(t)$$

Also, because

$$X(s) = X_1(s) + X_2(s)e^{-2s}$$

We can write

$$x(t) = x_1(t) + x_2(t-2)$$

$$= (2e^{-t} - e^{-2t})u(t) + 5\left[e^{-(t-2)} - e^{-2(t-2)}\right]u(t-2)$$

EXERCISE E4.3

Find the Laplace transform of the signal illustrated in Fig. 4.5.

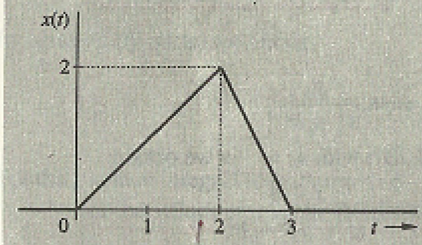


Figure 4.5

ANSWER

$\frac{1}{s^2}(1 - 3e^{-2s} + 2e^{-3s})$

EXERCISE E4.4

Find the inverse Laplace transform of

$$X(s) = \frac{3e^{-2s}}{(s-1)(s+2)}$$

ANSWER

$[e^{t-2} - e^{-2(t-2)}]u(t-2)$

4.2-2 Frequency Shifting

The frequency-shifting property states that if

$$x(t) \Longleftrightarrow X(s)$$

then

$$x(t)e^{s_0 t} \Longleftrightarrow X(s - s_0) \tag{4.23}$$

Observe the symmetry (or duality) between this property and the time-shifting property (4.19a).

Proof. x(t)e^(s0t) = x(t)e^(s0t)e^(-st) = x(t)e^-(s-s0)t X(s-s0)

$$\mathcal{L}[x(t)e^{s_0 t}] = \int_{0^-}^{\infty} x(t)e^{s_0 t}e^{-st}\,dt = \int_{0^-}^{\infty} x(t)e^{-(s-s_0)t}\,dt = X(s - s_0)$$

EXAMPLE 4.6

Derive pair 9a in Table 4.1 from pair 8a and the frequency-shifting property.

Pair 8a is

$$\cos bt\, u(t) \Longleftrightarrow \frac{s}{s^2 + b^2}$$

From the frequency-shifting property [Eq. (4.23)] with $s_0 = -a$ we obtain

$$e^{-at} \cos bt\, u(t) \Longleftrightarrow \frac{s + a}{(s + a)^2 + b^2}$$

$s - s_0$
$s - (-a)$

EXERCISE E4.5

Derive pair 6 in Table 4.1 from pair 3 and the frequency-shifting property.

We are now ready to consider the two of the most important properties of the Laplace transform: time differentiation and time integration.

4.2-3 The Time-Differentiation Property[†]

The time-differentiation property states that if

$$x(t) \Longleftrightarrow X(s)$$

then

$$\frac{dx}{dt} \Longleftrightarrow sX(s) - x(0^-) \tag{4.24a}$$

Repeated application of this property yields

$$\frac{d^2x}{dt^2} \Longleftrightarrow s^2X(s) - sx(0^-) - \dot{x}(0^-) \tag{4.24b}$$

$$\frac{d^nx}{dt^n} \Longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - s^{n-2}\dot{x}(0^-) - \cdots - x^{(n-1)}(0^-)$$

$$= s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-) \tag{4.24c}$$

where $x^{(r)}(0^-)$ is d^rx/dt^r at $t = 0^-$.

[†]The dual of the time-differentiation property is the frequency-differentiation property, which states that

$$tx(t) \Longleftrightarrow -\frac{d}{ds}X(s)$$

Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

Proof.

$$\mathcal{L}\left[\frac{dx}{dt}\right] = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st}\, dt$$

Integrating by parts, we obtain

$$\mathcal{L}\left[\frac{dx}{dt}\right] = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}\, dt$$

For the Laplace integral to converge [i.e., for $X(s)$ to exist], it is necessary that $x(t)e^{-st} \to 0$ as $t \to \infty$ for the values of s in the ROC for $X(s)$. Thus,

$$\mathcal{L}\left[\frac{dx}{dt}\right] = -x(0^-) + sX(s)$$

Repeated application of this procedure yields Eq. (4.24c). $s^n X(s) - \sum_{k=1}^{n} s^{n-k} x^{(k-1)}(0^-)$

EXAMPLE 4.7

Find the Laplace transform of the signal $x(t)$ in Fig. 4.6a by using Table 4.1 and the time-differentiation and time-shifting properties of the Laplace transform.

Figures 4.6b and 4.6c show the first two derivatives of $x(t)$. Recall that the derivative at a point of jump discontinuity is an impulse of strength equal to the amount of jump [see Eq. (1.27)]. Therefore

$$\frac{d^2x}{dt^2} = \delta(t) - 3\delta(t-2) + 2\delta(t-3)$$

The Laplace transform of this equation yields

$$\mathcal{L}\left(\frac{d^2x}{dt^2}\right) = \mathcal{L}\left[\delta(t) - 3\delta(t-2) + 2\delta(t-3)\right]$$

Using the time-differentiation property Eq. (4.24b), the time-shifting property (4.19a), and the facts that $x(0^-) = \dot{x}(0^-) = 0$, and $\delta(t) \Longleftrightarrow 1$, we obtain

$$s^2X(s) - 0 - 0 = 1 - 3e^{-2s} + 2e^{-3s}$$

Therefore

$$X(s) = \frac{1}{s^2}(1 - 3e^{-2s} + 2e^{-3s})$$

which confirms the earlier result in Exercise E4.3.

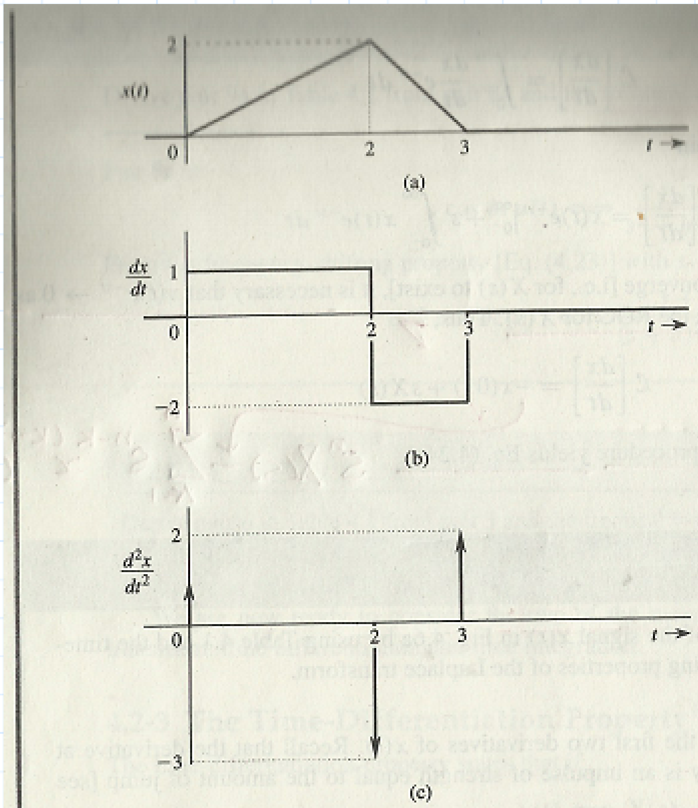


Figure 4.6 Finding the Laplace transform of a piecewise-linear function by using the time-differentiation property.

4.2-4 The Time-Integration Property

The time-integration property states that if

$$x(t) \Longleftrightarrow X(s)$$

then†

$$\int_{0^-}^{t} x(\tau)\, d\tau \Longleftrightarrow \frac{X(s)}{s}$$

and

$$\int_{-\infty}^{t} x(\tau)\, d\tau \Longleftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\tau)\, d\tau}{s}$$

†The dual of the time-integration property is the frequency-integration property, which states that

$$\frac{x(t)}{t} \Longleftrightarrow \int_{s}^{\infty} X(z)\, dz$$

Proof. We define

$$g(t) = \int_{0^-}^{t} x(\tau)\,d\tau$$

so that

$$\frac{d}{dt}g(t) = x(t) \qquad \text{and} \qquad g(0^-) = 0$$

Now, if

$$g(t) \Longleftrightarrow G(s)$$

then

$$X(s) = \mathcal{L}\left[\frac{d}{dt}g(t)\right] = sG(s) - g(0^-) = sG(s)$$

Therefore

$$G(s) = \frac{X(s)}{s}$$

or

$$\int_{0^-}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{X(s)}{s}$$

To prove Eq. (4.26), observe that

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{0^-} x(\tau)\,d\tau + \int_{0^-}^{t} x(\tau)\,d\tau$$

Note that the first term on the right-hand side is a constant for $t \geq 0$. Taking the Laplace transform of the foregoing equation and using Eq. (4.25), we obtain

$$\int_{-\infty}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{\int_{-\infty}^{0^-} x(\tau)\,d\tau}{s} + \frac{X(s)}{s}$$

SCALING

The scaling property states that if

$$x(t) \Longleftrightarrow X(s)$$

then for $a > 0$

$$x(at) \Longleftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right) \tag{4.27}$$

The proof is given in Chapter 7. Note that a is restricted to positive values because if $x(t)$ is causal, then $x(at)$ is anticausal (is zero for $t \geq 0$) for negative a, and anticausal signals are not permitted in the (unilateral) Laplace transform.

Recall that $x(at)$ is the signal $x(t)$ time-compressed by the factor a, and $X(\frac{s}{a})$ is $X(s)$ expanded along the s-scale by the same factor a (see Section 1.2-2). The scaling property states that *time compression of a signal by a factor a causes expansion of its Laplace transform in the s-scale by the same factor. Similarly, time expansion $x(t)$ causes compression of $X(s)$ in the s-scale by the same factor.*

4.2-5 Time Convolution and Frequency Convolution

Another pair of properties states that if

$$x_1(t) \Longleftrightarrow X_1(s) \quad \text{and} \quad x_2(t) \Longleftrightarrow X_2(s)$$

then (*time-convolution property*)

$$x_1(t) * x_2(t) \Longleftrightarrow X_1(s)X_2(s) \tag{4.38}$$

and (*frequency-convolution property*)

$$x_1(t)x_2(t) \Longleftrightarrow \frac{1}{2\pi j}[X_1(s) * X_2(s)] \tag{4.39}$$

Observe the symmetry (or duality) between the two properties. Proofs of these properties are postponed to Chapter 7.

Equation (2.48) indicates that $H(s)$, the transfer function of an LTIC system, is the Laplace transform of the system's impulse response $h(t)$; that is, Linear time invariant continuous time

$$h(t) \Longleftrightarrow H(s) \tag{4.30}$$

If the system is causal, $h(t)$ is causal, and, according to Eq. (2.48), $H(s)$ is the unilateral Laplace transform of $h(t)$. Similarly, if the system is noncausal, $h(t)$ is noncausal, and $H(s)$ is the bilateral transform of $h(t)$.

We can apply the time-convolution property to the LTIC input–output relationship $y(t) = x(t) * h(t)$ to obtain

$$Y(s) = X(s)H(s) \tag{4.31}$$

The response $y(t)$ is the zero-state response of the LTIC system to the input $x(t)$. From Eq. (4.31) it follows that

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}[\text{zero-state response}]}{\mathcal{L}[\text{input}]} \tag{4.32}$$

This may be considered to be an alternate definition of the LTIC system transfer function $H(s)$. It is the *ratio of the transform of zero-state response to the transform of the input.*

EXAMPLE 4.8

Using the time-convolution property of the Laplace transform to determine $c(t) = e^{at}u(t)*e^{bt}u(t)$.

From Eq. (4.28), it follows that

$$C(s) = \frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

The inverse transform of this equation yields

$$c(t) = \frac{1}{a-b}(e^{at} - e^{bt})u(t)$$

TABLE 4.2 The Laplace Transform Properties

Operation	$x(t)$	$X(s)$
Addition	$x_1(t)+x_2(t)$	$X_1(s)+X_2(s)$
Scalar multiplication	$kx(t)$	$kX(s)$
Time differentiation	$\frac{dx}{dt}$	$sX(s)-x(0^-)$
	$\frac{d^2x}{dt^2}$	$s^2X(s)-sx(0^-)-\dot{x}(0^-)$
	$\frac{d^3x}{dt^3}$	$s^3X(s)-s^2x(0^-)-s\dot{x}(0^-)-\ddot{x}(0^-)$
	$\frac{d^nx}{dt^n}$	$s^nX(s)-\sum_{k=1}^{n}s^{n-k}x^{(k-1)}(0^-)$
Time integration	$\int_{0^-}^{t}x(\tau)\,d\tau$	$\frac{1}{s}X(s)$
	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\frac{1}{s}X(s)+\frac{1}{s}\int_{-\infty}^{0^-}x(t)\,dt$
Time shifting	$x(t-t_0)u(t-t_0)$	$X(s)e^{-st_0}$ $\quad t_0 \ge 0$
Frequency shifting	$x(t)e^{s_0t}$	$X(s-s_0)$
Frequency differentiation	$-tx(t)$	$\frac{dX(s)}{ds}$
Frequency integration	$\frac{x(t)}{t}$	$\int_{s}^{\infty}X(z)\,dz$
Scaling	$x(at),\ a \ge 0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Time convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi j}X_1(s)*X_2(s)$
Initial value	$x(0^+)$	$\lim_{s\to\infty} sX(s)$ $\quad (n > m)$
Final value	$x(\infty)$	$\lim_{s\to 0} sX(s)$ $\quad$ [poles of $sX(s)$ in LHP]

Initial and Final Values

In certain applications, it is desirable to know the values of $x(t)$ as $t \to 0$ and $t \to \infty$ [initial and final values of $x(t)$] from the knowledge of its Laplace transform $X(s)$. Initial and final value theorems provide such information.

The initial value theorem states that if $x(t)$ and its derivative dx/dt are both Laplace transformable, then

$$x(0^+) = \lim_{s\to\infty} sX(s) \tag{4.33}$$

provided the limit on the right-hand side of Eq. (4.33) exists.

The final value theorem states that if both $x(t)$ and dx/dt are Laplace transformable, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

provided $sX(s)$ has no poles in the RHP or on the imaginary axis. To prove these theorems, we use Eq. (4.24a)

$$sX(s) - x(0^-) = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st}\,dt$$

$$= \int_{0^-}^{0^+} \frac{dx}{dt} e^{-st}\,dt + \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st}\,dt$$

$$= x(t)\Big|_{0^-}^{0^+} + \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st}\,dt$$

$$= x(0^+) - x(0^-) + \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st}\,dt$$

Therefore

$$sX(s) = x(0^+) + \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st}\,dt$$

and

$$\lim_{s\to\infty} sX(s) = x(0^+) + \lim_{s\to\infty} \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st}\,dt$$

$$= x(0^+) + \int_{0^+}^{\infty} \frac{dx}{dt} \left(\lim_{s\to\infty} e^{-st}\right) dt$$

$$= x(0^+)$$

→ 0

Comment. The initial value theorem applies only if $X(s)$ is strictly proper $(M < N)$, because for $M \geq N$, $\lim_{s\to\infty} sX(s)$ does not exist, and the theorem does not apply. In such a case, we can still find the answer by using long division to express $X(s)$ as a polynomial in s plus a strictly proper fraction, where $M < N$. For example, by using long division, we can express

$$\frac{s^3 + 3s^2 + s + 1}{s^2 + 2s + 1} = (s+1) - \frac{2s}{s^2 + 2s + 1}$$

The inverse transform of the polynomial in s is in terms of $\delta(t)$, and its derivatives, which are zero at $t = 0^+$. In the foregoing case, the inverse transform of $s + 1$ is $\dot{\delta}(t) + \delta(t)$. Hence, the desired $x(0^+)$ is the value of the remainder (strictly proper) fraction, for which the initial value theorem applies. In the present case

$$x(0^+) = \lim_{s\to\infty} \frac{-2s^2}{s^2 + 2s + 1} = -2$$

To prove the final value theorem, we let $s \to 0$ in Eq. (4.24a) to obtain

$$\lim_{s\to 0}[sX(s) - x(0^-)] = \lim_{s\to 0} \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st}\,dt = \int_{0^-}^{\infty} \frac{dx}{dt}\,dt$$

$$= x(t)\Big|_{0^-}^{\infty} = \lim_{t\to\infty} x(t) - x(0^-)$$

a deduction that leads to the desired result, Eq. (4.34).

Comment. The final value theorem applies only if the poles of $X(s)$ are in the LHP (including $s = 0$). If $X(s)$ has a pole in the RHP, $x(t)$ contains an exponentially growing term and $x(\infty)$ does not exist. If there is a pole on the imaginary axis, then $x(t)$ contains an oscillating term and $x(\infty)$ does not exist. However, if there is a pole at the origin, then $x(t)$ contains a constant term, and hence, $x(\infty)$ exists and is a constant.

EXAMPLE 4.9

Determine the initial and final values of $y(t)$ if its Laplace transform $Y(s)$ is given by

$$Y(s) = \frac{10(2s+3)}{s(s^2+2s+5)}$$

Equations (4.33) and (4.34) yield

$$y(0^+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \frac{10(2s+3)}{(s^2+2s+5)} = 0$$

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{10(2s+3)}{(s^2+2s+5)} = 6$$

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