Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

Tables from Linear Systems and Signals 2nd ed by B.P. Lathi.

TABLE 4.1 A Short Table of (Unilateral) Laplace Transforms

No.	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$
9a	$e^{-at}\cos bt\, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at}\sin bt\, u(t)$	$\frac{b}{(s+a)^2+b^2}$
10a	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta - br\sin(\theta)}{s^2+2as+(a^2+b^2)}$
10b	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb}+\frac{0.5re^{-j\theta}}{s+a+jb}$
10c	$re^{-at}\cos(bt+\theta)\, u(t)$ $r=\sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$ $\theta=\tan^{-1}\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right)$ $b=\sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$
10d	$e^{-at}\left[A\cos bt+\frac{B-Aa}{b}\sin bt\right]u(t)$ $b=\sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$

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TABLE 4.2 The Laplace Transform Properties

Operation	$x(t)$	$X(s)$
Addition	$x_1(t) + x_2(t)$	$X_1(s) + X_2(s)$
Scalar multiplication	$kx(t)$	$kX(s)$
Time differentiation	$\frac{dx}{dt}$	$sX(s) - x(0^-)$
	$\frac{d^2x}{dt^2}$	$s^2X(s) - sx(0^-) - \dot{x}(0^-)$
	$\frac{d^3x}{dt^3}$	$s^3X(s) - s^2x(0^-) - s\dot{x}(0^-) - \ddot{x}(0^-)$
	$\frac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$
Time integration	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(t)\,dt$
Time shifting	$x(t-t_0)u(t-t_0)$	$X(s)e^{-st_0} \quad t_0 \geq 0$
Frequency shifting	$x(t)e^{s_0t}$	$X(s-s_0)$
Frequency differentiation	$-tx(t)$	$\frac{dX(s)}{ds}$
Frequency integration	$\frac{x(t)}{t}$	$\int_{s}^{\infty} X(z)\,dz$
Scaling	$x(at),\ a \geq 0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi j}X_1(s) * X_2(s)$
Initial value	$x(0^+)$	$\lim_{s\to\infty} sX(s) \quad (n > m)$
Final value	$x(\infty)$	$\lim_{s\to 0} sX(s)$ [poles of $sX(s)$ in LHP]

Notes from textbook:

3.11 The Laplace Transform

We have seen earlier that the output of a Linear Time Invariant (LTI) system can be expressed as the convolution of the input with the impulse response as show on the block diagram of Figure 3.29. This assumption is made provided that the convolution is expressed always in the time domain. The Laplace transform, on the other hand, converts the impulse response of the system from the time domain to the frequency domain.

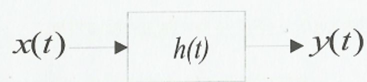


Figure 3.29: *A typical LTI system*

The Laplace Transform of a system can be calculated using the following formula; if $h(t)$ is the impulse response of a system, the Laplace Transform of $h(t)$ can be expressed from this formula.

$$H(s)=\int_0^\infty h(t)e^{-st}dt \qquad \text{(Equ.3.9)}$$

For a given $H(s)$, the Inverse Laplace Transform can be evaluated also by partial fraction expansion.

Example 3.9

$j := \sqrt{-1}$ $\quad \omega := 2 \cdot \pi$ $\quad i := j$

$s := j \cdot \omega$

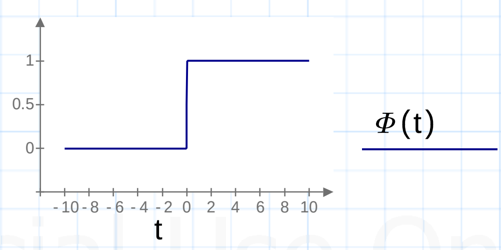
$a := 2$ 'a' is a constant
it is the a=2 in the solution's denominator (s+a) or (s+2)

$\Phi(t)$ unit step Prime built-in function

$$h(t) := e^{-a \cdot t} \cdot \Phi(t)$$

$$H(s) := \int_0^\infty h(t) \cdot e^{-s \cdot t}\, dt \rightarrow \frac{1}{s+2}$$ Laplace Transform - Answer

$H(s) = 0.046 - 0.144513j$



Example 3.10 - Inverse Laplace Transform

$$H(s) := \frac{2}{s+2}$$

Now to go back to the time domain solution for the inverse

The Lapalce transofrm looks like the solution in example 3.9, by inspection it looks like the H(s) is equal to

H(s) = 2 (1/s+2) => 2 (1/s + a)

So the time domain inverse Laplace is

2 e^-2t u(t)

$$H(s) := \frac{2}{s+2} \xrightarrow{\text{invlaplace}} 2 \cdot e^{-2 \cdot t}$$

Correct Answer.
Use the Evaluation Operator and fill in the label invlaplace

Example 3.11 - Poles and Zeros

clear(j)   clear(i)

Poles and zeros for system stability see textbook on systems

A system's transfer function is given below H(s) plot the poles and zeros

$$H(s) := \frac{(s+2)}{s \cdot (s+1) \cdot (s-1)}$$

Define the poles and zeros as a vector:

$p := 0..2$ number of poles is 3, since there are three 's' in the function
$z := 0..0$ 0 to 0 since there is only one '0'

From the function H(s) the poles and zeros are:
Poles: s(s + 1) (s - 1)
s = 0
s + 1 = 0
s - 1 = 0
s0=0, s1=-1, and s2= 1 by setting each term = 0 in the denominator

Zeros: (s+2)
s0 = -2 by setting the numerator term(s) equal zero

The pole and zero vectors are populated:

$ORIGIN := 0$

$j := \sqrt{-1}$

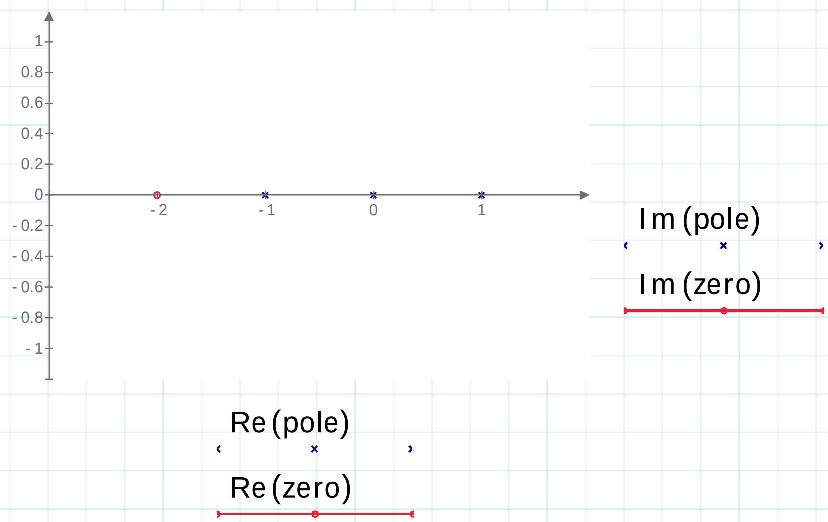
$pole := \begin{bmatrix} 0 + j \cdot 0 \\ -1 + j \cdot 0 \\ 1 + j \cdot 0 \end{bmatrix}$ $\quad pole = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ $\quad pole_1 = -1$

$zero := [-2 + j \cdot 0]$ $\quad zero = [-2]$ $\quad zero_0 = -2$

Now plot taking real and imaginary parts into consideration
Since s = j w , we have real and imaginary parts, so the poles and zeros are set similarly

Remember Prime/Mathcad has partial fractions function

The y axis is the imaginary axis in the plot, x axis the real part.



Example 3.12 - Poles and Zeros

$$H(s) := \frac{s^2 + \left(\frac{1}{2}\right)}{s^3 + s^2 + s + \frac{1}{2}}$$

Its a tedious task to get the roots of the equations above for the numerator and denominator.

So we use the 'polyroot' function in Prime/Mathcad

For polyroot a vector has to be created filling in the values for each power of and constant.

The top most element is the constant then starting with the variable, then variable squared,.....etc.

$$Z := \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

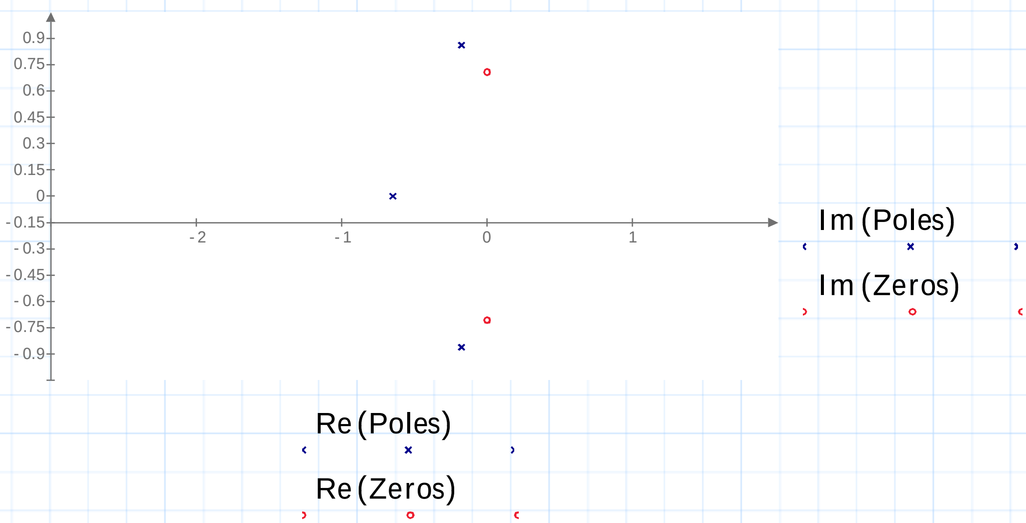
element 0 = 1/2 - constant
1 = 0 - s
2 = 1 - s^2

$Zeros := polyroots(Z)$ $\quad Zeros = \begin{bmatrix} -0.71j \\ 0.71j \end{bmatrix}$

$$P := \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

element 0 = 1/2 - constant
1 = 1 - s
2 = 1 - s^2
3 = 1 - s^3

$Poles := polyroots(P)$ $\quad Poles = \begin{bmatrix} -0.65 \\ -0.18 - 0.86j \\ -0.18 + 0.86j \end{bmatrix}$



Frequency Response in The Lapalce Transforms.

Example 3.13

$$H(s) := \frac{1}{s^3 + 2\cdot s^2 + 2\cdot s + 1}$$ System's transfer function

Plot the frequency response of the system and show the 3-dB:

$j := \sqrt{-1}$

$s := j\cdot\omega$

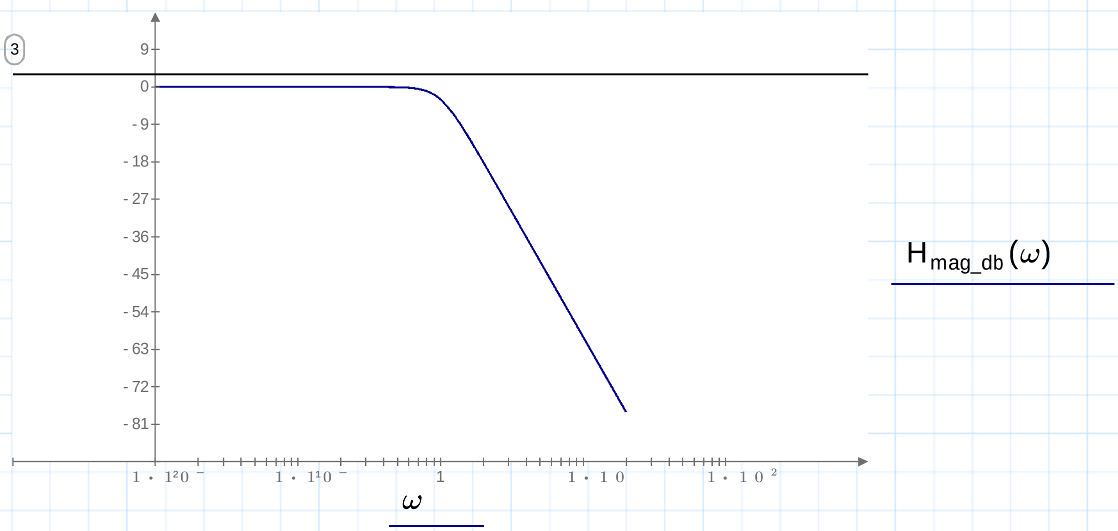
$\omega := 0.01, 0.02..20$ this line has to be placed after s = jw

Next setup the transfer function H(s) to H(jw):

$$H(\omega) := \frac{1}{(j\cdot\omega)^3 + 2\cdot(j\cdot\omega)^2 + 2\cdot(j\cdot\omega) + 1}$$

To plot the transfer function use the formula 20 log | H(s) | - note it is magnitude of |H(s)|:
Set plot in Logrithmic scale on x-axis w.

$$H_{mag\_db}(\omega) := 20\cdot\log\left(\left|\frac{1}{(j\cdot\omega)^3 + 2\cdot(j\cdot\omega)^2 + 2\cdot(j\cdot\omega) + 1}\right|\right)$$



Plot above the horizontal marker is set at 3dB.

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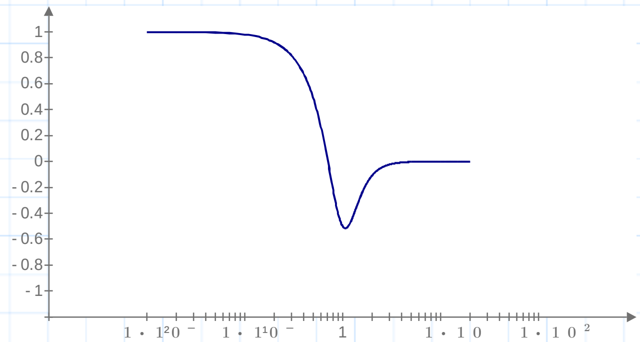
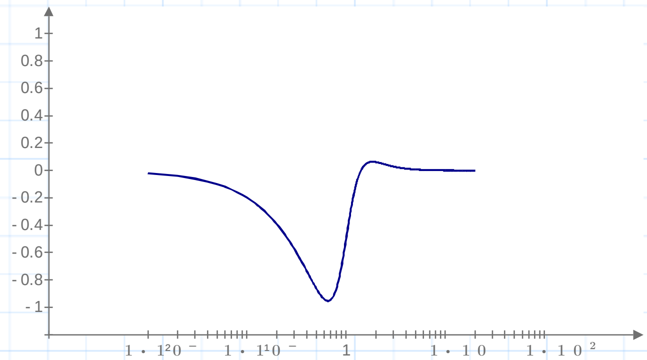
Next break the plot into the real part plot, and the phase in degrees plot.
Do NOT take the magnitude for the real, imaginary, and phase part.



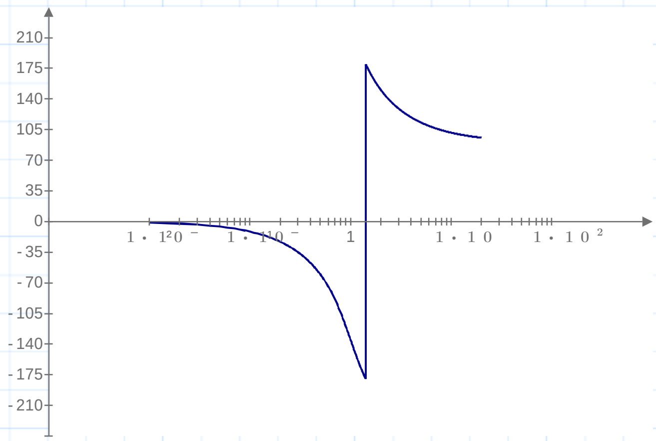
ω

$\mathrm{Re}(H(\omega))$



ω

$\mathrm{Im}(H(\omega))$



ω

$\arg(H(\omega))\cdot\left(\frac{180}{\pi}\right)$

Phase in degrees - multiply by 180/pi

Examples from Linear Systems and Signals 2nd Edition. Oxford. By B.P. Lathi.

Example 4.1

For a signal x(t) = e^-at u(t), find the Laplace transform X(s) and its region of convergence.

clear (s)   clear (X)

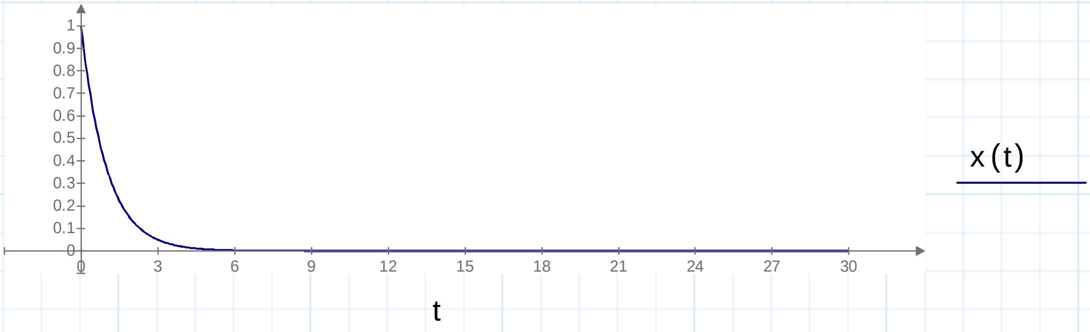
$j := \sqrt{-1}$ $\omega := 2 \cdot \pi$ $i := j$

$s := j \cdot \omega$

$a := 1$ 'a' is a constant
it is the a=1 in the solution's denominator (s+a) or (s+2)

$u(t) := if(|t| \geq 0, 1, 0)$ unit step function

$x(t) := e^{-a \cdot t} \cdot u(t)$ d(t) delta function is a Prime function

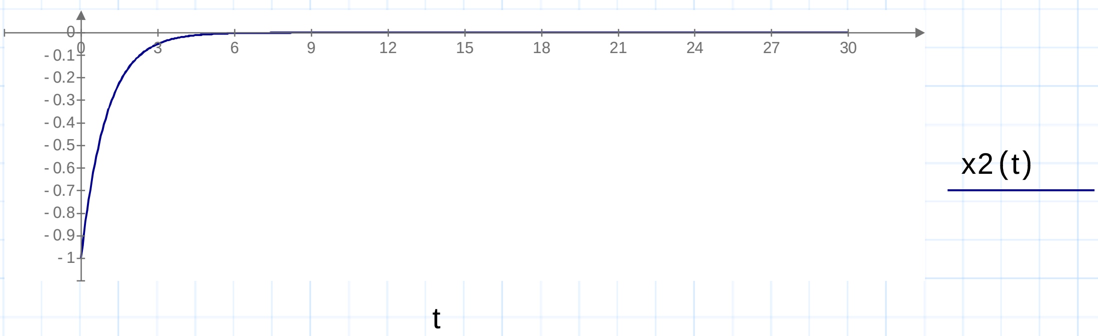


$$X(s) := \int_0^{\infty} x(t) \cdot e^{-s \cdot t} \, dt \rightarrow \frac{1}{s+1}$$ Laplace solution

There is a pole at s = -1, no zeros.
So the region of convergence is defined on the x-axis from -1 to infinty

Next a negative sign in x(t) and u(-t)

$x2(t) := -e^{-a \cdot t} \cdot u(-t)$ d(t) delta function is a Prime function



$$X2(s) := \int_0^{\infty} x2(t) \cdot e^{-s \cdot t} \, dt \rightarrow -\frac{1}{s+1}$$ Laplace solution

There is a pole at s = 1, no zeros.
-(s+1) = -s -1 ; s = 1
So the region of convergence is defined on the x-axis from 1 to -infinty

The valus of 'a' is a +ve of -ve integer value

Note: Page 385 - The Laplace transforms for the signal e^(-at) u(t) and -e^(at) u(t) are identical except for their region of convergence. Therefore for a given X(s) there may be more than one inverse transform, depending on the ROC region of occurence. In other words, unless the region of convergence is specified there is no one to one correspondence between X(s) and x(t). This fact increases complexity in using Laplace transform.

Example 4.2
Determine the Laplace transform of the following:
a). d(t) delta function
b). u(t) unit step function
c). cos w0 t u(t)

a)
$j := \sqrt{-1}$ $\omega := 2 \cdot \pi$ $i := j$ $s := j \cdot \omega$ $n := \omega$ $m := n$
clear (x) clear (X)
$x(t) := \delta(m, n)$ return 1 when m=n

$$X(s) := \int_0^{\infty} x(t) \cdot e^{-s \cdot t} \, dt$$

The transform tables show the Laplace transform of delta (t) = 1 for all s.

b).
clear (x) clear (X)
$u(t) := \text{if}(|t| \geq 0, 1, 0)$ unit step function

$x(t) := u(t)$

$$X(s) := \int_0^{\infty} x(t) \cdot e^{-s \cdot t} \, dt \rightarrow \frac{1}{s} - \frac{\lim_{t \to \infty} e^{-(s \cdot t)}}{s}$$

From tables the transform is 1/s for Re s > 0

c).
clear (x) clear (X) clear (x2)
$j := \sqrt{-1}$ $\omega_0 := 2 \cdot \pi$ $i := j$ $s := j \cdot \omega_0$
$u(t) := \text{if}(|t| \geq 0, 1, 0)$

$e\_cos(t) := \frac{1}{2} \cdot \left(e^{j \cdot \omega_0 \cdot t} + e^{-j \cdot \omega_0 \cdot t}\right)$ cosine term in exponential form

$x1(t) := \frac{1}{2} \cdot \left(e^{j \cdot \omega_0 \cdot t}\right) \cdot u(t)$ $x2(t) := \frac{1}{2} \cdot \left(e^{-j \cdot \omega_0 \cdot t}\right) \cdot u(t)$

From the tables no 8a:

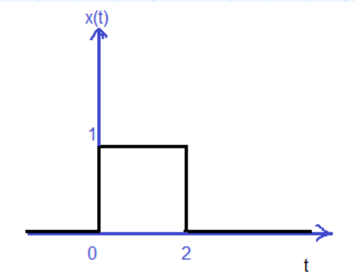
x1(t) = 1/2 (s/ (s^2 w0^2)
x2(t) = 1/2 (s/ (s^2 + w0^2)

$x(t) := x1(t) + x2(t)$

$X(s) := \int_0^{\infty} x(t) \cdot e^{-s \cdot t} \, dt$

X(s) = s / (s^2 + w0^2) Answer - when real part of s >0. See tables.

Exercise 4.2



a)
$j := \sqrt{-1}$ $\omega := 2 \cdot \pi$ $i := j$ $s := j \cdot \omega$
clear (x) clear (X)
$tt := 0, 1 .. 20$
$u(t) := if(0 \le t \le 2, 1, 0)$ return 1 when t is between 0 and 2

$x(t) := p(t)$

$X(s) := \int_0^{\infty} x(t) \cdot e^{-s \cdot t} \, dt$ cannot intergrate a conditional variable

Use the unit step function from 0 to 2
Start at 0 and subtract the rest starting at 2
t= 0 to 2 = 1, elsewhere 0
At t=0:
$x1(t) := u(0) = 1$

$$X1(s) := \int_0^{\infty} x1(t)\cdot e^{-s\cdot t}\,dt \rightarrow \frac{1}{s} - \frac{\lim_{t\rightarrow\infty} e^{-(s\cdot t)}}{s}$$

X1(s) = 1/s

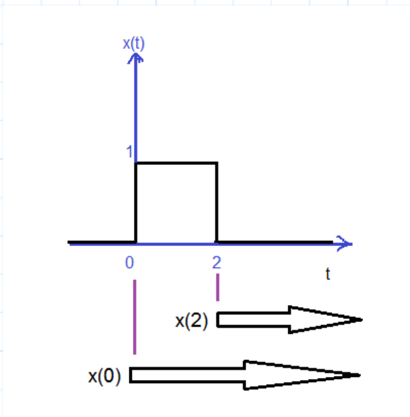
At t=2:

$$x2(t) := u(2) = 1$$

$$X2(s) := \int_0^{\infty} x2(t)\cdot e^{-s\cdot 2}\,dt \rightarrow e^{-2\cdot s}\cdot\infty$$ substitue t with 2 --> e^-s2

X2(s) is shifted to 2, its = 1/s at 0 not at 2 with the value of at 2 to infinity of e^-2s

X2(s) = 1/s (e^-2s)

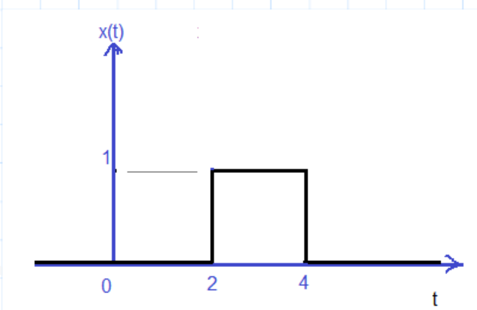


x(t --> 0-2) = x0 - x2
X(s) = X(0) - X(2)

X(s) = 1/s - 1/s(e^-2s)
X(s) = 1/s(1 - e^-2s)
for all s

Answer - check with your results.

b).



This signal is shiftet by t=2
Apply shift property from tables.

Shift t = 2 --> e-2s
X(s) = 1/s(1 - e^-2s) x Shift
X(s) = 1/s(1 - e^-2s)e^-2s
for all s

Answer - check with your results.

Exercise 4.3

Find the inverse Laplace transforms of the following:

$j := \sqrt{-1} \quad i := j \quad \omega := 2 \cdot \pi \quad s := j \cdot \omega$

a).

$$H1(s) := \frac{7 \cdot s - 6}{s^2 - s - 6} \xrightarrow{\text{invlaplace}} 7 \cdot e^{\frac{t}{2}} \cdot \left(\cosh\left(\frac{5 \cdot t}{2}\right) - \frac{\sinh\left(\frac{5 \cdot t}{2}\right)}{7} \right)$$

Prime/Mathcad solution above - not pleasant!
Apply PARTIAL FRACTIONS in Prime/Mathcad using function 'parfrac'.
To break the function into simpler parts

clear (s) Make sure s had been cleared cannot have s = jw for the parfrac evaluation

$$\frac{7 \cdot s - 6}{s^2 - s - 6} \xrightarrow{\text{parfrac}} \frac{4}{s+2} + \frac{3}{s-3}$$

Now apply Laplace transform to each term

$ss := j \cdot \omega$

$$H1a(s) := \frac{4}{s+2} \qquad H1a(s) \xrightarrow{\text{invlaplace}} \frac{4 \cdot \Delta(t)}{s+2}$$

From Laplace transform table - no 5
4[1/(s+2)] = 4e^-2t

$$H1b(s) := \frac{3}{s+2} \qquad H1b(s) \xrightarrow{\text{invlaplace}} \frac{3 \cdot \Delta(t)}{s+2}$$

From Laplace transform table - no 5
3[1/(s+2)] = 3e^-2t

$H1(s) := H1a(s) + H1b(s)$

$$H1(s) := \frac{4}{s+2} + \frac{3}{s+2}$$

With a unit step function u(t) as part of the input signal the transfer function H1(s) in exponential form now:
x(t) = (4e^-2t + 3e^3t)u(t) Answer - Inverse Laplace

b).

$$H2(s) := \frac{2 \cdot s^2 + 5}{s^2 + 3 \cdot s + 2} \xrightarrow{\text{invlaplace}} 2 \cdot \Delta(t) + 7 \cdot e^{-t} - 13 \cdot e^{-2 \cdot t}$$ - Prime solution

Apply PARTIAL FRACTIONS in Prime/Mathcad using function 'parfrac'.
To break the function into simpler parts

clear (s) Make sure s had been cleared it cannot be s = jw for the parfrac evaluation

$$\frac{2 \cdot s^2 + 5}{s^2 + 3 \cdot s + 2} \xrightarrow{\text{parfrac}} \frac{7}{s+1} - \frac{13}{s+2} + 2$$

Apply Laplace transform table no 5

7/(s+1) = 7 x (e^-at) a = 1 so 7 x (e^-t)

13/(s+2) = 13 x (e^-at) a = 2 so 13 x (e^-2t)

Apply Laplace transform table no 1

1 = d(t)
2 = 2d(t)

x(t) = [2d(t) + 7 (e^-at) - 13 (e^-2t)] u(t) with u(t) Answer Inverse Laplace

c).

$$H3(s) := \frac{6 \cdot (s + 34)}{s \cdot (s^2 + 10 \cdot s + 34)}$$

Apply PARTIAL FRACTIONS in Prime/Mathcad using function 'parfrac'.
To break the function into simpler parts

clear (s) Make sure s had been cleared it cannot be s = jw for the parfrac evaluation

$$\frac{6 \cdot (s + 34)}{s \cdot (s^2 + 10 \cdot s + 34)} \xrightarrow{\text{parfrac}} \frac{6}{s} - \frac{6 \cdot s + 54}{s^2 + 10 \cdot s + 34}$$

We proceed fresh using quadratic factors method:
Multiply both sides of equation by the denominator term

$$\frac{6(s + 34)}{s(s^2 + 10s + 34)} = \frac{k1}{s} + \frac{As + B}{(s^2 + 10s + 34)}$$

k1 = 6 from the partial fraction prior by Prime

6(s + 34) = 6(s^2 +10s + 34) + s(As + B)

Equating coefficients of s^2 and s on both sides

6s^2 + As^2
-6s^2 = As^2
A = -6

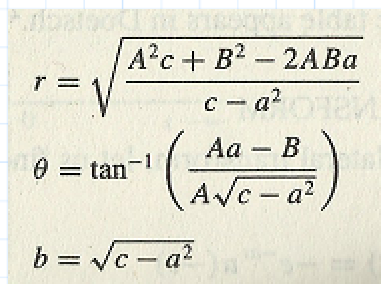
6s = 60s + Bs
Bs = s(-60+6) = -54s
B = -54

Now in the simpler form:

$$H3(s) := \frac{6}{s} + \frac{(-6 \cdot s - 54)}{s^2 + 10 \cdot s + 34}$$

Using transform table no 2 and 10c:

For 10c the parameters are:
A = -6, B = -54
a: 10 = 2a
a = 5
b = sqrt(c - a^2) = sqrt(34-25) = sqrt(9)
b = 3


$$r = \sqrt{\frac{A^2c + B^2 - 2ABa}{c - a^2}}$$
$$\theta = \tan^{-1}\left(\frac{Aa - B}{A\sqrt{c - a^2}}\right)$$
$$b = \sqrt{c - a^2}$$

$A := -6$ $B := -54$ $a := 5$ $b := 3$ $c := 34$

$$r := \sqrt{\left(\frac{A^2 \cdot c + B^2 - 2 \cdot A \cdot B \cdot a}{c - a^2}\right)} = 10$$

$r = 10$

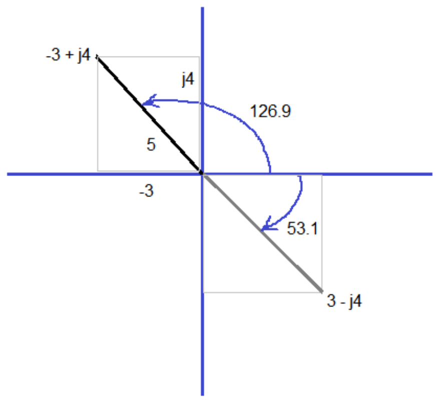
$A \cdot a - B = 24$ $A \cdot \sqrt{c - a^2} = -18$

24/-18= 4/-3 $\theta_1 := \mathrm{atan}\left(\frac{4}{-3}\right)$ $\theta_1 = -53.130102$ deg

Or plugging straight into the formula:

$$\theta := \operatorname{atan}\left(\frac{A \cdot a - B}{A \cdot \sqrt{c - a^2}}\right) = -53.1 \text{ deg}$$

Where is the angle -53.81 located?
Right now at the lower right quadrant at 3 - 4j but the angle is pointing to the other direction so we rotate it anticlockwise 180 degrees to the vector -3 + j4 its conjugate.



$$\theta_{pos} := 180 \text{ deg} + \theta = 126.9 \text{ deg}$$

Now forming the laplace inverse equation:

x(t) = [6 + 10e^-5t cos(3t +126.9 deg)] u(t) Answer - Inverse Laplace

The solution Prime provided Not the exact same.

$$H3(s) := \frac{6 \cdot (s + 34)}{s \cdot (s^2 + 10 \cdot s + 34)} \xrightarrow{\text{invlaplace}} 6 - 8 \cdot \sin(3 \cdot t) \cdot e^{-5 \cdot t} - 6 \cdot \cos(3 \cdot t) \cdot e^{-5 \cdot t}$$

$$H3(s) := \frac{6}{s} + \frac{(-6 \cdot s - 54)}{s^2 + 10 \cdot s + 34} \xrightarrow{\text{invlaplace}} 6 - 8 \cdot \sin(3 \cdot t) \cdot e^{-5 \cdot t} - 6 \cdot \cos(3 \cdot t) \cdot e^{-5 \cdot t}$$

d).

$$H4(s) := \frac{8 \cdot s + 10}{(s + 1) \cdot (s + 2)^3}$$

Apply PARTIAL FRACTIONS in Prime/Mathcad using function 'parfrac'.
To break the function into simpler parts

$$H4(s) := \frac{8\cdot s+10}{(s+1)\cdot(s+2)^3} \xrightarrow{\text{parfrac}} \frac{2}{s+1} - \frac{2}{s+2} - \frac{2}{(s+2)^2} + \frac{6}{(s+2)^3}$$

Clean results above ready to use

5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$

Use nos 5, 6, and 7 above to finish the solution, check to the correct answer generated by Prime/Mathcad.

$$H4(s) := \frac{2}{s+1} - \frac{2}{s+2} - \frac{2}{(s+2)^2} + \frac{6}{(s+2)^3}$$

$$H4(s) \xrightarrow{\text{invlaplace}} 2\cdot e^{-t} - 2\cdot e^{-2\cdot t} - 2\cdot t\cdot e^{-2\cdot t} + 3\cdot t^2\cdot e^{-2\cdot t}$$ Answer Inverse Laplace

Exercise E4.2

i).

$j := \sqrt{-1}$ $\omega := 2\cdot\pi$ $i := j$ $s := j\cdot\omega$

$x(t) := 10\cdot e^{-3\cdot t}\cdot\cos(4\cdot t + 53.13\ deg)$ Find the Laplace Transform

10a	$re^{-at}\cos(bt+\theta)u(t)$	$\frac{(r\cos\theta)s + (ar\cos\theta - br\sin(\theta))}{s^2 + 2as + (a^2+b^2)}$

Use transform no 10a from the table:

$r := 10$ $a := 3$ $b := 4$ $\theta := 53.13\ deg$

$$X(\omega) := \frac{r\cdot\cos(\theta)\cdot s + a\cdot r\cdot\cos(\theta) - b\cdot r\cdot\sin(\theta)}{s^2 + 2\cdot a\cdot s + (a^2 + b^2)}$$

$$X(\omega) := \frac{10\cdot\cos(\theta)\cdot s + 30\cdot\cos(\theta) - 40\cdot\sin(\theta)}{s^2 + 6\cdot s + 25}$$

$10\cdot\cos(\theta) = 6$ $30\cdot\cos(\theta) = 18$ $40\cdot\sin(\theta) = 32$

$$X(\omega) := \frac{6 \cdot s + 18 - 32}{s^2 + 6 \cdot s + 25}$$

$$X(\omega) := \frac{6 \cdot s - 14}{s^2 + 6 \cdot s + 25}$$ Answer

ii).
Find the inverse Laplace transform of the following:

a). (s + 17) / (s^2 + 4s -5)

clear (s)

$$X(\omega) := \frac{s + 17}{s^2 + 4 \cdot s - 5}$$

$$\frac{s + 17}{s^2 + 4 \cdot s - 5} \xrightarrow{\text{parfrac}} \frac{3}{s - 1} - \frac{2}{s + 5}$$

Apply no 5 in the list:

5	$e^{\lambda t}u(t)$	$\frac{1}{s - \lambda}$

x(t): 3e^(st) - 2e(-5st)

x(t): [3e^(st) - 2e(-5st)] u(t) Answer - Inverse Laplace

Using Prime/Mathcad:

$$\frac{3}{s - 1} - \frac{2}{s + 5} \xrightarrow{\text{invlaplace}} 3 \cdot e^{t} - 2 \cdot e^{-5 \cdot t}$$ Verifies Answer

b).

$j := \sqrt{-1}$ $\omega := 2 \cdot \pi$ $i := j$ $s := j \cdot \omega$
clear (s)

$$X(\omega) := \frac{3 \cdot s - 5}{(s + 1) \cdot (s^2 + 2 \cdot s + 5)}$$

$$\frac{3 \cdot s - 5}{(s + 1) \cdot (s^2 + 2 \cdot s + 5)} \xrightarrow{\text{parfrac}} \frac{2 \cdot s + 5}{s^2 + 2 \cdot s + 5} - \frac{2}{s + 1}$$

Apply Laplace transform number 10c to first term, and number 5 to second term.

$$\frac{2\cdot s+5}{s^2+2\cdot s+5}$$

10c	$re^{-at}\cos(bt+\theta)\,u(t)$	$\frac{As+B}{s^2+2as+c}$

$a:=1 \quad c:=5$

$A:=2 \quad B:=5$

$$r:=\sqrt{\left(\frac{A^2\cdot c+B^2-2\cdot A\cdot B\cdot a}{c-a^2}\right)}=2.5$$

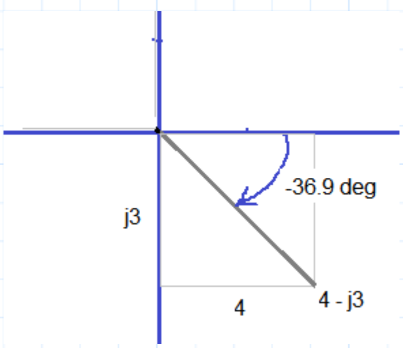
$$b:=\sqrt{\left(c-a^2\right)}=2$$

$$\theta:=\operatorname{atan}\left(\frac{A\cdot a-B}{A\cdot\sqrt{c-a^2}}\right)=-36.9\ \text{deg}$$

next verify correct direction of the angle

$A\cdot a-B=-3 \qquad A\cdot\sqrt{c-a^2}=4$

$$\theta:=\operatorname{atan}\left(\frac{-3}{4}\right)=-36.9\ \text{deg}$$



Correct.

x(t) = [2.5e^(-t) cos(2t-36.9deg) -2e^(-t)] u(t)   Answer - Inverse Laplace

Verify quadratice term with Prime/Mathcad:

$$\frac{2\cdot s+5}{s^2+2\cdot s+5}\xrightarrow{\text{invlaplace}}2\cdot e^{-t}\cdot\left(\cos(2\cdot t)+\frac{3\cdot\sin(2\cdot t)}{4}\right)$$

This instance for me the table solution is more suitable!

c).

$j := \sqrt{-1}$ $\omega := 2 \cdot \pi$ $i := j$ $s := j \cdot \omega$

clear (s)

$$X(\omega) := \frac{16 \cdot s + 43}{(s-2) \cdot (s+3)^2}$$

Expecting some combinations of from the table.

$$\frac{16 \cdot s + 43}{(s-2) \cdot (s+3)^2} \xrightarrow{\text{parfrac}} \frac{3}{s-2} - \frac{3}{s+3} + \frac{1}{(s+3)^2}$$

Prime partial fraction resulted with clear fractions, the time domain inverse transform expection is encouraging.

$\frac{3}{s-2}$ 3 e^(2t)

$\frac{3}{s+2}$ 3e^(-2t)

$\frac{1}{(s+3)^2}$ te^(-3t) from no 6 in table

x(t): [3 e^(2t) + 3e^(-2t) + te^(-3t)] u(t) Answer - Inverse Laplace Transform

Next notes on properties of Laplace transforms
from Signals and Systems 2nd ed by B.P. Lathi.

There are specific topics such as Bode Plots, Filters, Solutions of Differential and Integro-Differential Equations in the textbook in detail. These are specific to a course's content like Circuit Networks, Filters, Differential Equations, Controls,....., which you can continue on your own in context to those course.

The main objective:
1. to get started with Laplace for engineering problem solving
2. to get over the main hurdle in Laplace Transforms mathematics and using Prime/Mathcad.

4.2-4 The Time-Integration Property

The time-integration property states that if

$$x(t) \Longleftrightarrow X(s)$$

then†

$$\int_{0^-}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{X(s)}{s}$$

and

$$\int_{-\infty}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\tau)\,d\tau}{s}$$

†The dual of the time-integration property is the frequency-integration property, which states that

$$\frac{x(t)}{t} \Longleftrightarrow \int_{s}^{\infty} X(z)\,dz$$

Proof. We define

$$g(t) = \int_{0^-}^{t} x(\tau)\,d\tau$$

so that

$$\frac{d}{dt}g(t) = x(t) \quad \text{and} \quad g(0^-) = 0$$

Now, if

$$g(t) \Longleftrightarrow G(s)$$

then

$$X(s) = \mathcal{L}\left[\frac{d}{dt}g(t)\right] = sG(s) - g(0^-) = sG(s)$$

Therefore

$$G(s) = \frac{X(s)}{s}$$

or

$$\int_{0^-}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{X(s)}{s}$$

To prove Eq. (4.26), observe that

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{0^-} x(\tau)\,d\tau + \int_{0^-}^{t} x(\tau)\,d\tau$$

Reached the limit

Note that the first term on the right-hand side is a constant for $t \geq 0$. Taking the Laplace transform of the foregoing equation and using Eq. (4.25), we obtain

$$\int_{-\infty}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{\int_{-\infty}^{0^-} x(\tau)\,d\tau}{s} + \frac{X(s)}{s}$$

SCALING

The scaling property states that if

$$x(t) \Longleftrightarrow X(s)$$

then for $a > 0$

$$x(at) \Longleftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right) \quad (4.27)$$

The proof is given in Chapter 7. Note that a is restricted to positive values because if $x(t)$ is causal, then $x(at)$ is anticausal (is zero for $t \geq 0$) for negative a, and anticausal signals are not permitted in the (unilateral) Laplace transform.

Recall that $x(at)$ is the signal $x(t)$ time-compressed by the factor a, and $X(\frac{s}{a})$ is $X(s)$ expanded along the s-scale by the same factor a (see Section 1.2-2). The scaling property states that *time compression of a signal by a factor a causes expansion of its Laplace transform in the s-scale by the same factor. Similarly, time expansion x(t) causes compression of X(s) in the s-scale by the same factor.*

4.2-5 Time Convolution and Frequency Convolution

Another pair of properties states that if

$$x_1(t) \Longleftrightarrow X_1(s) \qquad \text{and} \qquad x_2(t) \Longleftrightarrow X_2(s)$$

then (*time-convolution property*)

$$x_1(t) * x_2(t) \Longleftrightarrow X_1(s)X_2(s) \tag{4.38}$$

and (*frequency-convolution property*)

$$x_1(t)x_2(t) \Longleftrightarrow \frac{1}{2\pi j}[X_1(s) * X_2(s)] \tag{4.39}$$

Observe the symmetry (or duality) between the two properties. Proofs of these properties are postponed to Chapter 7.

Equation (2.48) indicates that $H(s)$, the transfer function of an LTIC system, is the Laplace transform of the system's impulse response $h(t)$; that is,

linear time invariant continuous time

$$h(t) \Longleftrightarrow H(s) \tag{4.30}$$

If the system is causal, $h(t)$ is causal, and, according to Eq. (2.48), $H(s)$ is the unilateral Laplace transform of $h(t)$. Similarly, if the system is noncausal, $h(t)$ is noncausal, and $H(s)$ is the bilateral transform of $h(t)$.

We can apply the time-convolution property to the LTIC input–output relationship $y(t) = x(t) * h(t)$ to obtain

$$Y(s) = X(s)H(s) \tag{4.31}$$

The response $y(t)$ is the zero-state response of the LTIC system to the input $x(t)$. From Eq. (4.31) it follows that

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}[\text{zero-state response}]}{\mathcal{L}[\text{input}]} \tag{4.32}$$

This may be considered to be an alternate definition of the LTIC system transfer function $H(s)$. It is the *ratio of the transform of zero-state response to the transform of the input.*

EXAMPLE 4.8

Using the time-convolution property of the Laplace transform to determine $c(t) = e^{at}u(t)*e^{bt}u(t)$.

From Eq. (4.28), it follows that

$$C(s) = \frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

The inverse transform of this equation yields

$$c(t) = \frac{1}{a-b}(e^{at} - e^{bt})u(t)$$

TABLE 4.2 The Laplace Transform Properties

Operation	$x(t)$	$X(s)$
Addition	$x_1(t) + x_2(t)$	$X_1(s) + X_2(s)$
Scalar multiplication	$kx(t)$	$kX(s)$
Time differentiation	$\frac{dx}{dt}$	$sX(s) - x(0^-)$
	$\frac{d^2x}{dt^2}$	$s^2X(s) - sx(0^-) - \dot{x}(0^-)$
	$\frac{d^3x}{dt^3}$	$s^3X(s) - s^2x(0^-) - s\dot{x}(0^-) - \ddot{x}(0^-)$
	$\frac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$
Time integration	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(t)\,dt$
Time shifting	$x(t-t_0)u(t-t_0)$	$X(s)e^{-st_0} \quad t_0 \geq 0$
Frequency shifting	$x(t)e^{s_0t}$	$X(s-s_0)$
Frequency differentiation	$-tx(t)$	$\frac{dX(s)}{ds}$
Frequency integration	$\frac{x(t)}{t}$	$\int_s^{\infty} X(z)\,dz$
Scaling	$x(at), a \geq 0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi j}X_1(s) * X_2(s)$
Initial value	$x(0^+)$	$\lim_{s\to\infty} sX(s) \quad (n > m)$
Final value	$x(\infty)$	$\lim_{s\to 0} sX(s) \quad$ [poles of $sX(s)$ in LHP]

Initial and Final Values

In certain applications, it is desirable to know the values of $x(t)$ as $t \to 0$ and $t \to \infty$ [initial and final values of $x(t)$] from the knowledge of its Laplace transform $X(s)$. Initial and final value theorems provide such information.

The initial value theorem states that if $x(t)$ and its derivative dx/dt are both Laplace transformable, then

$$x(0^+) = \lim_{s\to\infty} sX(s) \tag{4.33}$$

provided the limit on the right-hand side of Eq. (4.33) exists.

Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

The final value theorem states that if both $x(t)$ and dx/dt are Laplace transformable, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \quad (4.3)$$

provided $sX(s)$ has no poles in the RHP or on the imaginary axis. To prove these theorems, we use Eq. (4.24a)

$$\begin{aligned}
sX(s) - x(0^-) &= \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st}\,dt \\
&= \int_{0^-}^{0^+} \frac{dx}{dt} e^{-st}\,dt + \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st}\,dt \\
&= x(t)\Big|_{0^-}^{0^+} + \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st}\,dt \\
&= x(0^+) - x(0^-) + \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st}\,dt
\end{aligned}$$

Therefore

$$sX(s) = x(0^+) + \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st}\,dt$$

and

$$\begin{aligned}
\lim_{s\to\infty} sX(s) &= x(0^+) + \lim_{s\to\infty} \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st}\,dt \\
&= x(0^+) + \int_{0^+}^{\infty} \frac{dx}{dt} \left(\lim_{s\to\infty} e^{-st}\right) dt \\
&= x(0^+)
\end{aligned}$$

Comment. The initial value theorem applies only if $X(s)$ is strictly proper ($M < N$), because for $M \geq N$, $\lim_{s\to\infty} sX(s)$ does not exist, and the theorem does not apply. In such a case, we can still find the answer by using long division to express $X(s)$ as a polynomial in s plus a strictly proper fraction, where $M < N$. For example, by using long division, we can express

$$\frac{s^3 + 3s^2 + s + 1}{s^2 + 2s + 1} = (s+1) - \frac{2s}{s^2 + 2s + 1}$$

The inverse transform of the polynomial in s is in terms of $\delta(t)$, and its derivatives, which are zero at $t = 0^+$. In the foregoing case, the inverse transform of $s+1$ is $\dot{\delta}(t) + \delta(t)$. Hence, the desired $x(0^+)$ is the value of the remainder (strictly proper) fraction, for which the initial value theorem applies. In the present case

$$x(0^+) = \lim_{s\to\infty} \frac{-2s^2}{s^2 + 2s + 1} = -2$$

To prove the final value theorem, we let $s \to 0$ in Eq. (4.24a) to obtain

$$\begin{aligned}
\lim_{s\to 0}[sX(s) - x(0^-)] &= \lim_{s\to 0} \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st}\,dt = \int_{0^-}^{\infty} \frac{dx}{dt}\,dt \\
&= x(t)\Big|_{0^-}^{\infty} = \lim_{t\to\infty} x(t) - x(0^-)
\end{aligned}$$

a deduction that leads to the desired result, Eq. (4.34).

Comment. The final value theorem applies only if the poles of $X(s)$ are in the LHP (including $s = 0$). If $X(s)$ has a pole in the RHP, $x(t)$ contains an exponentially growing term and $x(\infty)$ does not exist. If there is a pole on the imaginary axis, then $x(t)$ contains an oscillating term and $x(\infty)$ does not exist. However, if there is a pole at the origin, then $x(t)$ contains a constant term, and hence, $x(\infty)$ exists and is a constant.

EXAMPLE 4.9

Determine the initial and final values of $y(t)$ if its Laplace transform $Y(s)$ is given by

$$Y(s) = \frac{10(2s+3)}{s(s^2+2s+5)}$$

Equations (4.33) and (4.34) yield

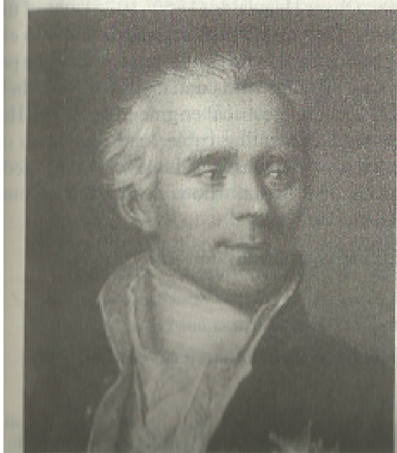
$$y(0^+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \frac{10(2s+3)}{(s^2+2s+5)} = 0$$

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{10(2s+3)}{(s^2+2s+5)} = 6$$

Next a short introduction to Laplace and Heaviside.
Laplace met Napolean (French Military Leader)
Heaviside was ignored for several of his discoveries.

A HISTORICAL NOTE:
MARQUIS PIERRE-SIMON DE LAPLACE (1749–1827)

The Laplace transform is named after the great French mathematician and astronomer Laplace, who first presented the transform and its applications to differential equations in a paper published in 1779.

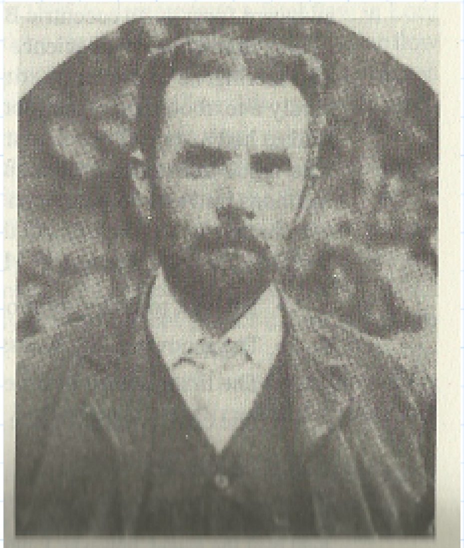


Pierre-Simon de Laplace

Laplace developed the foundations of potential theory and made important contribu to special functions, probability theory, astronomy, and celestial mechanics. In his *Exposit du système du monde* (1796), Laplace formulated a nebular hypothesis of cosmic origin tried to explain the universe as a pure mechanism. In his *Traité de mécanique céleste* (*cele mechanics*), which completed the work of Newton, Laplace used mathematics and physics subject the solar system and all heavenly bodies to the laws of motion and the principle gravitation. Newton had been unable to explain the irregularities of some heavenly bod in desperation, he concluded that God himself must intervene now and then to prevent catastrophes as Jupiter eventually falling into the sun (and the moon into the earth) as predi by Newton's calculations. Laplace proposed to show that these irregularities would corr themselves periodically and that a little patience—in Jupiter's case, 929 years—would everything returning automatically to order; thus there was no reason why the solar and the ste systems could not continue to operate by the laws of Newton and Laplace to the end of time

Laplace presented a copy of *Mécanique céleste* to Napoleon, who, after reading the boo took Laplace to task for not including God in his scheme: "You have written this huge book the system of the world without once mentioning the author of the universe." "Sire," Lapl retorted, "I had no need of that hypothesis." Napoleon was not amused, and when he repea this reply to another great mathematician-astronomer, Louis de Lagrange, the latter remar "Ah, but that is a fine hypothesis. It explains so many things."[5]

Napoleon, following his policy of honoring and promoting scientists, made Laplace the min ister of the interior. To Napoleon's dismay, however, the new appointee attempted to bring "th spirit of infinitesimals" into administration, and so Laplace was transferred hastily to the sena



Oliver Heaviside

OLIVER HEAVISIDE (1850–1925)

Although Laplace published his transform method to solve differential equations in 1779, the method did not catch on until a century later. It was rediscovered independently in a rather awkward form by an eccentric British engineer, Oliver Heaviside (1850–1925), one of the tragic figures in the history of science and engineering. Despite his prolific contributions to electrical engineering, he was severely criticized during his lifetime and was neglected later to the point that hardly a textbook today mentions his name or credits him with contributions. Nevertheless, his studies had a major impact on many aspects of modern electrical engineering. It was Heaviside who made transatlantic communication possible by inventing cable loading, but no one even mentions him as a pioneer or an innovator in telephony. It was Heaviside who suggested the use of inductive cable loading, but the credit is given to M. Pupin, who was not even responsible for building the first loading coil.† In addition, Heaviside was[6]

- The first to find a solution to the distortionless transmission line.
- The innovator of lowpass filters.
- The first to write Maxwell's equations in modern form.
- The codiscoverer of rate energy transfer by an electromagnetic field.

†Heaviside developed the theory for cable loading, George Campbell built the first loading coil, and the telephone circuits using Campbell's coils were in operation before Pupin published his paper. In the legal fight over the patent, however, Pupin won the battle because of his shrewd self-promotion and the poor legal support for Campbell.

Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

- An early champion of the now-common phasor analysis.
- An important contributor to the development of vector analysis. In fact, he essentially created the subject independently of Gibbs.[7]
- An originator of the use of operational mathematics used to solve linear integro-differential equations, which eventually led to rediscovery of the ignored Laplace transform.
- The first to theorize (along with Kennelly of Harvard) that a conducting layer (the Kennelly–Heaviside layer) of atmosphere exists, which allows radio waves to follow earth's curvature instead of traveling off into space in a straight line.
- The first to posit that an electrical charge would increase in mass as its velocity increases, an anticipation of an aspect of Einstein's special theory of relativity.[8] He also forecast the possibility of superconductivity.

Heaviside was a self-made, self-educated man. Although his formal education ended with elementary school, he eventually became a pragmatically successful mathematical physicist. He began his career as a telegrapher, but increasing deafness forced him to retire at the age of 24. He then devoted himself to the study of electricity. His creative work was disdained by many professional mathematicians because of his lack of formal education and his unorthodox methods.

Heaviside had the misfortune to be criticized both by mathematicians, who faulted him for lack of rigor, and by men of practice, who faulted him for using too much mathematics and thereby confusing students. Many mathematicians, trying to find solutions to the distortionless transmission line, failed because no rigorous tools were available at the time. Heaviside succeeded because he used mathematics not with rigor, but with insight and intuition. Using his much maligned operational method, Heaviside successfully attacked problems that the rigid mathematicians could not solve, problems such as the flow of heat in a body of spatially varying conductivity. Heaviside brilliantly used this method in 1895 to demonstrate a fatal flaw in Lord Kelvin's determination of the geological age of the earth by secular cooling; he used the same flow of heat theory as for his cable analysis. Yet the mathematicians of the Royal Society remained unmoved and were not the least impressed by the fact that Heaviside had found the answer to problems no one else could solve. Many mathematicians who examined his work dismissed it with contempt, asserting that his methods were either complete nonsense or a rehash of known ideas.[6]

Sir William Preece, the chief engineer of the British Post Office, a savage critic of Heaviside, ridiculed Heaviside's work as too theoretical and, therefore, leading to faulty conclusions. Heaviside's work on transmission lines and loading was dismissed by the British Post Office and might have remained hidden, had not Lord Kelvin himself publicly expressed admiration for it.[6]

Heaviside's operational calculus may be formally inaccurate, but in fact it anticipated the operational methods developed in more recent years.[9] Although his method was not fully understood, it provided correct results. When Heaviside was attacked for the vague meaning of his operational calculus, his pragmatic reply was, "Shall I refuse my dinner because I do not fully understand the process of digestion?"

Heaviside lived as a bachelor hermit, often in near-squalid conditions, and died largely unnoticed, in poverty. His life demonstrates the persistent arrogance and snobbishness of the intellectual establishment, which does not respect creativity unless it is presented in the strict language of the establishment.

EoF - End of File