DIGITAL FILTERS (IIR) EQUIVALENT TO LINEAR CLASSICS

Basics

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The aim of this work is to give a simple and concise (but not exhaustive) presentation of the argument to all who are interested in it. We've created this worksheet to implement digital filters (IIR) similar to the classics ones for DSP applications. We assume that the reader already knows both the concepts and the use of the Laplace transform in the study of linear and invariant lumped electrical circuits and everything about the z-transform. However, we report about the latter, some concepts and some useful examples in the worksheets dedicated to this argument. The paper therefore, consists of an initial part which refers to the z-transform. Subsequently (worksheet 2) we depict linear and time invariant active filters and the transfer functions thereof, namely: two filters of the first order (low pass and high pass) and four of the second order (low pass, high pass, band pass and band reject). For each of them, (worksheet: 3, 4, 5, 6, 7, 8, 9, 10), we report the impulse response, the unit step response and the Bode diagrams. Subsequently, again for each of them, we calculate the corresponding transfer function in z domain using the two common transformations s->z i.e. the linear and the bilinear. For each of them follows the corresponding difference equation and the Bode plots together with those of the linear system for comparison. For each filter we derive the parameters to implement the digital filter with a given DSP.

CONTENT

This Worksheet

Other Worksheets:

INTRODUCTION

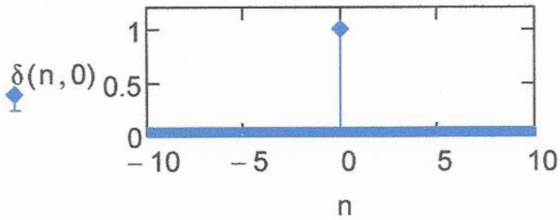
The most common signals used when dealing with signal processing are those here illustrated:

$$\text{Unitary pulse: } \delta(\nu, k) := \begin{cases} 1 & \text{if } \nu = k \ , \\ 0 & \text{otherwise} \end{cases} \qquad (1.1)$$

which is an abstract mathematical entity.
In electronics we use very short voltage pulses of limited amplitude in agreement with the DSP data.

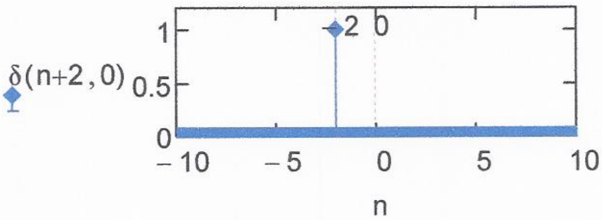
$n := 0 .. 99$

Unitary pulse:



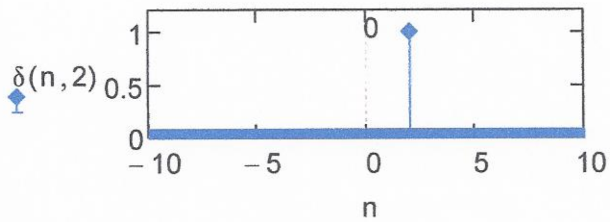
(fig. 1.0.1)

Unit impulse in advance.



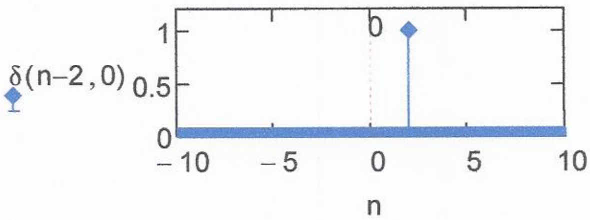
(fig. 1.0.2)

Unit impulse delayed.

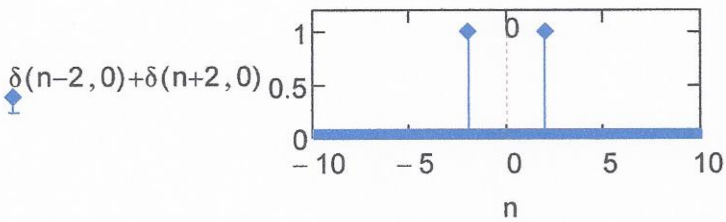


(fig. 1.0.3)

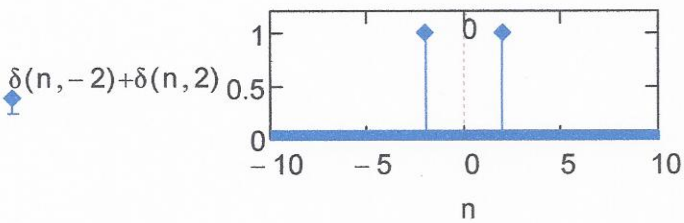
Unit impulse delayed.



(fig. 1.0.4)



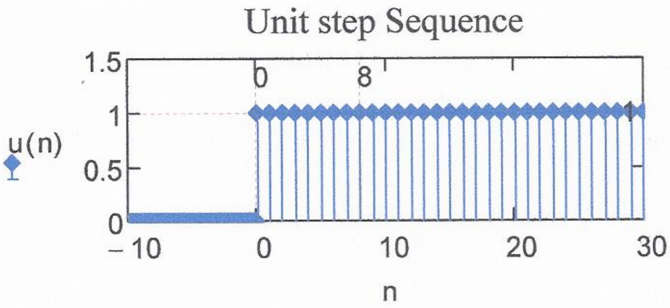
(fig. 1.0.5)



(fig. 1.0.6)

$$\text{Unitary step: } \quad u(n) := \begin{cases} \sum_{k=0}^{n} \delta(n,k) & \text{if } n \geq 0 \ , \\ 0 & \text{otherwise} \end{cases} \qquad (1.2)$$

$$\text{Generic sequence: } x_o(n) = \sum_{k=-\infty}^{\infty} \left(x(k) \cdot \delta(k, n-k)\right) \qquad (1.3)$$



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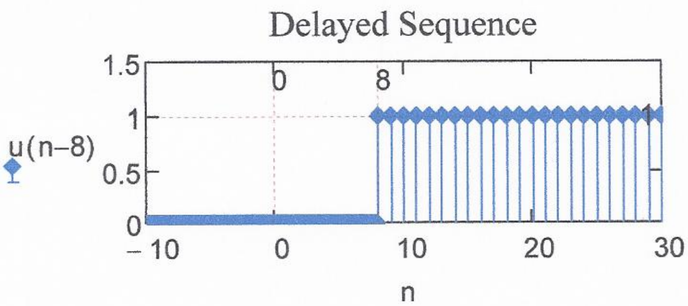
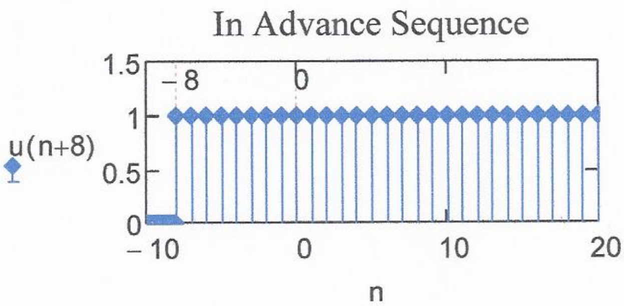


(fig. 1.0.8)



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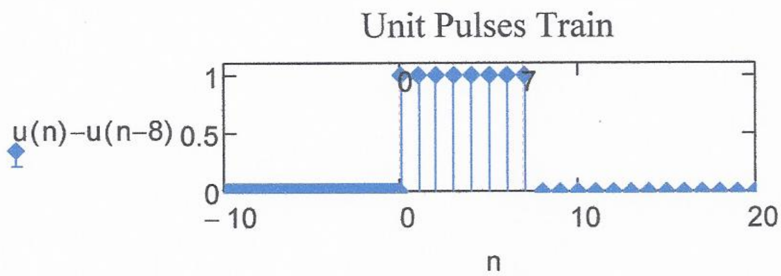

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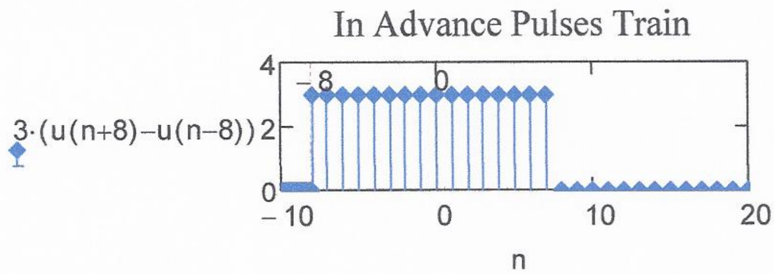
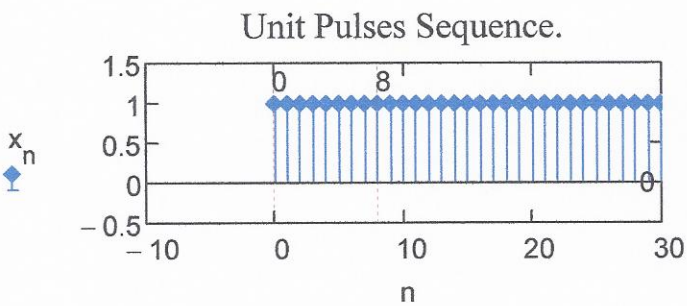


(fig. 1.0.11)

$\nu := 0 .. 100$ $\qquad x_\nu := u(\nu)$



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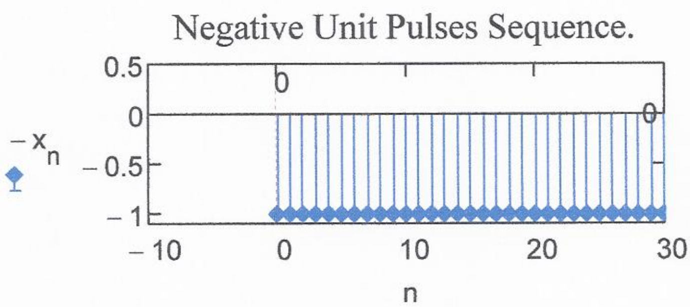


(fig. 1.0.13)

§1.1) The Z transform of a generic causal sequence h_n,

(non causal: unitary pulse response $h_n \neq 0$ **for n<0)** is so defined:

$$H(z) = \int_{-\infty}^{\infty} \phi(t)\cdot z^{-t}\,dt = \sum_{n=0}^{\infty} \left(h_n\cdot z^{-n}\right), \qquad (1.1.1)$$

where: $\phi(t) = \sum_{n=0}^{\infty} (h(n)\cdot\Delta(t-n))$ and h(n) are the sample values of h(t) at t=0,1,2...n, and $\Delta(t)$ is the well known Dirac pulse.

Example 1): $h(n) = \frac{4}{5}\cdot\left(1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}+...\right)$,

(geometric series)

z transform: $$H(z) = \sum_{n=0}^{\infty}\left(h_n\cdot z^{-n}\right) = \frac{4}{5}\cdot\left(1+\frac{z^{-1}}{2}+\frac{z^{-2}}{4}+\frac{z^{-3}}{8}+...+\frac{z^{-n}}{2^n}+...\right),\ q = \frac{z^{-1}}{2}$$

$$H(z) = \frac{4}{5}\cdot\sum_{n=0}^{\infty}\left(\frac{z^{-n}}{2^n}\right) = \frac{4}{5}\cdot\lim_{N\to\infty}\frac{1-\left(\frac{z^{-1}}{2}\right)^N}{1-\frac{z^{-1}}{2}}.$$

$$H(z) = \frac{4}{5}\cdot\frac{2}{2-z^{-1}}$$

Example 2): $h(n) = \frac{4}{5}\cdot\left(1+0.75+0.75^2+0.75^3+...+0.75^n+...\right)$,

$$q = \frac{3}{4}\cdot z^{-1},\quad H(z) = \frac{4}{5}\cdot\sum_{n=0}^{\infty}\left[\left(\frac{3}{4}\right)^n\cdot z^{-n}\right],$$

$$\frac{4}{5}\cdot\sum_{n=0}^{\infty}\left[\left(\frac{3}{4}\right)^n\cdot z^{-n}\right] = \frac{4}{5}\cdot\lim_{N\to\infty}\frac{\left[1-\left(0.75\cdot z^{-1}\right)^N\right]}{1-0.75\cdot z^{-1}},$$

$$\lim_{N\to\infty}\frac{\left[1-\left(0.75\cdot z^{-1}\right)^N\right]}{1-0.75\cdot z^{-1}} = \frac{4}{5}\cdot\frac{1}{\frac{3}{4}\cdot z^{-1}-1}$$

$$H(z) = \frac{16\cdot z}{20\cdot z-15} = \frac{4}{5}\cdot\frac{4}{4-3\cdot z^{-1}}$$

Example 3): $h(n) = \frac{4}{5}\cdot\left[1+0.75+0.75^2+0.75^3+...+0.75^n-\left(1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}\right)\right]$,

$$H(z) = \frac{4}{5}\cdot\left[\sum_{n=0}^{\infty}\left[\left(\frac{3}{4}\right)^n\cdot z^{-n}\right]+\sum_{n=0}^{\infty}\left(\frac{z^{-n}}{2^n}\right)\right],$$

for the linearity of the z transform, namely

$$Z\left(a\cdot x1_n+b\cdot x2_n\right) = a\cdot Y1(z)+b\cdot Y2(z),$$

we can write:

$$H(z) = \frac{4}{5}\cdot\left(\frac{4}{4-3\cdot z^{-1}} - \frac{2}{2-z^{-1}}\right) = \frac{4}{5}\cdot\frac{2\cdot z^{-1}}{3\cdot z^{-2}-10\cdot z^{-1}+8},$$

,

$$H(z) = \frac{4}{5}\cdot\frac{2\cdot z^{-1}}{3\cdot z^{-2}-10\cdot z^{-1}+8}$$

In this worksheet and all others we will use the operator **ztrans**.
Examples of Z transform:

1) $n \text{ ztrans} \rightarrow \frac{z}{(z-1)^2},$

2) $\frac{1}{2^n} \text{ ztrans} \rightarrow \frac{2\cdot z}{2\cdot z-1},$

3) $k := k \quad k\cdot n \text{ ztrans} \rightarrow \frac{k\cdot z}{(z-1)^2},$

4) $k := k, \; k\cdot n^6 \left| \begin{array}{l}\text{ztrans}\\ \text{collect}, k\end{array}\right. \rightarrow \frac{z^6+57\cdot z^5+302\cdot z^4+302\cdot z^3+57\cdot z^2+z}{(z-1)^7}\cdot k,$

5) $k\cdot\left(-n+n^2-n^3+n^4\right) \text{ ztrans} \rightarrow \frac{2\cdot k\cdot z\cdot\left(5\cdot z^2+5\cdot z+2\right)}{(z-1)^5},$

6) $a\cdot(n-k)+b\cdot(n+k) \left| \begin{array}{l}\text{ztrans}\\ \text{collect}, z, k\end{array}\right. \rightarrow \frac{z\cdot[a+b+k\cdot(a-b)]-k\cdot z^2\cdot(a-b)}{z^2-2\cdot z+1},$

7) $a\cdot(n-k+m)+b\cdot(n+k+m) \left| \begin{array}{l}\text{ztrans}\\ \text{simplify}\\ \text{collect}, k, m, z\end{array}\right. \rightarrow \left[-\frac{z\cdot(a-b)}{z-1}\right]\cdot k+\frac{z\cdot(a+b)}{z-1}\cdot m+\frac{z\cdot(a+b)}{z^2-2\cdot z+1}.$

8) $\sin(\omega\cdot n) \text{ ztrans} \rightarrow \frac{z\cdot\sin(\omega)}{z^2-2\cdot\cos(\omega)\cdot z+1},$

9) $\cos(\omega\cdot n) \text{ ztrans} \rightarrow \frac{z\cdot(z-\cos(\omega))}{z^2-2\cdot\cos(\omega)\cdot z+1},$

10) $e^{-(\omega1\cdot n)}\cdot\sin(\omega1\cdot n) \left| \begin{array}{l}\text{ztrans}\\ \text{simplify}\\ \text{factor}\end{array}\right. \rightarrow \frac{z\cdot e^{\omega1}\cdot\sin(\omega1)}{e^{2\cdot\omega1}\cdot z^2-2\cdot e^{\omega1}\cdot\cos(\omega1)\cdot z+1},$

11) $k\cdot a^n \text{ ztrans} \rightarrow -\frac{k\cdot z}{a-z}$

12) $k\cdot e^{-a\cdot n} \text{ ztrans} \rightarrow \frac{k\cdot z}{z-e^{-a}},$

$$13)\qquad k\cdot n\cdot e^{-a\cdot n}\ \text{ztrans} \rightarrow \frac{k\cdot z\cdot e^{a}}{\left(z\cdot e^{a}-1\right)^{2}},$$

$$14)\qquad 1-e^{a\cdot n}\ \text{ztrans} \rightarrow -\frac{z\cdot\left(e^{a}-1\right)}{\left(z-e^{a}\right)\cdot(z-1)},$$

$$15)\qquad k\cdot n\cdot \alpha^{n}\ \text{ztrans} \rightarrow \frac{\alpha\cdot k\cdot z}{(\alpha-z)^{2}},$$

$$16)\qquad \sinh(\alpha\cdot n)\ \left|\begin{array}{l}\text{ztrans}\\ \text{simplify}\\ \text{collect}, e^{\alpha}, z\end{array}\right. \rightarrow \frac{z\cdot\left(e^{2\cdot\alpha}-1\right)}{e^{\alpha}\cdot\left(2\cdot z^{2}+2\right)-z\cdot\left(2\cdot e^{2\cdot\alpha}+2\right)},$$

$$17)\qquad \cosh(\alpha\cdot n)\ \left|\begin{array}{l}\text{ztrans}\\ \text{simplify}\end{array}\right. \rightarrow -\frac{z\cdot\cosh(\alpha)-z^{2}}{z^{2}-2\cdot\cosh(\alpha)\cdot z+1},$$

$$18)\qquad (-1)^{n+1}\cdot\frac{a^{n}}{n}\ \left|\begin{array}{l}\text{ztrans}\\ \text{simplify}\end{array}\right. \rightarrow \text{ztrans}\left[\frac{e^{\pi\cdot(n+1)\cdot j}}{n}, n, \frac{z}{a}\right] \qquad \log\left(1+a\cdot z^{-1}\right)\ |a|<|z|,$$

$$19)\qquad \frac{a\cdot\left(a^{n}+b^{n}\right)}{a-b}\ \text{ztrans} \rightarrow -\frac{a\cdot z\cdot(a+b-2\cdot z)}{(a-b)\cdot(a-z)\cdot(b-z)},$$

$$20)\qquad \frac{a\cdot\left(a^{n}-b^{n}\right)}{a-b}\ \text{ztrans} \rightarrow \frac{a\cdot z}{(a-z)\cdot(b-z)}$$

$$21)\qquad \frac{n}{2\cdot n+1}\ \text{ztrans} \rightarrow \left|\begin{array}{ll}\dfrac{z+\sqrt{z}\cdot\operatorname{atanh}\left(z^{\frac{-1}{2}}\right)-z^{\frac{3}{2}}\cdot\operatorname{atanh}\left(z^{\frac{-1}{2}}\right)}{2\cdot z-2} & \text{if}\ \ 1<|z|\\ \text{undefined} & \text{otherwise}\end{array}\right.$$

§1.2) The inverse Z transform of H(z)

It is a contour integration of $F(z) = z^{n-1} H(z)$ on a counterclockwise contour Γ traced on the Gauss plane, enclosing all the poles of F(z):

$$h_n = \frac{1}{2\cdot\pi\cdot j} \oint_\Gamma z^{n-1} H(z)\, d\xi , \qquad (1.2.1)$$

Expand to see some classical examples

§1.2.1) RESIDUES METHOD

For a *rational function* $F(z,n) = z^{n-1} H(z) = \frac{A(z)}{B(z)}$ whose poles are known, the integral can be analytically calculated with the residues method.

Stated that "m_k" is the order of a single pole z_k , "μ" the number of such poles and P is the total number of poles:

the contour integral: $$h_n = \frac{1}{2\cdot\pi\cdot j} \oint_\Gamma F(z,n)\, dz = \sum_{k=0}^{P-1} (\mathrm{Res}(F(z,n))), \qquad (1.2.1.1)$$

If F(z,n) is a rational function with P poles of order m_k, with $\sum_{k=0}^{P-1} m_k = P$,

$$F(z,n) = \frac{A(z)}{B(z)} = \frac{A(z)}{(z-p_0)^{m_0}\cdot(z-p_1)^{m_1}\cdot\ldots\cdot(z-p_{q-1})^{m_{P-1}}}, \qquad (1.2.1.2)$$

the theorem gives the residue for the pole p_k of order m_k:

$$\boxed{\mathrm{Res}(F(z,n)) = \lim_{z\to p_k} \left[\frac{1}{(m_k-1)!} \cdot \frac{\partial^{m_k-1}}{\partial z^{m_k-1}} \left[(z-p_k)^{m_k} \cdot (F(z,n)) \right] \right]}, \qquad (1.2.1.3)$$

(If the system has 6 poles, for example, with 1 one of third order and the other (3) of first order, the sum o the residues is composed by the sum of the residues of the poles of the first order plus the residue of the pole of third order, namely it results the sum of four residues.)

For simple poles ($m_k=1$, ∀k) we have:

$$\boxed{\sum_{k=0}^{P-1} (\mathrm{Res}(F(z,n))) = \sum_{k=0}^{P-1} \left[\lim_{z\to p_k} \left[(z-p_k)\cdot(F(z,n)) \right] \right]}. \qquad (1.2.1.4)$$

If F(z,n) is a rational function $F(z,n) = \frac{A(z)}{B(z)} = \frac{A(z)}{(z-z_1)\cdot(z-z_2)\cdot\ldots\cdot(z-z_n)}$ (1.2.1.5)

with all simple poles, then:

$$\boxed{h_n = \sum_{k=0}^{P-1} \lim_{z\to p_k} \frac{A(z)}{\frac{\partial}{\partial z} B(z)}} \qquad (1.2.$$

Expand to see some examples

Example 1) *two simple poles*:

Let us compute the inverse Z transform of the given function: $H3(z) := \dfrac{z}{(z-0.75)\cdot(z+0.5)}$, (1.2.1.7)

$$F3(z,n) = z^{n-1}\cdot H(z) \quad (1.2.1.8)$$

$$h3_n = \sum_{j=0}^{P}\left[\lim_{z\to p_j}\left[\left(z-p_j\right)\cdot(F3(z,n))\right]\right] \quad (1.2.1.9)$$

Creation of the coefficient's vector of the polynomial denominator:

$z := z$ $n := n$ $F3(z,n) := \dfrac{z^n}{(z-0.75)\cdot(z+0.5)}$ $P = 2$ (1.2.1.10)

$$v := \text{denom}(F3(z,n)) \text{ coeffs}, z \rightarrow \begin{pmatrix} -3.0 \\ -2.0 \\ 8.0 \end{pmatrix} \quad (1.2.1.11)$$

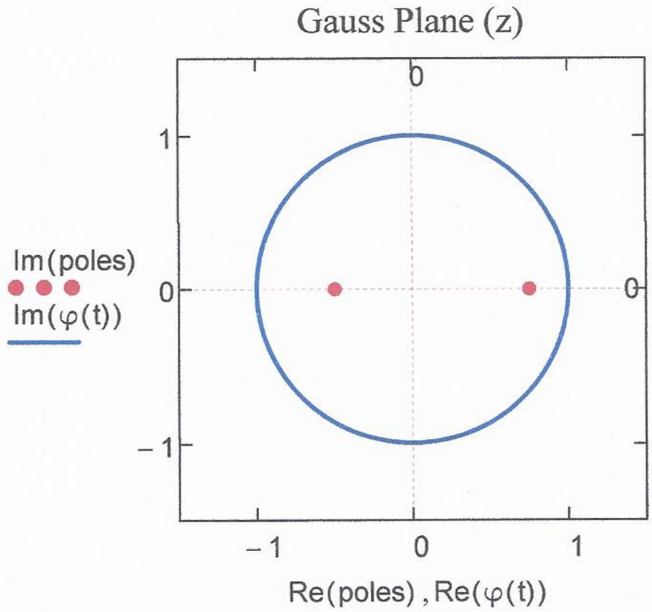
Search of the poles of F3(z,n): poles := polyroots(v) ,

$poles^T = (-0.5 \;\; 0.75)$ $p := poles$

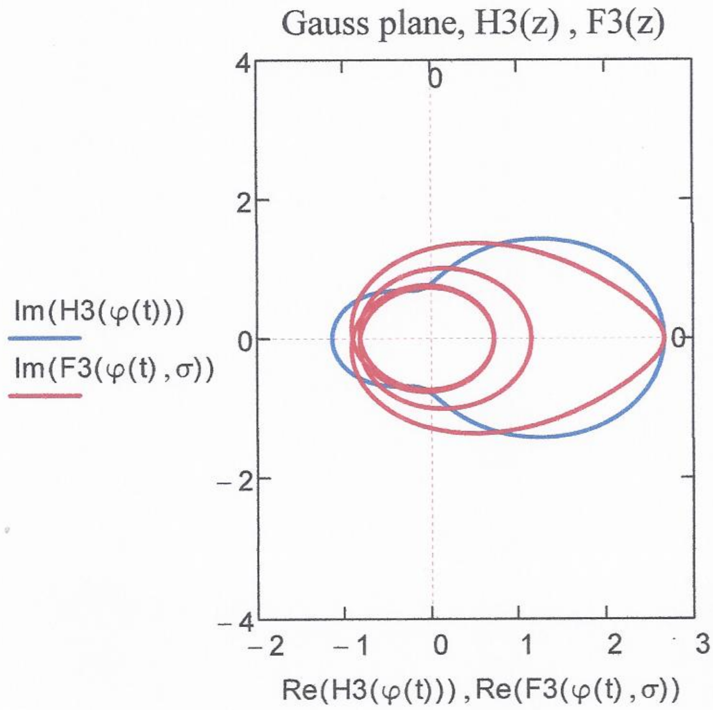
$$h3_n = \lim_{z\to p_0}\left[\left(z-p_0\right)\cdot F3(z,n)\right] + \lim_{z\to p_1}\left[\left(z-p_1\right)\cdot F3(z,n)\right] \quad (1.2.1.12)$$

Representation of the contour of integration on the Gaussian plane of the complex variable z.
Radius of the circle that encloses all the poles of F3 (z,v): $r := \text{ceil}(\max(|poles|))\cdot 1.0,\;\; r = 1$

$\xi(t) := r\cdot\cos(t)$ $\psi(t) := r\cdot\sin(t)$ $\varphi(t) := \xi(t) + j\cdot\psi(t)$ $\sigma := 6$ (1.2.1.13)



(fig. 1.2.1.1)



(fig. 1.2.1.2)

$$h3_n = \left[\left[\lim_{z \to 0.75}\left[(z-0.75)\cdot\left[z^{n-1}\cdot\frac{z}{(z-0.75)\cdot(z+0.5)}\right]\right] \cdots \atop + \lim_{z \to -0.5}\left[(z+0.5)\cdot\left[z^{n-1}\cdot\frac{z}{(z-0.75)\cdot(z+0.5)}\right]\right]\right]\right] \qquad (1.2.1.14)$$

Finally we get the sequence sought:

$M1 := 100$

$$h3_n = \left(\frac{0.75^n}{0.75+0.5}\right) + \left[\frac{(-0.5)^n}{-0.5-0.75}\right] \qquad (1.2.1.15)$$

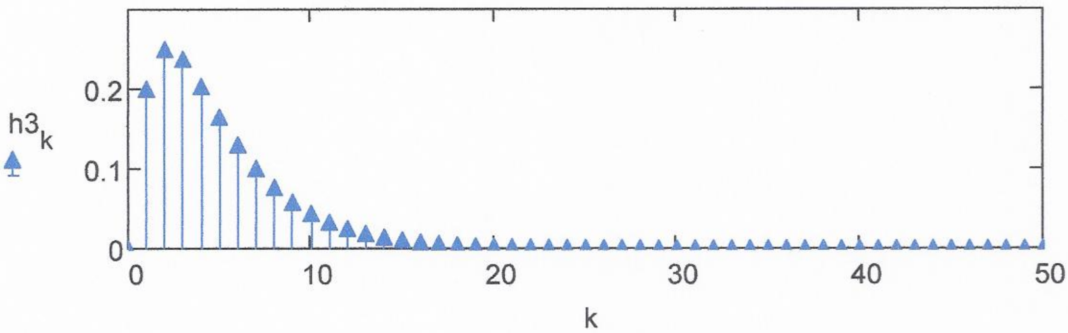
$n := 0 .. M1 - 1$

$$h3_n := \frac{4}{5}\cdot\left(0.75^n - 0.5^n\right) \qquad (1.2.1.16)$$

$h3^T =$

0	0.2	0.25	0.238	0.203	0.165	...

Energy $E := \sum_{k=0}^{rows(h3)-1} \left(\left|h3_k\right|\right)^2 \qquad E = 0.268 \qquad (1.2.1.17)$



(fig. 1.2.1.3)

Example 2) *c.c. poles*:

Let us compute the inverse Z transform of the given rational function:

$\alpha := 0.3561$

$$H4(z) := \frac{1 + 2\cdot z^{-1} + z^{-2}}{1 - z^{-1} + \alpha\cdot z^{-2}}, \qquad (1.2.1.18)$$

$P = 3 \qquad F4(z) = z^{n-1}\cdot H4(z) \quad \Rightarrow \quad F4(z,n) := \frac{z^n\cdot(z+1)^2}{z\cdot\left(z^2 - z + \alpha\right)} \qquad (1.2.1.19)$

$$h4_n = \sum_{k=0}^{2}\left[\lim_{z \to p_k}\left[\left(z - p_k\right)\cdot\left(F4(z)\right)\right]\right] \qquad (1.2.1.20)$$

$\alpha := \alpha$

$z := z \qquad n := n \qquad v := \text{denom}(F4(z,n)) \text{ coeffs}, z \rightarrow \begin{pmatrix} 0 \\ 3561.0 \\ -10000.0 \\ 10000.0 \end{pmatrix}$ (1.2.1.21)

Search of the poles of F4(z,n): $\text{poles4} := \text{polyroots}(v) \quad \text{poles4}^T = (0 \quad 0.5-0.326j \quad 0.5+0.326j)$

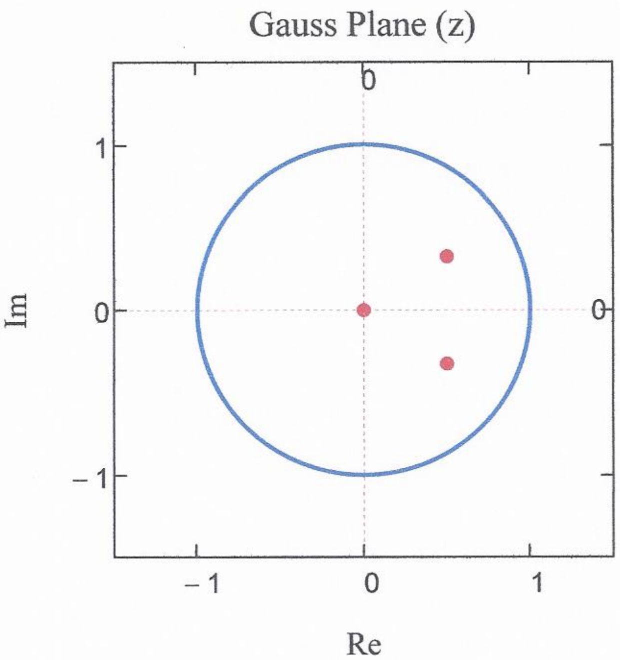
,

$\xi(t) := r\cdot\cos(t) \qquad \psi(t) := r\cdot\sin(t) \qquad \varphi(t)(t) := \xi(t) + j\cdot\psi(t)$ (1.2.1.22)

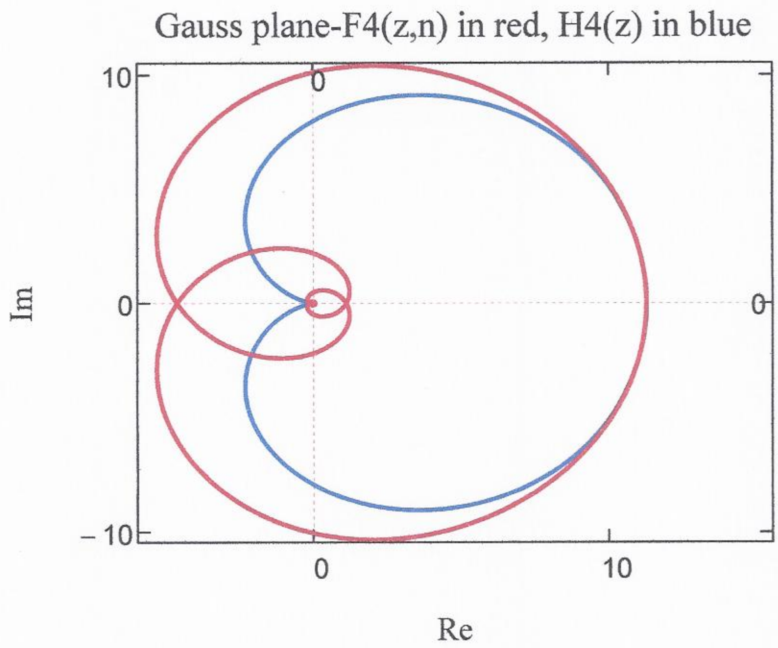
Representation of the contour of integration on the Gaussian plane of the complex variable z.

Radius of the circle that encloses all the poles of F4 (z): $r := \text{ceil}(\max(|\text{poles}|))\cdot 1.0$, (1.2.1.23)

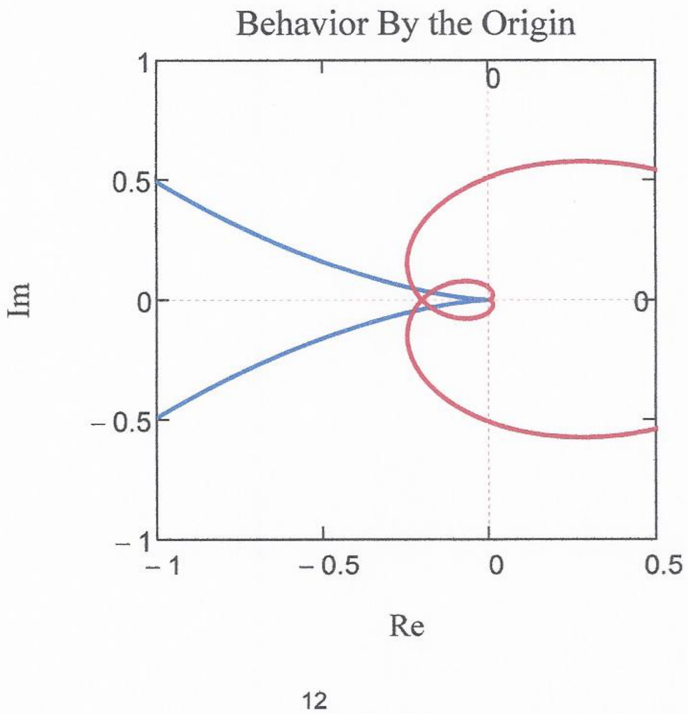
$t_0 := 0 \quad t_{fin} := 2\cdot\pi \qquad \sigma = 6$



(fig. 1.2.1.4)



(fig. 1.2.1.5)



(fig. 1.2.1.6)

$$poles4_0 = 0 \qquad poles4_1 = 0.5 - 0.326j \qquad poles4_2 = 0.5 + 0.326j$$

$$p0 := poles4_0 \qquad p1 := \frac{\sqrt{1-4\cdot\alpha}}{2} + \frac{1}{2} \qquad p2 := \frac{1}{2} - \frac{\sqrt{1-4\cdot\alpha}}{2} \tag{1.2.1.24}$$

$$p0 = 0 \qquad p1 = 0.5 + 0.326j \qquad p2 = 0.5 - 0.326j$$

The sum of the residues gives:

$$h4_n = \lim_{z \to 0}\left[z\cdot z^{n-1}\cdot\frac{(z+1)^2}{(z-p1)\cdot(z-p2)}\right] \ldots + \left[\lim_{z \to p2}\left[z^{n-1}\cdot\frac{(z+1)^2}{(z-p1)}\right] \ldots + \lim_{z \to p1}\left[z^{n-1}\cdot\frac{(z+1)^2}{(z-p2)}\right]\right] \tag{1.2.1.25}$$

$$\lim_{z \to 0}\left[z\cdot z^{n-1}\cdot\frac{(z+1)^2}{(z-p1)\cdot(z-p2)}\right] = \frac{\delta(n,0)}{p1\cdot p2} \tag{1.2.1.26}$$

$$\frac{1}{p1\cdot p2} = \left(\frac{\sqrt{1-4\cdot\alpha}}{2} + \frac{1}{2}\right)\cdot\left(\frac{1}{2} - \frac{\sqrt{1-4\cdot\alpha}}{2}\right) = \frac{1}{\alpha} \qquad \frac{1}{\alpha} = 2.808 \tag{1.2.1.27}$$

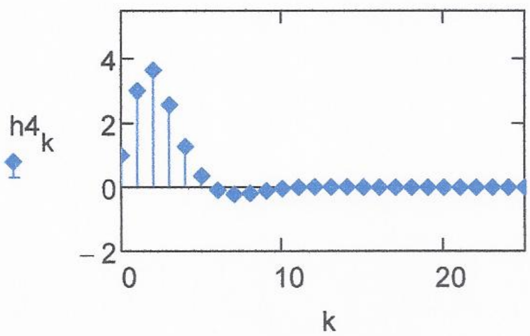
$$\lim_{z \to p1}\left[z^{n-1}\cdot\frac{(z+1)^2}{(z-p2)}\right] = \frac{p1^{n-1}\cdot(p1+1)^2}{p1-p2} \qquad \lim_{z \to p2}\left[z^{n-1}\cdot\frac{(z+1)^2}{(z-p1)}\right] = \frac{p2^{n-1}\cdot(p2+1)^2}{p2-p1}$$

and finally:

$$h4_\nu := \frac{1}{\alpha}\cdot\delta(\nu,0) + \frac{p1^{\nu-1}\cdot(p1+1)^2}{p1-p2} + \frac{p2^{\nu-1}\cdot(p2+1)^2}{p2-p1} \tag{1.2.1.28}$$

$h4^T =$

	0	1	2	3	4	5
0	1	3	3.644	2.576	1.278	...



(fig. 1.2.1.7)

Example 3) *one second order pole*:

Let us compute the inverse Z transform of the given rational function :

$$\alpha_2 := 1\ ,\ \beta_2 := \sqrt{2} \qquad H5(z) := \alpha_2 \cdot \frac{\left(z^{-1}+1\right)^2}{\left(z^{-1}+\beta_2\right)^2}, \tag{1.2.1.29}$$

$$F5(z,n) = z^{n-1} \cdot H5(z) \quad \Rightarrow \quad F5(z,n) := \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot \left(\beta_2 \cdot z + 1\right)^2} \tag{1.2.1.30}$$

$$h5_n = \text{Res}(F5(z,n),p0) + \text{Res}(F5(z,n),p1) \tag{1.2.1.31}$$

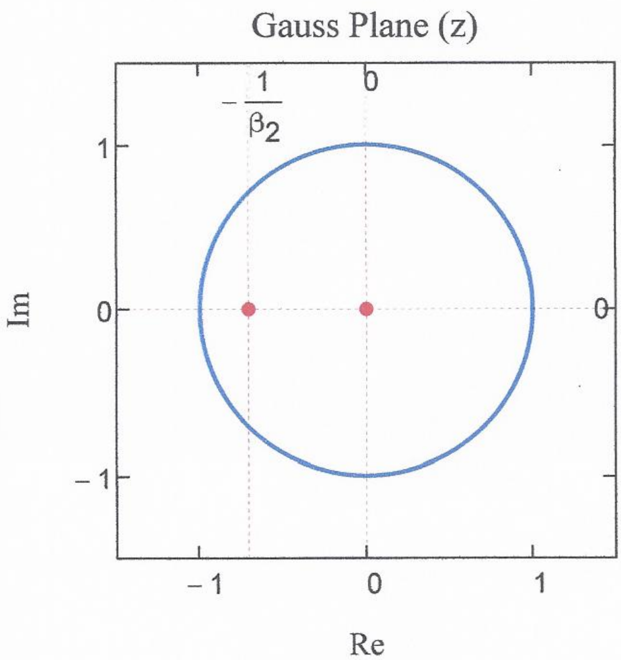
$$h5_n = \lim_{z \to 0} \left[z \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot \left(\beta_2 \cdot z + 1\right)^2} \right] + \lim_{z \to p1} \frac{\partial}{\partial z} \left[(z-p1)^2 \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot \left(\beta_2 \cdot z + 1\right)^2} \right] \tag{1.2.1.32}$$

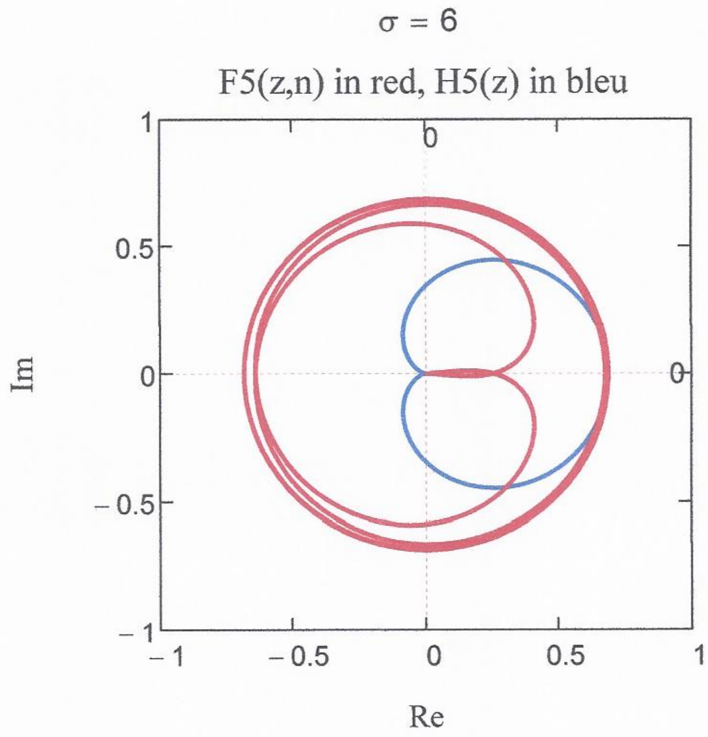
$\alpha_2 := \alpha_2 \qquad \beta_2 := \beta_2$

Search of the poles of F5(z,n): , $\text{poles5} := z \cdot \left(\beta_2 \cdot z + 1\right)^2 \text{ solve}, z \rightarrow \begin{pmatrix} 0 \\ -\frac{1}{\beta_2} \\ -\frac{1}{\beta_2} \end{pmatrix},$ (1.2.1.33)

$$p0 := 0 \qquad p1 := -\frac{1}{\beta_2} \qquad F5(z,n) = \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{(z-p0) \cdot (z-p1)^2} \tag{1.2.1.34}$$

The function has a second order pole= $-\frac{1}{\beta_2}$





(fig. 1.2.1.8) (fig. 1.2.1.9)

$$\mathrm{Res}(F5(z,n),p0) = \lim_{z \to 0} \left[z \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot \left(\beta_2 \cdot z + 1\right)^2} \right] = \lim_{z \to 0} \left[\frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{\left(\beta_2 \cdot z + 1\right)^2} \right] = \alpha_2 \cdot \delta(n,0) \quad (1.2.1.35)$$

$$\mathrm{Res}(F5(z,n),p1) = \lim_{z \to p1} \frac{\partial}{\partial z} \left[(z - p1)^2 \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot \left(\beta_2 \cdot z + 1\right)^2} \right] = \lim_{z \to p1} \frac{\partial}{\partial z} \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z} \quad (1.2.1.36)$$

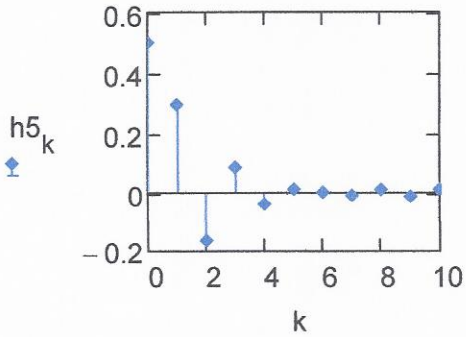
$n := n$

$$\frac{\partial}{\partial z} \left[\left(z + \frac{1}{\beta_2}\right)^2 \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot \left(\beta_2 \cdot z + 1\right)^2} \right] \text{ simplify, max } \rightarrow \frac{\alpha_2 \cdot z^{n-2} \cdot (z+1) \cdot (n + z + n \cdot z - 1)}{\beta_2^{\,2}} \quad (1.2.1.37)$$

$$\mathrm{Res}(F5(z,n),p1) = \lim_{z \to p1} \frac{\alpha_2 \cdot z^{n-2} \cdot (z+1) \cdot [n + z \cdot (1+n) - 1]}{\beta_2^{\,2}} \quad (1.2.1.38)$$

$$\lim_{z \to -\frac{1}{\beta_2}} \frac{\alpha_2 \cdot z^{n-2} \cdot (z+1) \cdot [n + z \cdot (1+n) - 1]}{\beta_2^{\,2}} \text{ simplify, max } \rightarrow -\frac{\alpha_2 \cdot \left(\beta_2 - 1\right) \cdot \left(-\frac{1}{\beta_2}\right)^n \cdot \left(n + \beta_2 - n \cdot \beta_2 + 1\right.}{\beta_2^{\,2}}$$

$$\boxed{h5_n := \alpha_2 \cdot \delta(n,0) - \frac{\alpha_2 \cdot \left(\beta_2 - 1\right) \cdot \left(-\frac{1}{\beta_2}\right)^n \cdot \left[n + \beta_2 \cdot (1-n) + 1\right]}{\beta_2^{\,2}}} \quad (1.2.1.39)$$



(fig. 1.2.1.10)

$$\alpha_2 \cdot \frac{\left(z^{-1} + 1\right)^2}{\left(z^{-1} + \beta_2\right)^2} \left| \begin{array}{l} \text{invztrans} \\ \text{simplify} \\ \text{collect}, \left(-\frac{1}{\beta_2}\right)^n \end{array} \right. \rightarrow \frac{\alpha_2 \cdot \left(n + n \cdot \beta_2^{\,2} - \beta_2^{\,2} - 2 \cdot n \cdot \beta_2 + 1\right)}{\beta_2^{\,2}} \cdot \left(-\frac{1}{\beta_2}\right)^n + \alpha_2 \cdot \delta(n,0)$$

(1.2.1.40)

Example 4) *one second order pole*:
Let us compute the inverse Z transform of the given rational function :

$\alpha_2 := 1$, $\beta_2 := \sqrt{2}$ $$H55(z) := \alpha_2 \cdot \frac{\left(z^{-1}+1\right)^2}{\left(z^{-1}+\beta_2\right)^3}, \qquad (1.2.1.41)$$

$$F55(z,n) = z^{n-1} \cdot H55(z) \Rightarrow \quad F55(z,n) := \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{\left(\beta_2 \cdot z+1\right)^3} \qquad (1.2.1.42)$$

$\alpha_2 := \alpha_2$ $\beta_2 := \beta_2$

Search of the poles of F55(z,n): , $poles55 := \left(\beta_2 \cdot z+1\right)^3 \text{ solve}, z \rightarrow \begin{pmatrix} -\frac{1}{\beta_2} \\ -\frac{1}{\beta_2} \\ -\frac{1}{\beta_2} \end{pmatrix}$, (1.2.1.43)

The function has a third order pole= $-\frac{1}{\beta_2}$

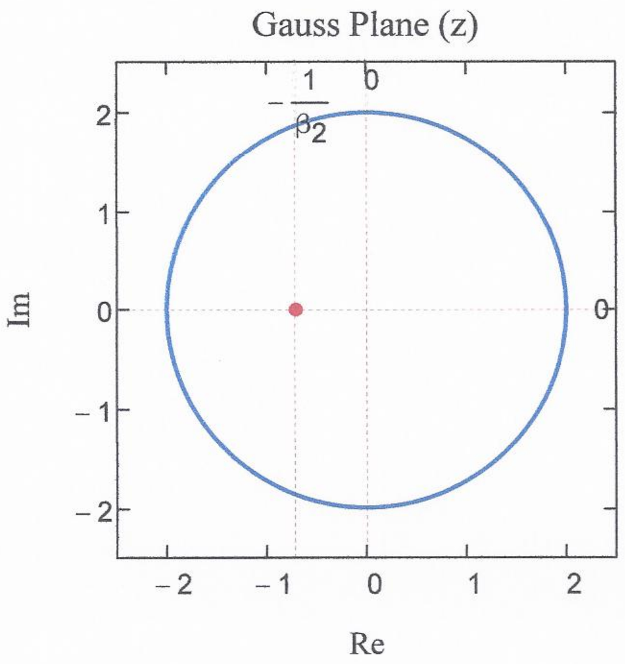
Representation of the contour of integration on the Gaussian plane of the complex variable z.

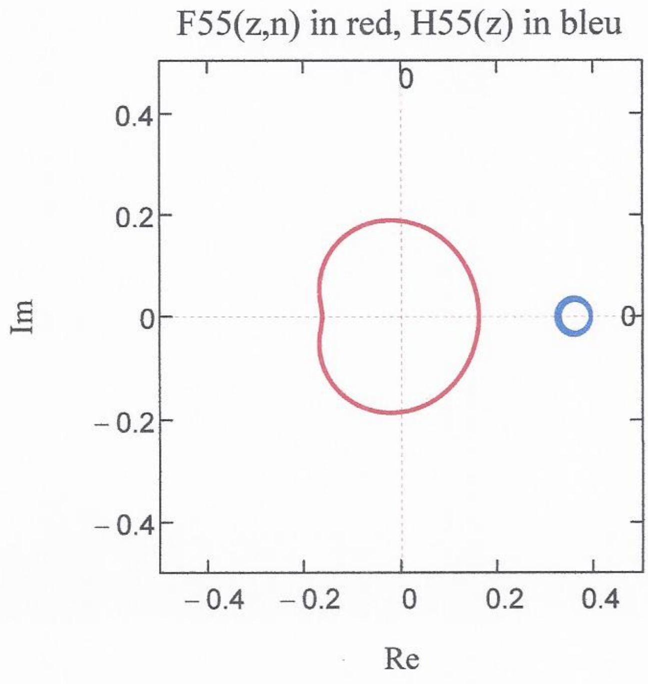
Radius of the circle that encloses all the poles of F55 (z,v): $r := \text{ceil}(\max(|poles55|)) \cdot 1.0$, $r = 2$

$$\xi(t) := r \cdot \cos(t) \qquad \psi(t) := r \cdot \sin(t) \qquad (1.2.1.44)$$

$$\varphi(t) := \xi(t) + j \cdot \psi(t)$$

$\sigma := 0$





(fig. 1.2.1.8) (fig. 1.2.1.9)

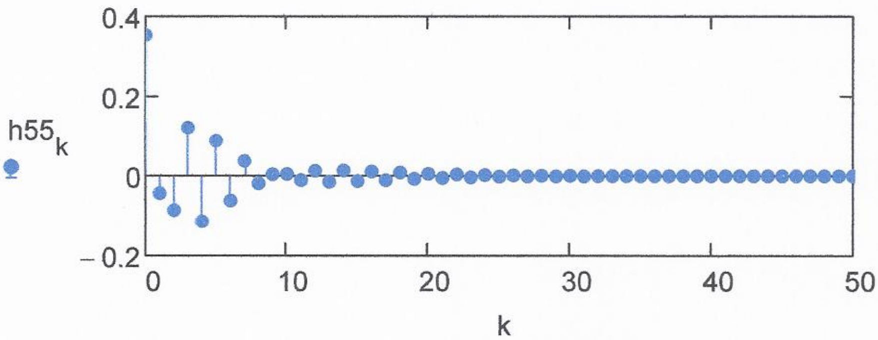
$$h55_n = \mathrm{Res}(F5(z,n),p0) = \lim_{z \to -\frac{1}{\beta_2}} \left[\frac{1}{2} \cdot \frac{\partial^2}{\partial z^2} \left[\left(z + \frac{1}{\beta_2} \right)^3 \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{(\beta_2 \cdot z + 1)^3} \right] \right] \quad (1.2.1.45)$$

$$h55_n = \frac{1}{2} \cdot \alpha_2 \cdot \left(\frac{1}{\beta_2} \right)^3 \cdot \lim_{z \to -\frac{1}{\beta_2}} \frac{\partial^2}{\partial z^2} \left[z^n \cdot (z+1)^2 \right] \quad (1.2.1.46)$$

$\alpha_2 := \alpha_2$ $\beta_2 := \beta_2$ $z := z$

$$h55_n := \frac{1}{2} \cdot \alpha_2 \cdot \left(\frac{1}{\beta_2} \right)^3 \cdot \lim_{z \to -\frac{1}{\beta_2}} \frac{\partial^2}{\partial z^2} \left[z^n \cdot (z+1)^2 \right] \text{ simplify, max } \rightarrow \frac{\alpha_2 \cdot \left(-\frac{1}{\beta_2} \right)^n \cdot \left(n^2 \cdot \beta_2^2 - 2 \cdot n^2 \cdot \beta_2 + n^2 \right.}{2 \cdot \beta_2^3}$$

$$h55_n := \frac{\alpha_2 \cdot \left(-\frac{1}{\beta_2} \right)^n \cdot \left(n^2 \cdot \beta_2^2 - 2 \cdot n^2 \cdot \beta_2 + n^2 - n \cdot \beta_2^2 - 2 \cdot n \cdot \beta_2 + 3 \cdot n + 2 \right)}{2 \cdot \beta_2^3} \quad (1.2.1.47)$$



Verify:

$$\alpha_2 \cdot \frac{\left(z^{-1} + 1 \right)^2}{\left(z^{-1} + \beta_2 \right)^3} \ \Bigg| \begin{array}{l} \text{invztrans} \\ \text{simplify, max} \end{array} \rightarrow \frac{\alpha_2 \cdot \left(-\frac{1}{\beta_2} \right)^n \cdot \left(n^2 \cdot \beta_2^2 - 2 \cdot n^2 \cdot \beta_2 + n^2 - n \cdot \beta_2^2 - 2 \cdot n \cdot \beta_2 + 3 \cdot n + 2 \right)}{2 \cdot \beta_2^3}$$

(1.2.1.48)

Expand to see some examples

§1.2.2) POWER SERIES EXPANSION METHOD.

We now consider a generic Z transform of a causal sequence, corresponding to a rational function, like th (IIR):

$$H(z) = \frac{\sum_{k=0}^{N} \left(b_k \cdot z^{-k}\right)}{\sum_{k=0}^{M} \left(a_k \cdot z^{-k}\right)} , \qquad (1.2.2.1)$$

The Division of the two polynomials at the numerator and denominator generate the series:

$$H(z) = \sum_{n=0}^{\infty} \left(h_n \cdot z^{-n}\right), \qquad (1.2.2.1')$$

whose coefficients are given by the following algorithm:

$$\text{with: } h_0 = \frac{b_0}{a_0}, \quad h_n = \frac{1}{a_0} \cdot \left[b_n - \sum_{i=1}^{n} \left(h_{n-i} \cdot a_i\right) \right] \qquad (1.2.2.2)$$

$$\text{where for N=2: } H(z) = \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{a_0 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} \qquad (1.2.2.3)$$

$$\text{or } H(z) = \frac{b_0 \cdot z^2 + b_1 \cdot z + b_2}{a_0 \cdot z^2 + a_1 \cdot z + a_2} \qquad (1.2.2.3')$$

$$H(z) = \sum_{n=0}^{\infty} \left(h_n \cdot z^{-n}\right),$$

$$\text{with: } h_0 = \frac{b_0}{a_0}, \quad h_n = \frac{1}{a_0} \cdot \left[b_n - \sum_{i=1}^{n} \left(h_{n-i} \cdot a_i\right) \right] . \qquad (1.2.2.2')$$

Expand to see some examples

Example 1):

$$n := 1..M1 \qquad H6(z) := \frac{z^2 - 1.618 \cdot z + 4.0}{z^2 - 1.5161 \cdot z + 0.878} , \qquad (1.2.2.4)$$

$b_n := 0$, $a_n := 0$,

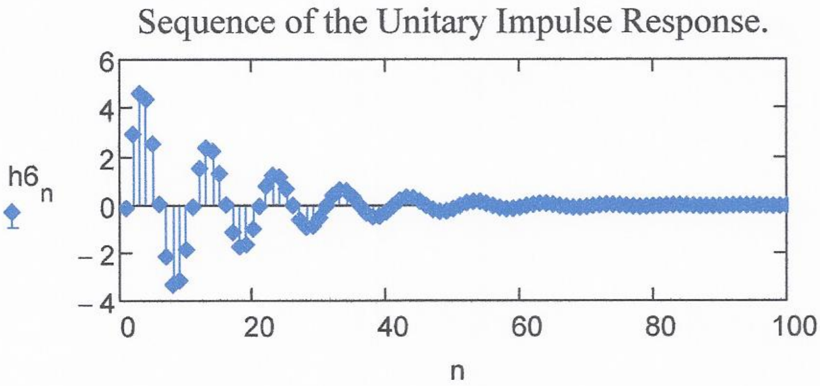
$$\text{rows}(b) = 101 \qquad \beta := \text{numer}(H6(z)) \text{ coeffs} \rightarrow \begin{pmatrix} 2.0e7 \\ -8.09e6 \\ 5.0e6 \end{pmatrix}, \qquad \mu := \text{rows}(\beta), \qquad (1.2.2.5)$$

$$i := 0..\mu - 1 \qquad b_i := \beta_{\mu-i-1}$$

$rows(\beta) = 3$ $\quad\quad \alpha := denom(H6(z))\ coeffs \rightarrow \begin{pmatrix} 4.39e6 \\ -7.5805e6 \\ 5.0e6 \end{pmatrix}, \quad nr := rows(\alpha)$

$j := 0..\, nr - 1 \quad\quad a_j := \alpha_{nr-j-1}$

$$h6_0 := \frac{b_0}{a_0}, \quad h6_n := \frac{1}{a_0}\cdot\left[b_n - \sum_{i=1}^{n} \left(h6_{n-i}\cdot a_i\right)\right]. \quad (1.2.2.7)$$



(fig. 1.2.2.1)

Example 2):

$$H7(z) := \frac{1 + 2\cdot z^{-1} + z^{-2}}{1 - z^{-1} + 0.3561\cdot z^{-2}} \qquad (1.2.2.8)$$

Numerator $N := 2$ Denominator $M := 2$

$N1 := N + M$ $n := 1..M1$ $i := 0..N1$ $M1 = 100$ $k := 0..M1$

Numerator

$b_n := 0.0$

$b_0 := 1$

$b_1 := 2$

$b_2 := 1$

Denominator

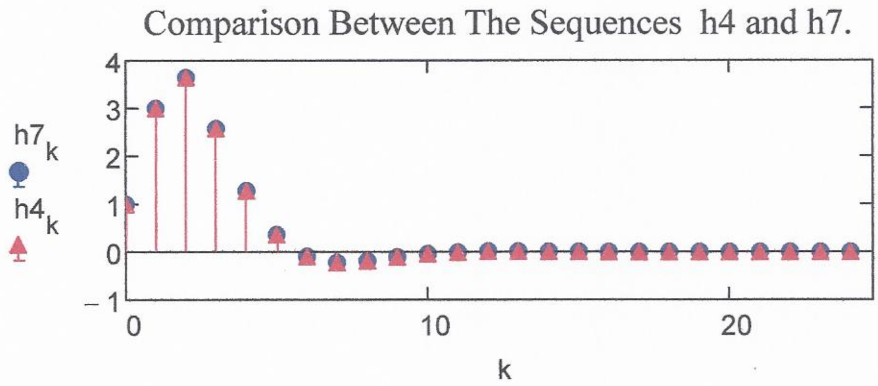
$a_n := 0.0$

$a_0 := 1$

$a_1 := -1$

$a_2 := 0.3561$

$$N1 = 4 \quad h7_0 := \frac{b_0}{a_0} \qquad h7_n := \frac{1}{a_0}\cdot\left[b_n - \sum_{i=1}^{n}\left(h7_{n-i}\cdot a_i\right)\right] \qquad (1.2.2.9)$$



(fig. 1.2.2.2)

Expand to see some examples

§1.2.3) PARTIAL FRACTION EXPANSION METHOD.

This analytic method consists in expanding H(z) in a sum of simple partial fractions. The inverse transform then found using the elementary table of Z transform.

If we suppose N=M and the poles of H(z) are of the first order, then

$$H(z) = \frac{\sum_{k=0}^{N} \left(b_k \cdot z^{-k}\right)}{\sum_{k=0}^{M} \left(a_k \cdot z^{-k}\right)} \qquad ((1.2.2.1))$$

is expanded as:

$$H(z) = B_0 + \sum_{k=1}^{M} \left(C_k \cdot \frac{z}{z-p_k}\right), \text{ where } B_0 = \frac{b_N}{a_N}. \qquad (1.2.3.1)$$

if N<M we have:

$$H(z) = \sum_{k=1}^{M} \left(C_k \cdot \frac{z}{z-p_k}\right) \qquad (1.2.3.2)$$

where:

$$C_k = \lim_{z \to p_k} \left[\frac{H(z)}{z} \cdot (z-p_k)\right]. \qquad (1.2.3.3)$$

If H(z) has one or more multiple-order poles (coincident) (for an m^{th}-order pole at $z = p_i$), then,

$$H(z) = B_0 + \sum_{k=1}^{M} \left(C_k \cdot \frac{z}{z-p_k}\right) + \sum_{k=1}^{m} \frac{D_k}{\left(z-p_i\right)^k}, \qquad (1.2.2.4)$$

where:

$$D_k = \frac{1}{\left(m_k-k\right)!} \cdot \lim_{z \to p_i} \frac{\partial^{m_k-k}}{\partial z^{m_k-k}} \left[\left(z-p_i\right)^{m_k} \cdot \frac{H(z)}{z}\right]. \qquad (1.2.2.5)$$

Expand to see some examples

Example 1) (two real and distinct poles): $H8(z) = \dfrac{z^{-1}}{1 - \dfrac{z^{-1}}{4} - \dfrac{3}{8} \cdot z^{-2}}$, (1.2.2.6)

H8(z) can be written also: $H8(z) = \dfrac{z}{z^2 - \dfrac{z}{4} - \dfrac{3}{8}} = \dfrac{z}{\left(z - \dfrac{3}{4}\right) \cdot \left(z + \dfrac{1}{2}\right)}$, it has a zero at the origin and two simpl poles in $p_0 = \dfrac{3}{4}$, and $p_1 = \dfrac{-1}{2}$.

Since the numerator order is less than the order of denominator we can write the partial fraction in this form

$$H8(z) = \frac{z}{\left(z-\frac{3}{4}\right)\cdot\left(z+\frac{1}{2}\right)} = \frac{C_0\cdot z}{z-\frac{3}{4}} + \frac{C_1\cdot z}{z+\frac{1}{2}}, \qquad (1.2.2.7)$$

so:

$$C_0 = \lim_{z\to p_0}\left[\frac{H8(z)}{z}\cdot(z-p_0)\right],$$

$$C_0 = \lim_{z\to p_0}\left[\frac{H8(z)}{z}\cdot(z-p_0)\right] = \lim_{z\to p_0}\left[\frac{1}{\left(z-\frac{3}{4}\right)\cdot\left(z+\frac{1}{2}\right)}\cdot\left(z-\frac{3}{4}\right)\right] = \lim_{z\to \frac{3}{4}}\left[\frac{1}{\left(z+\frac{1}{2}\right)}\right] = \frac{4}{5},$$

$$C_1 = \lim_{z\to p_1}\left[\frac{H8(z)}{z}\cdot(z-p_1)\right],$$

$$C_1 = \lim_{z\to p_1}\left[\frac{H8(z)}{z}\cdot(z-p_1)\right] = \lim_{z\to p_1}\left[\frac{1}{\left(z-\frac{3}{4}\right)\cdot\left(z+\frac{1}{2}\right)}\cdot\left(z+\frac{1}{2}\right)\right] = \lim_{z\to -\frac{1}{2}}\left[\frac{1}{\left(z-\frac{3}{4}\right)}\right] = -\frac{4}{5},$$

$$H8(z) = \frac{4}{5}\cdot\left(\frac{z}{z-\frac{3}{4}} - \frac{z}{z+\frac{1}{2}}\right), \qquad (1.2.2.8)$$

we know that: $a := a \quad z := z, \quad \frac{z}{z-a}$ invztrans $\to a^n, \quad \frac{z}{z+a}$ invztrans $\to (-a)^n$

and ultimately:

$$h8(n) := \frac{4}{5}\cdot\left[\left(\frac{3}{4}\right)^n - \left(-\frac{1}{2}\right)^n\right]. \qquad (1.2.2.9)$$

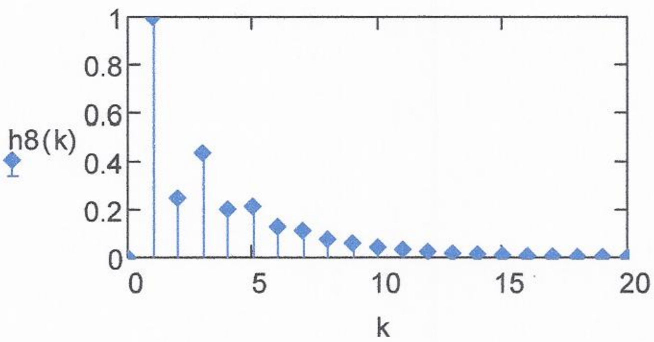


fig.(1.2.3.1)

$$\frac{z^{-1}}{1-\frac{z^{-1}}{4}-\frac{3}{8}\cdot z^{-2}} \text{ invztrans } \to \frac{4\cdot\left(\frac{3}{4}\right)^n}{5} - \frac{4\cdot\left(-\frac{1}{2}\right)^n}{5} \qquad (1.2.2.10)$$

Example 2) (first order complex conjugate poles): $H9(z) = \frac{1+a\cdot z^{-1}+z^{-2}}{1-z^{-1}+b\cdot z^{-2}}, \quad a := 2, \; b := \frac{1}{e}$

or:

$$H9(z) := \frac{z^2+a\cdot z+1}{z^2-z+b} \qquad (1.2.2.11)$$

$a := a \qquad b := b$

$$v := \text{denom}(H9(z)) \text{ coeffs}, z \rightarrow \begin{pmatrix} e^{-1} \\ -1 \\ 1 \end{pmatrix} \qquad (1.2.2.12)$$

$a := a \qquad b := b$

$$\text{poles3} := \text{polyroots}(v) \qquad \text{poles3}^T = (0.5 - 0.343j \quad 0.5 + 0.343j)$$

$$p_0 := \text{poles3}_0 \qquad p_0 = 0.5 - 0.343j \qquad p_1 := \text{poles3}_1 \qquad p_1 = 0.5 + 0.343j$$

$$\xi(t) := r \cdot \cos(t) \qquad \psi(t) := r \cdot \sin(t) \qquad \zeta(t) := \xi(t) + j \cdot \psi(t) \qquad (1.2.2.13)$$

$$t_0 := 0 \quad t_{fin} := 2 \cdot \pi$$

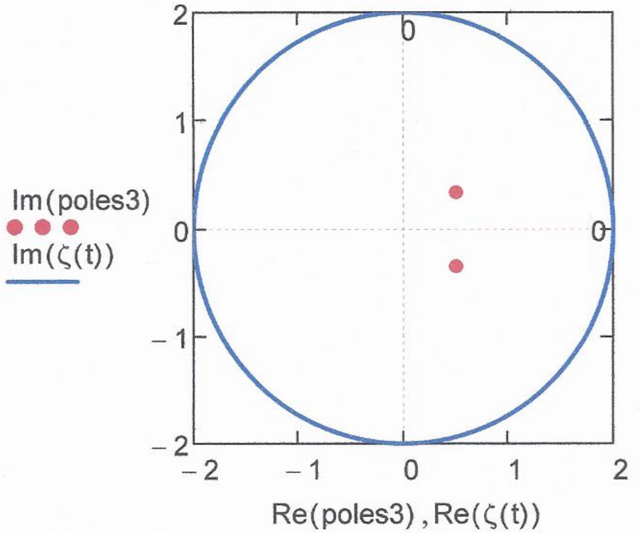


fig.(1.2.3.2)

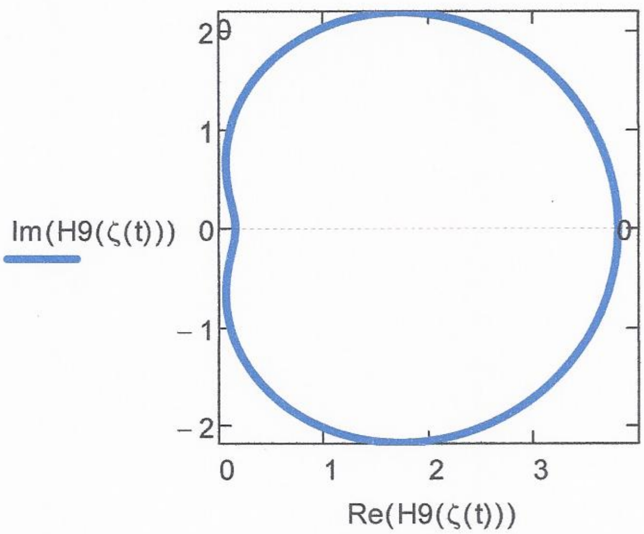


fig.(1.2.3.3)

so, we can write:

$$H9(z) = \frac{z^2 + 2 \cdot z + 1}{(z - p_0) \cdot (z - p_1)}. \qquad (1.2.2.14)$$

$$p_0 = 0.5 - 0.343j$$

$$p_1 = 0.5 + 0.343j$$

Since numerator and denominator are of the same order, the partial fraction expansion has the form:

$$\frac{H9(z)}{z} = \frac{B_0}{z} + \frac{C_1}{z - p_0} + \frac{C_2}{z - \overline{p_0}}, \quad B_0 = \frac{b_N}{a_N} = \frac{1}{b} = e, \qquad (1.2.2.15)$$

$$B_0 := e, \quad C1 = \lim_{z \to p_0} \left[\frac{H9(z)}{z} \cdot (z - p_0) \right],$$

$$C1 = \lim_{z \to p_0} \left[\frac{z^2 + 2 \cdot z + 1}{z \cdot (z - p_0) \cdot (z - p_1)} \cdot (z - p_0) \right] \qquad (1.2.2.16)$$

$$(p_0 - \overline{p_0}) = -0.687j$$

$$p_0 := p_0 \qquad C_1 := \lim_{z \to p_0} \left[\frac{z^2 + 2\cdot z + 1}{z\cdot\left(z - \overline{p_0}\right)} \right] \rightarrow \frac{{p_0}^2 + 2\cdot p_0 + 1}{p_0\cdot\left(p_0 - \overline{p_0}\right)} \tag{1.2.2.17}$$

$$C_1 = -0.859 + 5.62j$$

$$p_1 := p_1 \qquad p_0 := p_0$$

$$C_2 := \lim_{z \to \overline{p_0}} \left[\frac{z^2 + 2.0\cdot z + 1}{z\cdot\left(z - p_0\right)} \right] \rightarrow -\frac{\left(\overline{p_0}\right)^2 + 2.0\cdot\overline{p_0} + 1.0}{p_0\cdot\overline{p_0} - 1.0\cdot\left(\overline{p_0}\right)^2} \tag{1.2.2.18}$$

$$C_2 = -0.859 - 5.62j$$

$$H9(z) = e + \frac{C_1\cdot z}{z - p_0} + \frac{C_2\cdot z}{z - \overline{p_0}} \tag{1.2.2.19}$$

$$e \text{ invztrans } \rightarrow e\cdot\delta(n, 0)$$

$$\frac{C_1\cdot z}{z - p_0} \text{ invztrans } \rightarrow \frac{{p_0}^{n-1}\cdot\left(p_0 + 1\right)^2}{p_0 - \overline{p_0}} \tag{1.2.2.20}$$

$$\frac{C_2\cdot z}{z - \overline{p_0}} \text{ invztrans } \rightarrow -\frac{\left(\overline{p_0}\right)^{n-1}\cdot\left(\overline{p_0} + 1\right)^2}{p_0 - \overline{p_0}} \tag{1.2.2.21}$$

$$k := 0 .. 99 \qquad h9_k := e\cdot\delta(k, 0) + \frac{{p_0}^{k-1}\cdot\left(p_0 + 1\right)^2}{p_0 - \overline{p_0}} - \frac{\left(\overline{p_0}\right)^{k-1}\cdot\left(\overline{p_0} + 1\right)^2}{p_0 - \overline{p_0}} \tag{1.2.2.22}$$

$h9^T =$

1	3	3.632	2.528	1.192	0.262	-0.176	...

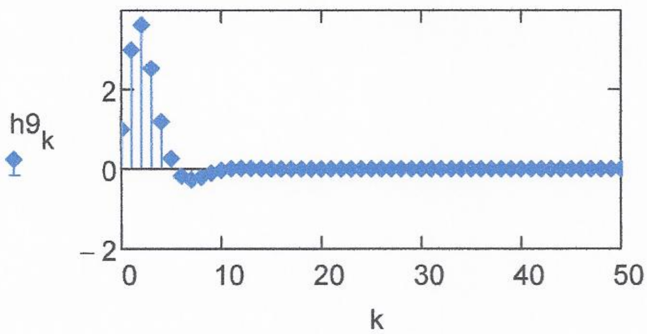


fig.(1.2.3.4)

Example 3) (H10(z) has a second order pole)**:** $H10(z) = \dfrac{z^2}{z^3 - \dfrac{5\cdot z^2}{2} + 2\cdot z - \dfrac{1}{2}} = \dfrac{z^2}{\left(z - \dfrac{1}{2}\right)\cdot(z-1)^2}$, Poles:

$$p_0 := \frac{1}{2}$$ First order,

$$p_1 := 1$$ Second order.

$$\frac{H10(z)}{z} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-1} + \frac{C}{(z-1)^2} \qquad (1.2.2.23)$$

remembering that $C_k = \lim_{z \to p_k} \left[\frac{H10(z)}{z}\cdot(z-p_k)\right]$,

$$A = \lim_{z \to p_0} \left[\frac{H10(z)}{z}\cdot(z-p_0)\right] = \lim_{z \to \frac{1}{2}} \frac{\left(z-\frac{1}{2}\right)\cdot z^2}{z\cdot\left(z-\frac{1}{2}\right)\cdot(z-1)^2} = \frac{\left(\frac{1}{2}\right)^2}{\frac{1}{2}\cdot\left(\frac{-1}{2}\right)^2} = 2, \quad (1.2.2.24)$$

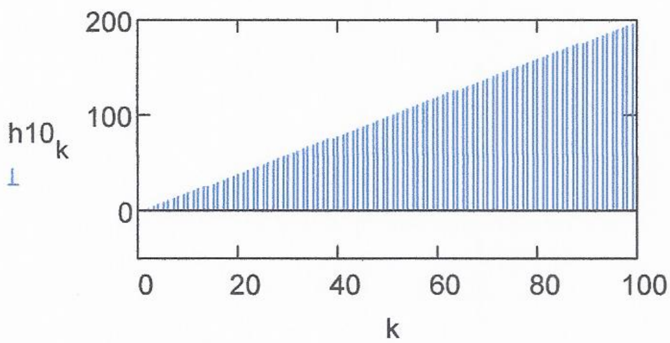
$$B = \lim_{z \to p_1} \frac{\partial}{\partial z}\left[\frac{H10(z)}{z}\cdot(z-p_1)^2\right] = \lim_{z \to 1} \frac{\partial}{\partial z}\left[\frac{z^2}{z\cdot\left[\left(z-\frac{1}{2}\right)\cdot(z-1)^2\right]}\cdot(z-1)^2\right] = -2, \quad (1.2.2.25)$$

$$C = \lim_{z \to p_1} \left[\frac{H10(z)}{z}\cdot(z-p_1)^2\right] = \lim_{z \to p_1} \left[\frac{z^2}{z\cdot\left[\left(z-\frac{1}{2}\right)\cdot(z-1)^2\right]}\cdot(z-1)^2\right] = 2. \quad (1.2.2.26)$$

$$H10(z) = \frac{2\cdot z}{z-\frac{1}{2}} - \frac{2\cdot z}{z-1} + \frac{2\cdot z}{(z-1)^2}, \qquad (1.2.2.2$$

$$\frac{2\cdot z}{z-\frac{1}{2}} \text{ invztrans} \to 2\cdot\left(\frac{1}{2}\right)^n, \qquad \frac{2\cdot z}{z-1} \text{ invztrans} \to 2, \qquad \frac{2\cdot z}{(z-1)^2} \text{ invztrans} \to 2\cdot n$$

$$h10_n := 2\cdot\left[\left(\frac{1}{2}\right)^n + n - 1\right], \; n \ge 0 \qquad (1.2.2.28$$



n := n

fig.(1.2.3.5)

test: $$2\cdot\left[\left(\frac{1}{2}\right)^n + n - 1\right] \text{ ztrans} \to \frac{2\cdot z^2}{(z-1)^2\cdot(2\cdot z-1)} \qquad (1.2.2.29)$$

Expand to see some examples

§1.2.4) USING THE "INVZTRANS" OPERATOR.

We will use this operator in all worksheets.
Examples:

$$n := n \quad \frac{z\cdot(z+1)}{(z-1)^2} \;\Big|\; \begin{array}{l}\text{invztrans}\\ \text{simplify}\end{array} \rightarrow 2\cdot n+1 \qquad (1.2.4.1)$$

$$n := n \quad \left[-\frac{2\cdot z}{(z-1)^3}\right] \;\Big|\; \begin{array}{l}\text{invztrans}\\ \text{simplify}\\ \text{factor}\end{array} \rightarrow -n\cdot(n-1) \qquad (1.2.4.2)$$

$$a := a \quad \left(\frac{a\cdot z}{a\cdot z-1}\right) \;\Big|\; \begin{array}{l}\text{invztrans}\\ \text{simplify}\end{array} \rightarrow \left(\frac{1}{a}\right)^n \qquad (1.2.4.3)$$

$$k := k \quad \left[\left(-\frac{z}{z-1}\right)\cdot k+\frac{z}{(z-1)^2}\right] \;\Big|\; \begin{array}{l}\text{invztrans}\\ \text{simplify}\end{array} \rightarrow n-k \qquad (1.2.4.4)$$

$$k := k \quad \left[-\frac{z\cdot(k-1)-k\cdot z^2}{z^2-2\cdot z+1}\right] \;\Big|\; \begin{array}{l}\text{invztrans}\\ \text{simplify}\end{array} \rightarrow k+n \qquad (1.2.4.5)$$

$m := m \quad k := k \quad a := a \quad b := b$

$$\frac{z\cdot[a+b+k\cdot(a-b)]-k\cdot z^2\cdot(a-b)}{z^2-2\cdot z+1} \;\Big|\; \begin{array}{l}\text{invztrans}\\ \text{simplify}\\ \text{collect}, a, b\end{array} \rightarrow (n-k)\cdot a+(k+n)\cdot b \qquad (1.2.4.6)$$

$m := m \quad k := k \quad a := a \quad b := b$

$$\left[\left[-\frac{z\cdot(a-b)}{z-1}\right]\cdot k+\frac{z\cdot(a+b)}{z-1}\cdot m+\frac{z\cdot(a+b)}{z^2-2\cdot z+1}\right] \;\Big|\; \begin{array}{l}\text{invztrans}\\ \text{simplify}\\ \text{collect}, a, b\end{array} \rightarrow (m-k+n)\cdot a+(k+m+n)\cdot b \qquad (1.2.4.7)$$

§1.2.5) Transformation $s \rightarrow z$, $\Rightarrow F(s) \rightarrow G(z)$

1) First transformation ($s \rightarrow z$).

Place $z^{-1} = e^{-s \cdot T_s} = \sum_{k=0}^{\infty} \left[\frac{(-s \cdot T_s)^k}{k!} \right]$, $(s = \sigma + j \cdot \omega)$ and truncate the power series to the first order,

namely; $z^{-1} = 1 - s \cdot T_s$, from which

$$s = \frac{1 - z^{-1}}{T_s}, \qquad (1.2.5.1)$$

Although, it is a very inaccurate approximation, nevertheless using this approximation and notwithstandin the previously mentioned defect, the results produced are very acceptable, as will be seen later) .

The previous transformation is a conformal mapping because it is analytical:

Now place $z = x + j \cdot y$ and consider the function $f(x, y, T_s) := \frac{1 - (x + j \cdot y)^{-1}}{T_s}$. To see if it is analytical or not, we apply the Cauchy-Riemann conditions:

$$\frac{\partial}{\partial y} f(x, y, T_s) \rightarrow \frac{j}{T_s \cdot (x + y \cdot j)^2} \quad , \quad j \cdot \left(\frac{\partial}{\partial x} f(x, y, T_s) \right) \rightarrow \frac{j}{T_s \cdot (x + y \cdot j)^2},$$

$$\text{hence } \frac{\partial}{\partial y} f(x, y, T_s) = j \cdot \frac{\partial}{\partial x} f(x, y, T_s).$$

moreover it satisfies the condition:

$$\lim_{z \rightarrow z_0} \left| \frac{\partial}{\partial z} f(z) \right| \neq 0$$

$$z := z, \ z_0 := z_0, \ T_s := T_s, \quad \lim_{z \rightarrow z_0} \left| \frac{\partial}{\partial z} \frac{1 - z^{-1}}{T_s} \right| \rightarrow \begin{cases} \infty & \text{if } z_0 = 0 \\ \dfrac{\text{signum}(z_0, 0)^2}{z_0^2 \cdot |T_s|} & \text{if } z_0 \neq 0 \end{cases}$$

this result let the previous transformation be a conformal mapping (as requested by the theorem thereof).

Complex plane representation of s and z.

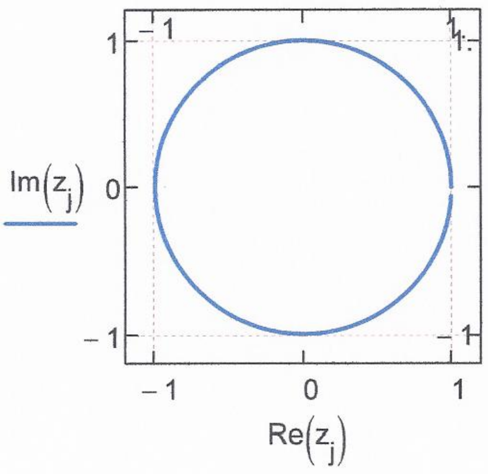
$N := 100 \qquad j := 0..N-1 \quad \rho := 1.0$

$$\theta_j := \frac{2 \cdot \pi}{N} \cdot j \qquad \text{rows}(\theta) = 100$$

$$z_j := \rho \cdot e^{j \cdot \theta_j}$$

$$T_s := 1 \qquad s_j := \frac{1-(z_j)^{-1}}{T_s}$$

z-plane



s-plane

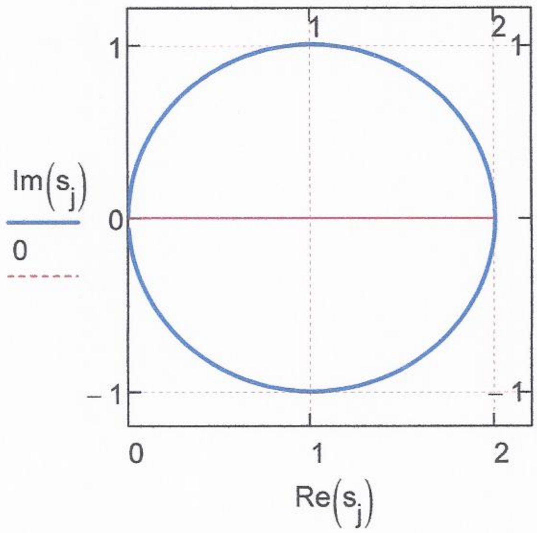


fig.(1.2.5.1)

fig.(1.2.5.2)

Example: knowing the function f(t), we search the corresponding sequence via Laplace transformatior

The function needs some parameters: $\mu := 1000$, $\beta := \frac{1}{\mu}$, $\alpha := 2\cdot\pi\cdot\mu$,

$$\text{the function is: } f00(t) := -t\cdot\alpha^2\cdot e^{-t\cdot\alpha}\cdot\Phi(t)\ , \tag{1.2.5.2}$$

Sampling frequency: $\mu_0 := 100\cdot\mu$, $\tau_0 := \frac{1}{\mu_0}$, $\tau_0 = 1\times10^{-5}$

$$t := 0, 0+\frac{2\cdot\beta}{N} .. 2\cdot\beta$$

$$\frac{\beta}{\tau_0\cdot N} = 1$$

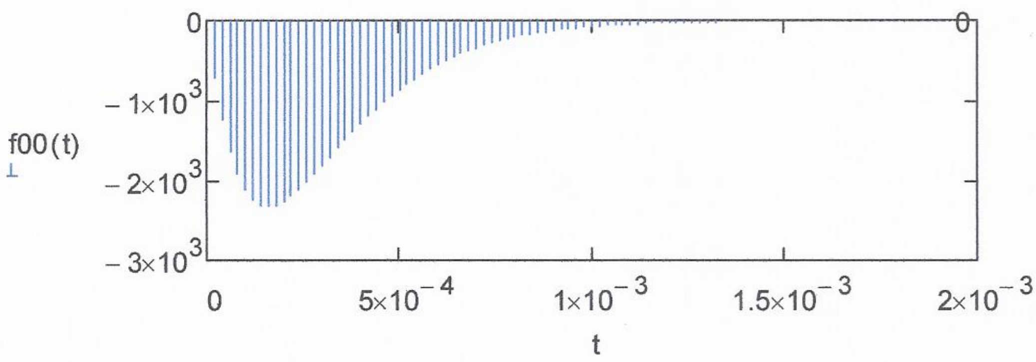


fig.(1.2.5.3)

Laplace transform of the given function:

$t := t \qquad T_s := \tau_0 \qquad s := s \qquad \rho := \rho \qquad \alpha := \alpha$

$$-t\cdot\alpha^2\cdot e^{-t\cdot\alpha}\cdot\Phi(t) \text{ laplace}, t \rightarrow -\frac{\alpha^2}{(\alpha+s)^2} \tag{1.2.5.3}$$

$$F(s) = \mathcal{L}\left(-t\cdot\alpha^2\cdot e^{-t\cdot\alpha}\cdot\Phi(t)\right) = \frac{-\alpha^2}{(s+\alpha)^2} \quad (1.2.5.4)$$

$$F6(s) := \frac{-\alpha^2}{(s+\alpha)^2} \quad (1.2.5.5)$$

Z transform applying the given approximation:

$\tau_0 := \tau_0$ $\quad s := s$ $\quad z := z$ $\quad \alpha := \alpha$

$$G6(z) := \frac{-\alpha^2}{(s+\alpha)^2} \left| \begin{array}{l} \text{substitute}, s = \dfrac{1-z^{-1}}{\tau_0} \\ \text{collect}, z \end{array} \right. \rightarrow -\frac{\alpha^2\cdot\tau_0^2\cdot z^2}{\left(\alpha^2\cdot\tau_0^2 + 2\cdot\alpha\cdot\tau_0 + 1\right)\cdot z^2 + \left(-2\cdot\alpha\cdot\tau_0 - 2\right)\cdot z + 1}$$

$$\text{or} \quad G6(z) = \frac{-\left(\tau_0^2\cdot\alpha^2\right)}{\left(\tau_0^2\cdot\alpha^2 + 2\cdot\tau_0\cdot\alpha + 1\right) - 2\cdot\left(\tau_0\cdot\alpha + 1\right)\cdot z^{-1} + z^{-2}} \quad (1.2.5.6)$$

Define the following parameters:

$$A1 := -1\cdot\tau_0^2\cdot\alpha^2 \qquad B1 := 2\cdot\left(\tau_0\cdot\alpha + 1\right) \qquad C1 := \tau_0^2\cdot\alpha^2 + 2\cdot\tau_0\cdot\alpha + 1 \quad (1.2.5.7)$$

$$A1 = -3.948\times 10^{-3} \qquad B1 = 2.126 \qquad C1 = 1.13$$

$$G6(z) := \frac{A1}{z^{-2} - B1\cdot z^{-1} + C1} \quad (1.2.5.8)$$

Plotting magnitude in dB and phase of the two functions F(s) and G(z):

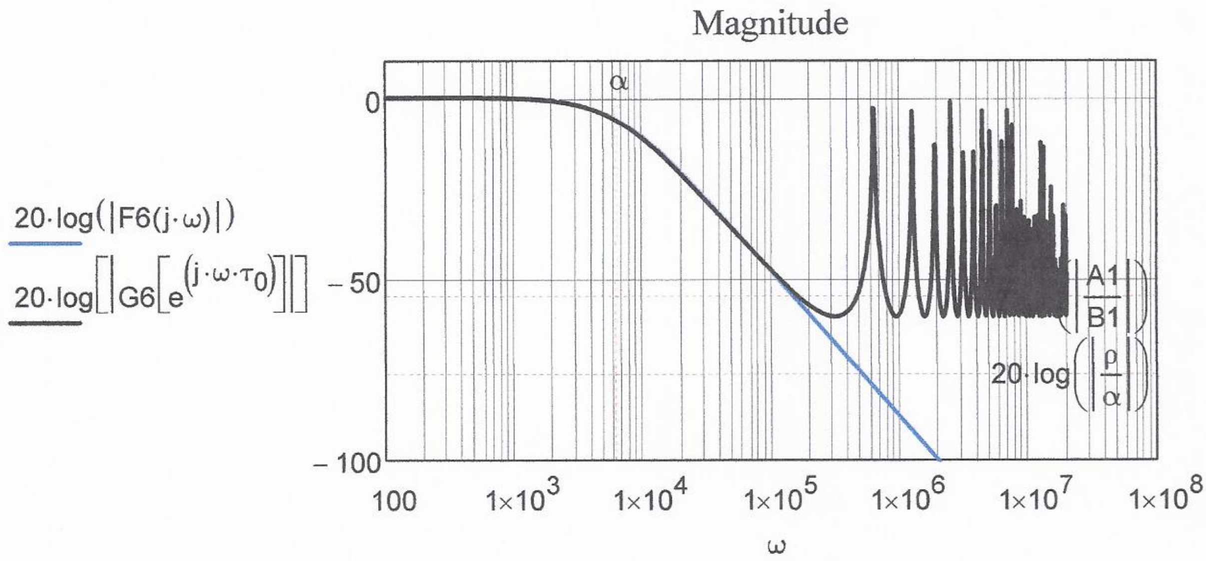


fig.(1.2.5.4)

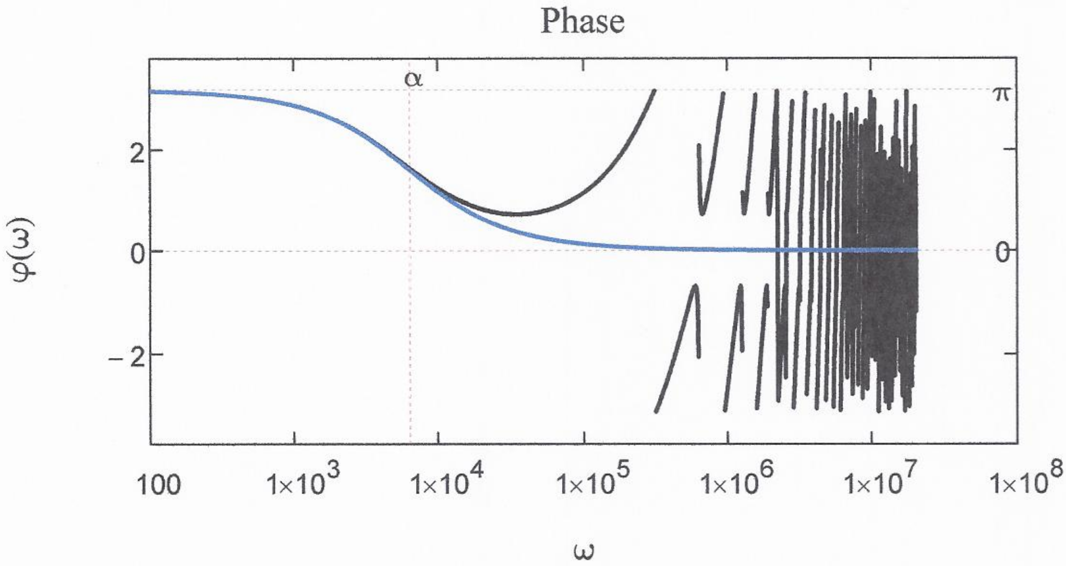


fig.(1.2.5.5)

Poles calculation of G(z):

$$v := \text{denom}(G6(z)) \text{ coeffs}, z \rightarrow \begin{pmatrix} 1 \\ -2\cdot\alpha\cdot\tau_0 - 2 \\ \alpha^2\cdot\tau_0^{\,2} + 2\cdot\alpha\cdot\tau_0 + 1 \end{pmatrix} \qquad (1.2.5.9)$$

$\text{poles6} := \text{polyroots}(v)$ $\qquad \text{poles6}^T = (0.941 \quad 0.941)$

$p_0 := \text{poles6}_0$ $\qquad p_0 = 0.941$ $\qquad p_1 := \text{poles6}_1$ $\qquad p_1 = 0.941$

$t := t$

$r := \text{ceil}(\max(|\text{poles6}|))\cdot 1.0$

$\xi(t) := r\cdot\cos(t)$ $\qquad r = 2$ $\qquad |p|$

$\psi(t) := r\cdot\sin(t)$

$\zeta(t) := \xi(t) + j\cdot\psi(t)$

$t_0 := 0 \quad t_{fin} := 2\cdot\pi \qquad t := t_0, t_0 + \frac{t_{fin} - t_0}{100} .. t_{fin}$

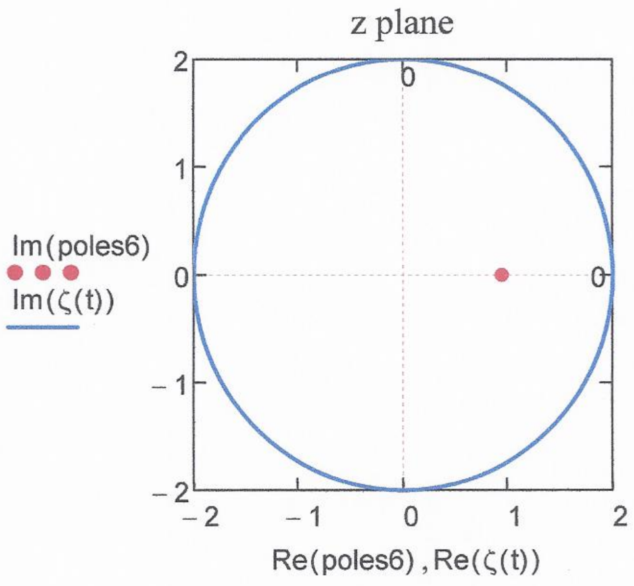


fig.(1.2.5.6)

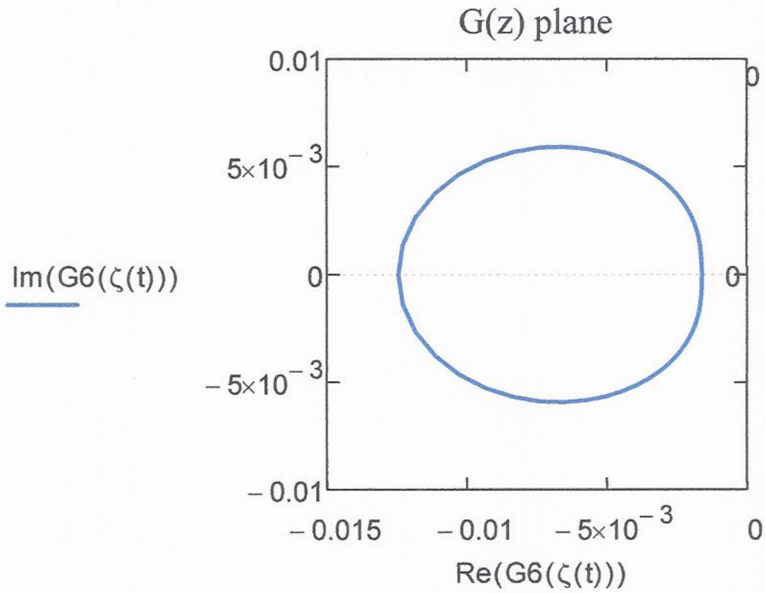


fig.(1.2.5.7)

Calculation of the sequence corresponding to G6(z):

$$out_n := \frac{A1}{z^{-2} - B1\cdot z^{-1} + C1} \begin{vmatrix} z := z \quad A1 := A1 \quad B1 := B1 \\ \text{invztrans} \\ \text{simplify}, \max \end{vmatrix} \rightarrow \qquad (1.2.5.10)$$

parameters definition:

$$\gamma 1 := \frac{B1 - \sqrt{B1^2 - 4\cdot C1}}{C1} \qquad \gamma 2 := \frac{B1 + \sqrt{B1^2 - 4\cdot C1}}{C1} \qquad (1.2.5.11)$$

$$\sigma 1 := 2\cdot C1 - B1^2 + B1\cdot\sqrt{B1^2 - 4\cdot C1} \qquad \sigma 2 := B1^2 - 2\cdot C1 + B1\cdot\sqrt{B1^2 - 4\cdot C1} \qquad (1.2.5.12)$$

searched sequence $out_n := \left(\frac{1}{2}\right)^n \cdot \frac{A1}{\left(C1^2\cdot\sqrt{B1^2 - 4\cdot C1}\right)} \cdot \left(\gamma 1^{n-1}\cdot\sigma 1 + \gamma 2^{n-1}\cdot\sigma 2\right)$ (1.2.5.13)

$$n := 0 .. N - 1$$

The approximation , as shown by the plot below, reduces the amplitude of the given function, although it looks very like to the original one. Hence, the amplitude of the original function is scaled down.

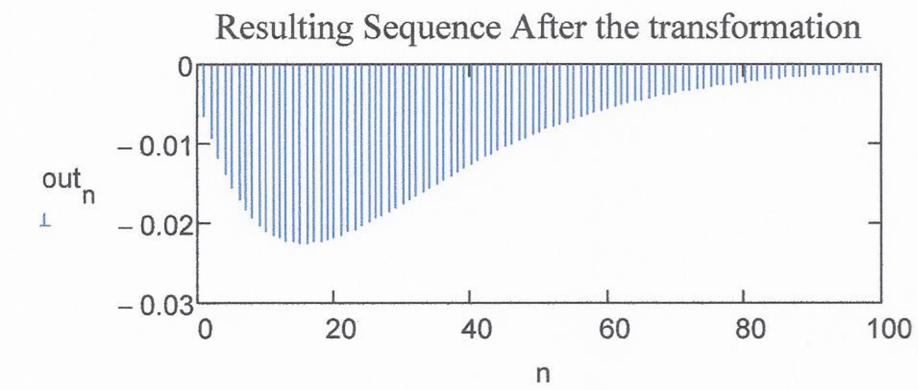


fig.(1.2.5.8)

$$t := 0, 0 + \frac{2\cdot\beta}{N} .. 2\cdot\beta$$

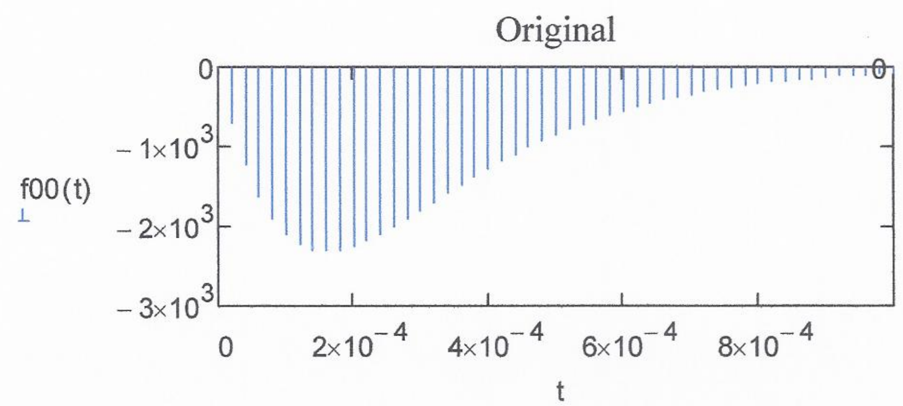


fig.(1.2.5.9)

2) Bilinear transformation:

$$s = \left(\frac{1}{\tau_0}\cdot\ln(z)\right) \approx \left[\frac{2}{\tau_0}\cdot\left[\frac{z-1}{z+1} + \frac{(z-1)^3}{3\cdot(z+1)^3} + \dots\right]\right] \qquad (1.2.5.14)$$

(See: **R. Isermann**, *Digitale Regelsysteme*, 3.2-74 pag.:32) in a first approximation one can write:

$$s \approx \left[\frac{2}{\tau_0}\cdot\left(\frac{z-1}{z+1}\right)\right]$$

Bilinear transformation: $s = \frac{2}{\tau_0}\cdot\frac{1-z^{-1}}{1+z^{-1}}$ (1.2.5.15)

The previous transformation is a conformal mapping because it is analytical, in fact:

place $(z = x+j\cdot y)$, define $f(x,y,\tau_0) := \frac{2}{\tau_0}\cdot\frac{1-(x+j\cdot y)^{-1}}{1+(x+j\cdot y)^{-1}}$,

To see if it is analytical or not, we apply the Cauchy-Riemann conditions:

$x := x \qquad y := y \qquad \tau_0 := \tau_0$

$$\frac{\partial}{\partial y}f(x,y,\tau_0) \text{ simplify, max } \rightarrow \frac{4j}{\tau_0\cdot(x+1+y\cdot j)^2} \ ,$$

$$j\cdot\frac{\partial}{\partial x}f(x,y,\tau_0) \text{ simplify, max } \rightarrow \frac{4j}{\tau_0\cdot(x+1+y\cdot j)^2},$$

hence $\frac{\partial}{\partial y}f(x,y,\tau_0) = j\cdot\frac{\partial}{\partial x}f(x,y,\tau_0)$.

$$\lim_{z\to z_0}\left|\frac{\partial}{\partial z}\left(\frac{2}{\tau_0}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right)\right| \ \begin{vmatrix}\text{simplify} \\ \text{assume}, z_0 \neq 0\end{vmatrix} \rightarrow \frac{4}{\left(|z_0+1|\right)^2\cdot|\tau_0|}$$

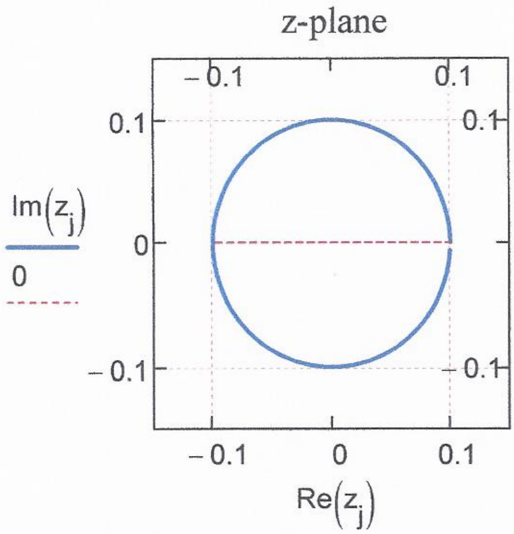
this result let the previous transformation be a conformal mapping (as requested by the theorem thereof

Bilinear transformation s and z planes:

$N := 100 \qquad j := 0 .. N-1 \qquad \rho := 0.1 \qquad \theta_j := \frac{2\cdot\pi}{N}\cdot j \qquad \text{rows}(\theta) = 100$

$$z_j := \rho\cdot e^{j\cdot\theta_j}$$

$$s_j := 1\cdot\frac{1-(z_j)^{-1}}{1+(z_j)^{-1}} \qquad (1.2.5.16)$$



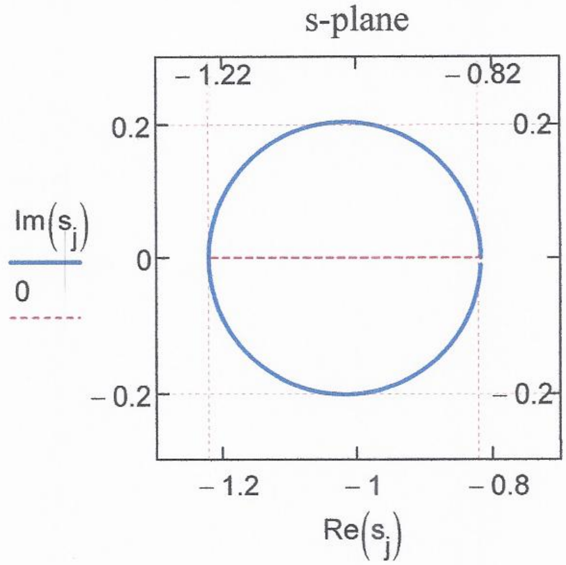


fig.(1.2.5.10)

fig.(1.2.5.11)

Z transform calculation of a given Laplace transform (function of s)

$\tau_0 := \tau_0 \qquad s := s \qquad z := z \qquad \alpha := \alpha$

$$G1(z) := \frac{-\alpha^2}{(s+\alpha)^2} \left| \begin{array}{l} \text{substitute}, s = \frac{2}{\tau_0}\cdot\frac{1-z^{-1}}{1+z^{-1}} \\ \text{collect}, z \\ \text{collect}, \rho, \alpha\cdot\tau_0^2 \end{array} \right. \rightarrow \tag{1.2.5.17}$$

Function of z $\quad G1(z) = \dfrac{-\left[\tau_0^2\cdot\alpha^2\cdot(z+1)^2\right]}{(\tau_0\cdot\alpha+2)^2\cdot z^2 + 2\cdot\left(\tau_0^2\cdot\alpha^2-4\right)\cdot z + (\tau_0\cdot\alpha-2)^2}$ (1.2.5.18)

or as a function of z^{-1}:

$$G1(z) = \frac{-\left(\tau_0^2\cdot\alpha^2\right)}{(\tau_0\cdot\alpha-2)^2}\cdot\frac{\left(z^{-1}+1\right)^2}{\frac{(\tau_0\cdot\alpha+2)^2}{(\tau_0\cdot\alpha-2)^2} + 2\cdot\frac{\left(\tau_0^2\cdot\alpha^2-4\right)}{(\tau_0\cdot\alpha-2)^2}\cdot z^{-1} + z^{-2}} \tag{1.2.5.19}$$

Magnitude and phase plots.
Define the following constants to simplify the calculation:

$$A11 := \frac{-\left(\tau_0^2\cdot\alpha^2\right)}{(\tau_0\cdot\alpha-2)^2} \qquad B11 := 2\cdot\frac{\left(\tau_0^2\cdot\alpha^2-4\right)}{(\tau_0\cdot\alpha-2)^2} \qquad C11 := \frac{(\tau_0\cdot\alpha+2)^2}{(\tau_0\cdot\alpha-2)^2} \tag{1.2.5.20}$$

$A11 = -1.052\times10^{-3} \qquad B11 = -2.13 \qquad C11 = 1.134$

$z := z \qquad A11 := A11 \qquad B11 := B11 \qquad C11 := C11$

results: $G1(z) := \dfrac{A11\cdot\left(z^{-1}+1\right)^2}{z^{-2}+B11\cdot z^{-1}+C11}$ (1.2.5.21)

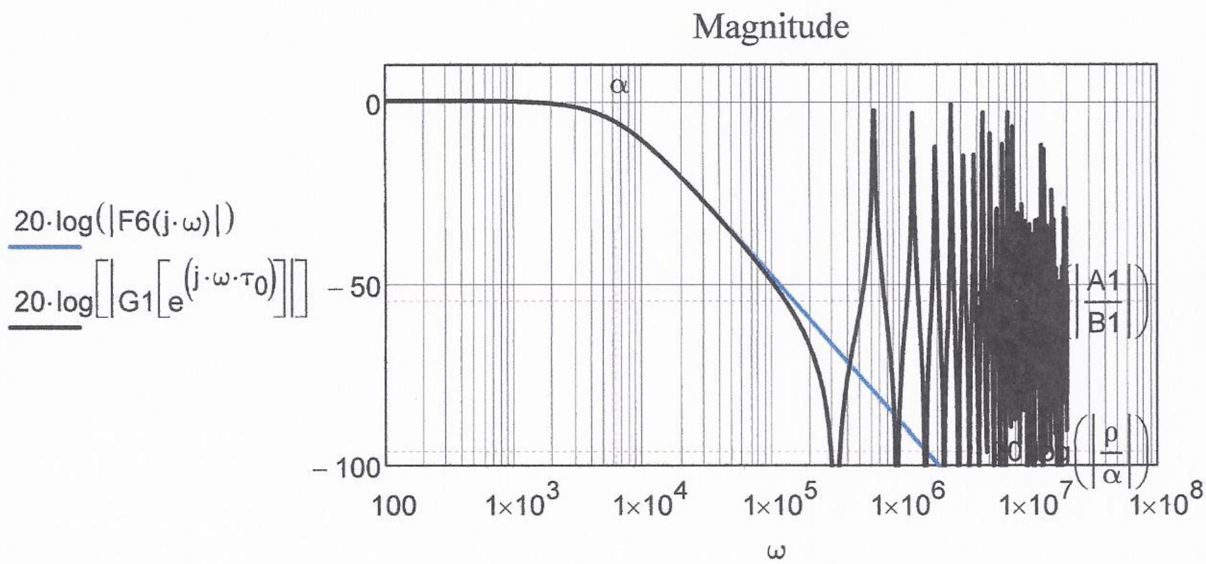


fig.(1.2.5.12)

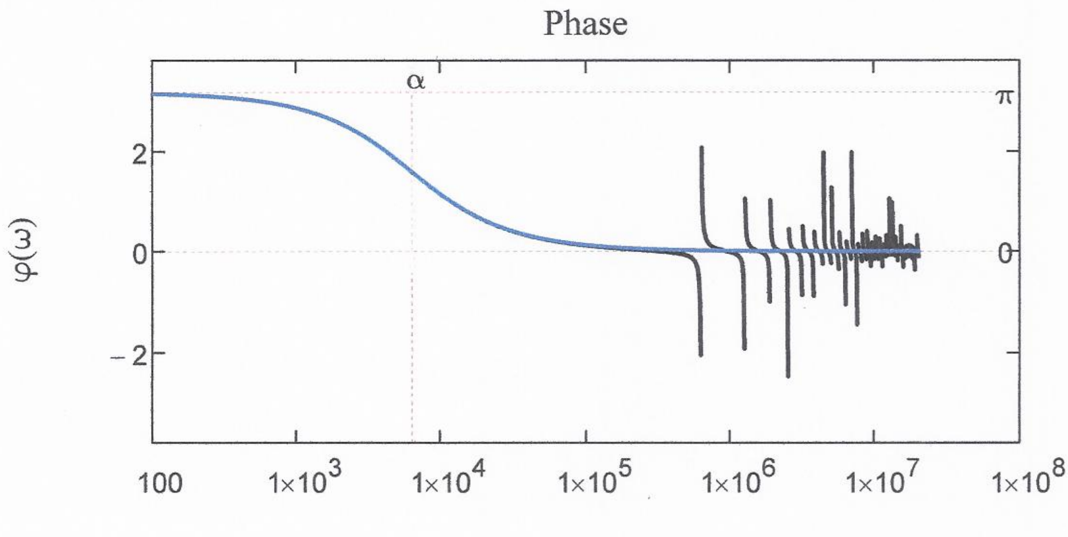


fig.(1.2.5.13)

Plot of G(z) as z moves on a circle

$z := z$ $A11 := A11$ $B11 := B11$ $C11 := C11$

$$v := \text{denom}(G1(z))\ \text{coeffs}, z \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ B11 \\ C11 \end{pmatrix} \quad (1.2.5.22)$$

search of the poles to represent on the z plane

$poles1 := \text{polyroots}(v)$ $poles1^T = (0 \quad 0 \quad 0.939 \quad 0.939)$

$p_0 := \text{poles1}_0 \qquad p_0 = 0 \qquad\qquad p_1 := \text{poles1}_1 \qquad p_1 = 0$

z moves on a circle $\quad r := \text{ceil}\left(\max\left(\left|\overrightarrow{\text{poles1}}\right|\right)\right)\cdot 1.0 \qquad r = 2$ (1.2.5.23)

$t := t$

$$\xi(t) := r\cdot\cos(t)$$

$$\psi(t) := r\cdot\sin(t)$$

$$\zeta(t) := \xi(t) + j\cdot\psi(t)$$

$$t := t_0, t_0 + \frac{t_{fin} - t_0}{1000} .. t_{fin}$$

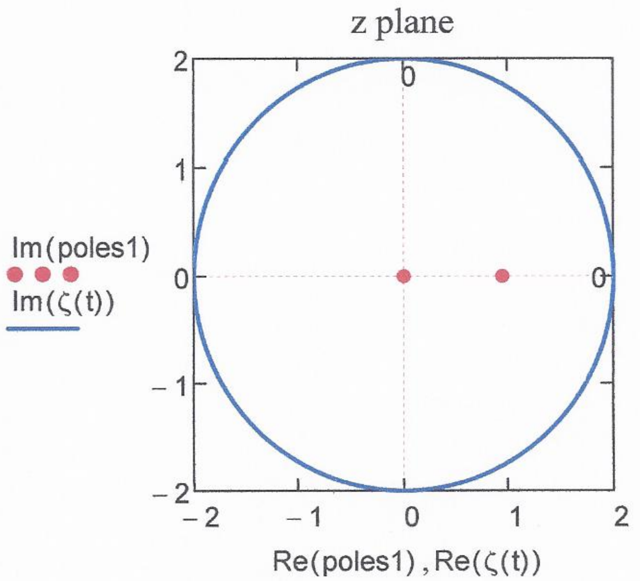


fig.(1.2.5.14)

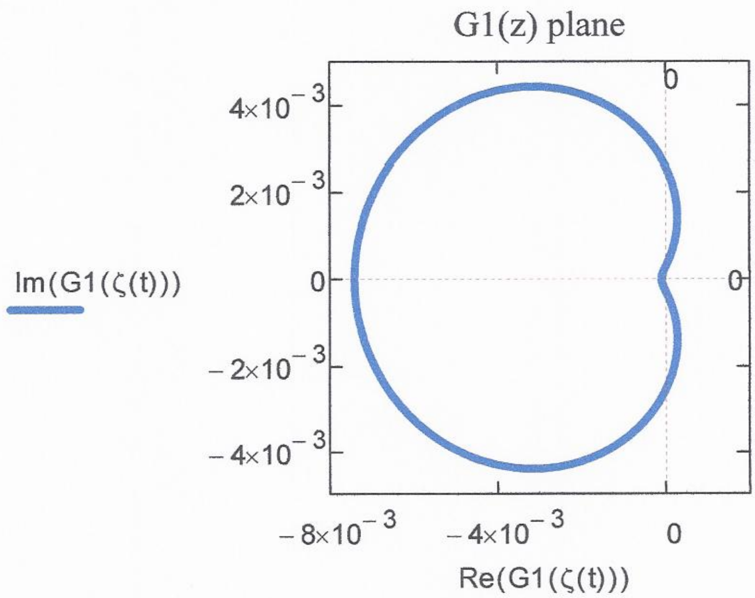


fig.(1.2.5.15)

Sequence calculation of the given z function:

$z := z \qquad A11 := A11 \qquad B11 := B11 \qquad C11 := C11$

$$\frac{A11\cdot\left(z^{-1}+1\right)^2}{z^{-2}+B11\cdot z^{-1}+C11} \left|\begin{array}{l}\text{invztrans}\\ \text{simplify}, \max\end{array}\right. \rightarrow \qquad (1.2.5.14)$$

Seek of simplicity define the following constants:

$$k1 := -\frac{B11-\sqrt{B11^2-4\cdot C11}}{C11} \qquad (1.2.5.15)$$

$$k2 := \frac{\left(\sqrt{B11^2-4\cdot C11}-B11\right)\cdot(B11-2\cdot C11) + -2\cdot C11\cdot(C11-1)}{C11^2\cdot\sqrt{B11^2-4\cdot C11}}$$

$$k3 := -\frac{B11+\sqrt{B11^2-4\cdot C11}}{C11}$$

$$k4 := \left[\frac{-\left(B11+\sqrt{B11^2-4\cdot C11}\right)\cdot(B11-2\cdot C11) - 2\cdot C11\cdot(C11-1)}{C11^2\cdot\sqrt{B11^2-4\cdot C11}}\right]$$

The resulting sequence is:

$$out1_n := A11\cdot\delta(n,0) + \left(\frac{1}{2}\right)^n \cdot A11\cdot\left(-k1^{n-1}\cdot k2 + k3^{n-1}\cdot k4\right) \qquad (1.2.5.16)$$

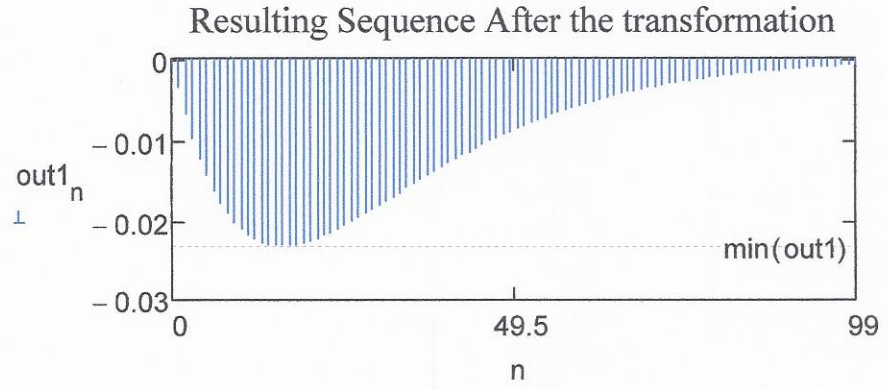


fig.(1.2.5.16)

$$t := 0, 0 + \frac{2\cdot\beta}{N} .. 2\cdot\beta$$

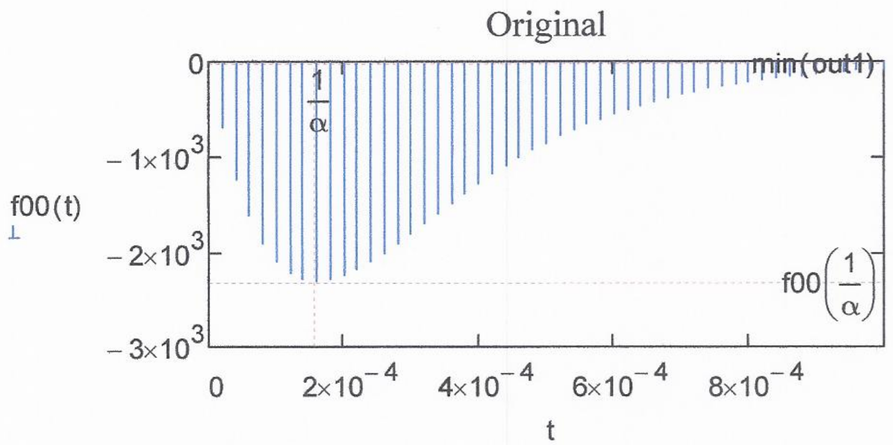


fig.(1.2.5.17)

$f00\left(\frac{1}{\alpha}\right) = -2311.455$ $\qquad \frac{\left|f00\left(\frac{1}{\alpha}\right)\right|}{|min(out1)|} = 1 \times 10^5$ $\qquad min(out1) = -0.023$

§1.3) DIFFERENCE EQUATIONS

We consider the following z transform, as a rational function:

$$H(z) = \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}}, \qquad (1.3.1)$$

moreover, since H(z) is defined as:

$$H(z) = \frac{Y(z)}{X(z)},$$

(for a causal system, X(z) is the z-transform of the input signal while Y(z) is the corresponding output), we can write:

$$Y(z) \cdot \left(1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}\right) = X(z) \cdot \left(b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}\right),$$

z inverse transforming we have:

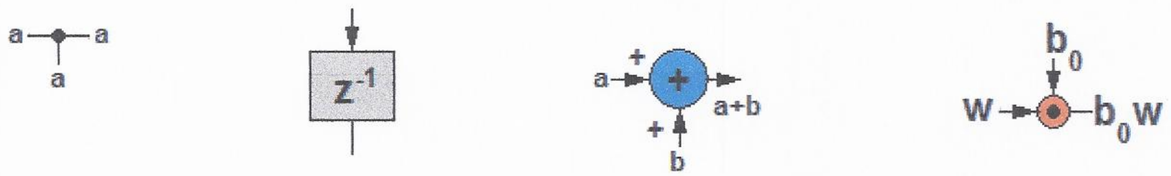
$$y(n) + a_1 \cdot y(n-1) + a_2 \cdot y(n-2) = b_0 \cdot x(n) + b_1 \cdot x(n-1) + b_2 \cdot x(n-2) .$$

The corresponding difference equation is,

$$y(n) = x(n) + b_1 \cdot x(n-1) + b_2 \cdot x(n-2) - \left(a_1 \cdot y(n-1) + a_2 \cdot y(n-2)\right), \; n>1. \qquad (1.3.2)$$

To these difference equation we must associate the initial conditions y(0) and y(1).

We can match to each difference equation a block diagram that depict the filter structure and the computational procedure for implementing the digital filter. The basic four elements of the realization structure are: branch, delay block, summing node, node multiplier, namely:



node | delay block | summing node | node multiplier

fig.(1.3.1)

So, for the difference equations defined above, we have the following flow or block diagrams:

DIRECT FORM

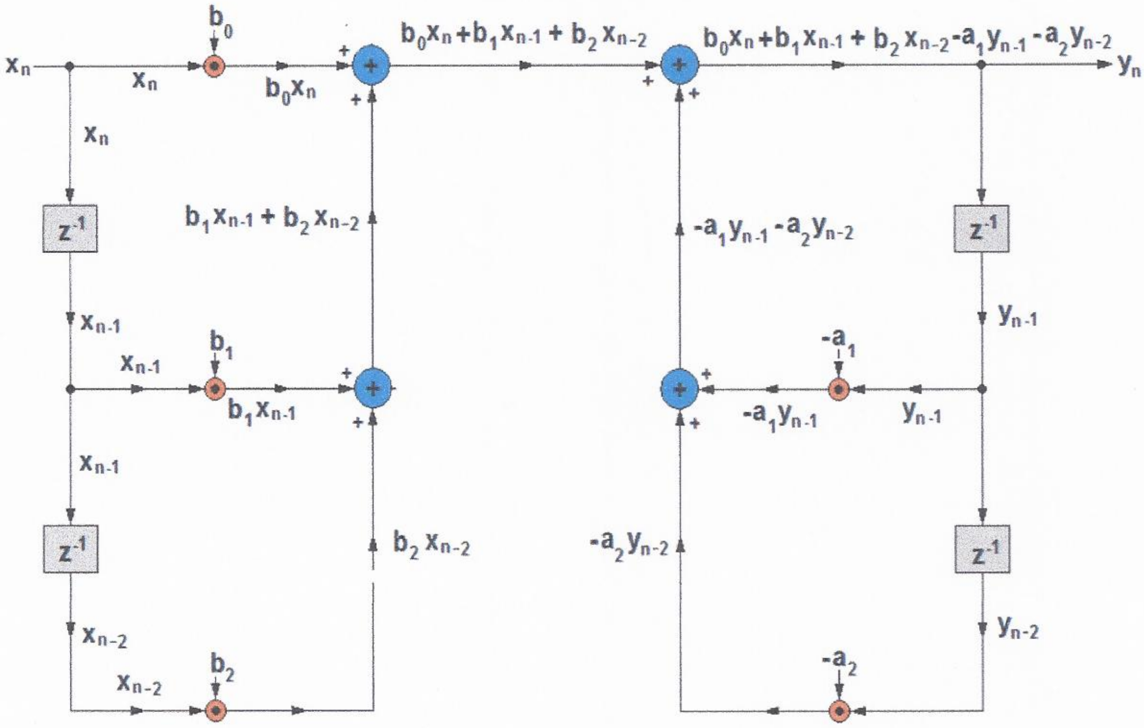


fig.(1.3.2)

Practical second order building block for IIR structure.
DIRECT FORM

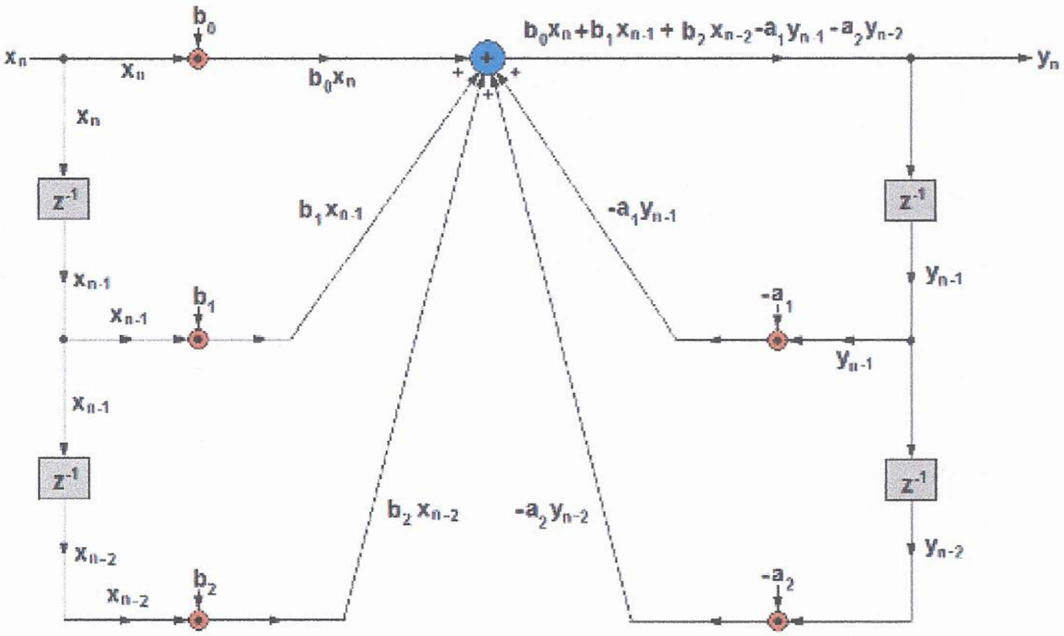
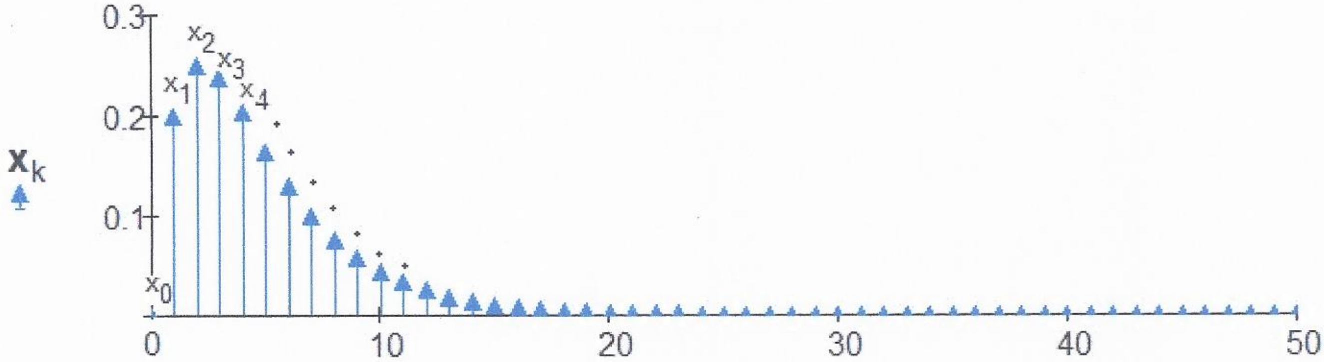


fig.(1.3.3)

Example of input sequence and computing of the corresponding output :



$x_0 \; x_1 \; x_2 \; x_3 \; x_4 \; \vdots \; \vdots \; \vdots$

$\downarrow \; \downarrow \; \downarrow \; \downarrow \; \downarrow$

$y_0 \; y_1 \; y_2 \; y_3 \; y_4 \; \vdots \; \vdots \; \vdots$

i. c. $\neq 0 \; \neq 0 \; \neq 0$

$$y_2 = b_0 x_2 + b_1 x_1 + b_2 x_0 - a_1 y_1 - a_2 y_0$$

0

$$y_3 = b_0 x_3 + b_1 x_2 + b_2 x_1 - a_1 y_2 - a_2 y_1$$

$$y_4 = b_0 x_4 + b_1 x_3 + b_2 x_2 - a_1 y_3 - a_2 y_2$$

$\vdots$

fig.(1.3.4)

DSP registers:

b_0	b_1	b_2	a_1	a_2	x_n	x_{n-1}	x_{n-2}	y_n	y_{n-1}	y_{n-2}

fig.(1.3.5)

A useful stability criterion for linear time invariant systems is that all bounded inputs produce bounded outputs, this is the so called BIBO (bounded input-bounded output) condition.

LTI system is said to be stable if and only if it satisfies the criterion:

$$\sum_{k=0}^{\infty} |h(k)| < \infty \quad \text{(IIR)}, \qquad (1.3.3)$$

where h(k) is the sequence of the impulse response of the system.

Special cases:

$x(n) = x(n-1) = x(n-2) = 0 \;\Rightarrow\; y(n) = -\left(a_1 \cdot y(n-1) + a_2 \cdot y(n-2)\right)$,

the resulting block diagram is below depicted:

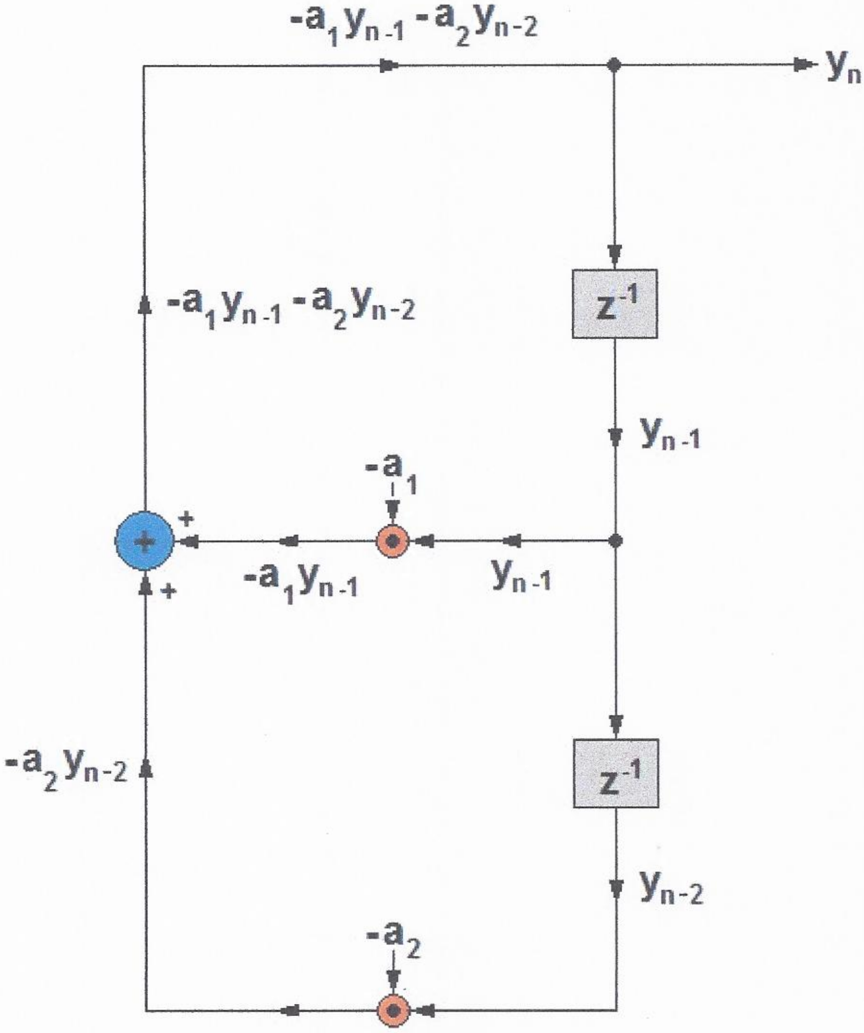


fig.(1.3.6)

For system stability the following conditions must be satisfied:

$$0 \le |a_2| < 1 \qquad |a_1| \le 1 + a_2$$

It is well known that a filter with a transfer function like this:

$$H(z) = \frac{1}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}}, \tag{1.3.4}$$

difference equation: $\quad y(k) = a_1 \cdot y(k-1) + a_2 \cdot y(k-2)$, (1.3.5)

is stable if and only if the parameters (a_1, a_2) lie inside the triangle drawn below.

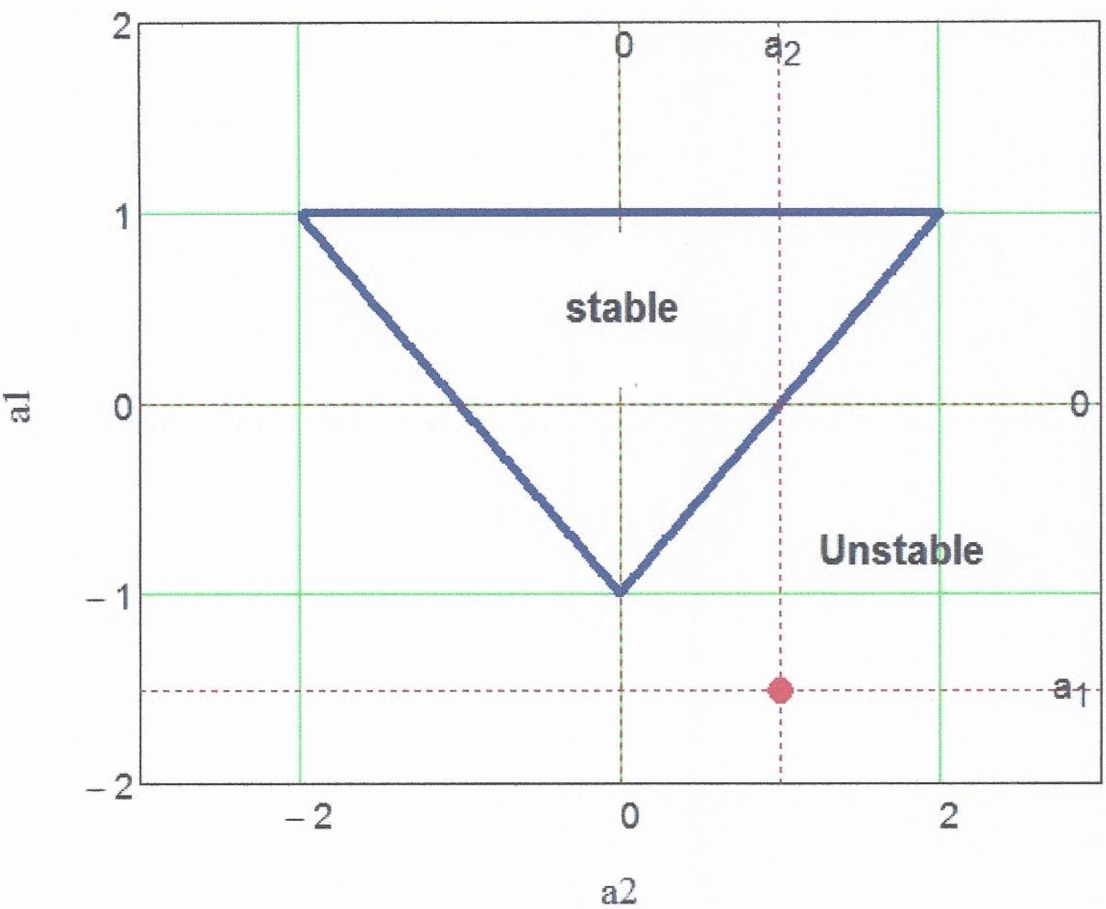


fig.(1.3.7)

Under some conditions this can be a digital oscillator.
Example: if we place:

$a_1 := -1.5 \qquad a_2 := 1.0$

$n := 2 .. M1 \qquad k1 := 0 .. 35$

and define the following recurrence relation:

$$u1(k1) := \begin{cases} -\left(a_1\, u1(k1-1) + a_2 \cdot u1(k1-2)\right) & \text{if } k1 > 1 \\ \text{otherwise} & \begin{cases} 1.5 & \text{if } k1 = 0 \\ 0.65 & \text{if } k1 = 1 \end{cases} \end{cases} \qquad (1.3.6)$$

Output Signal

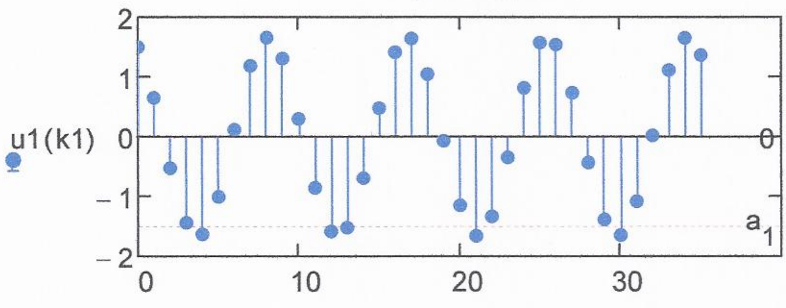


fig.(1.3.8)

Instead of using a recurrence relation , we can define an array, as follows, which is faster than the previous one, but with large expense of memory:

$$v_0 := 1.5 \qquad v_1 := 0.65$$
$$v_n := -\left(a_1 \cdot v_{n-1} + a_2 \cdot v_{n-2}\right) \qquad (1.3.7)$$

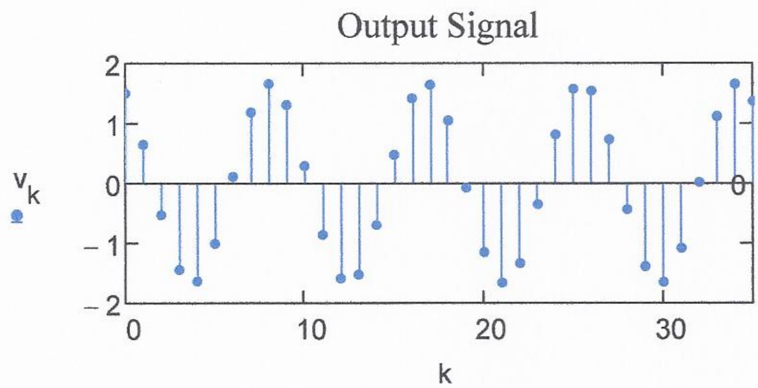


fig.(1.3.9)

Stability "Triangle"

for the previous example we have: Check_stability(a) = "Unstable"

while if we use some other coefficients the system become stable as we can see changing them.

$a_1 = -1.5 \qquad a_2 = 1 \qquad x := -3, -3 + 0.01 .. 3$

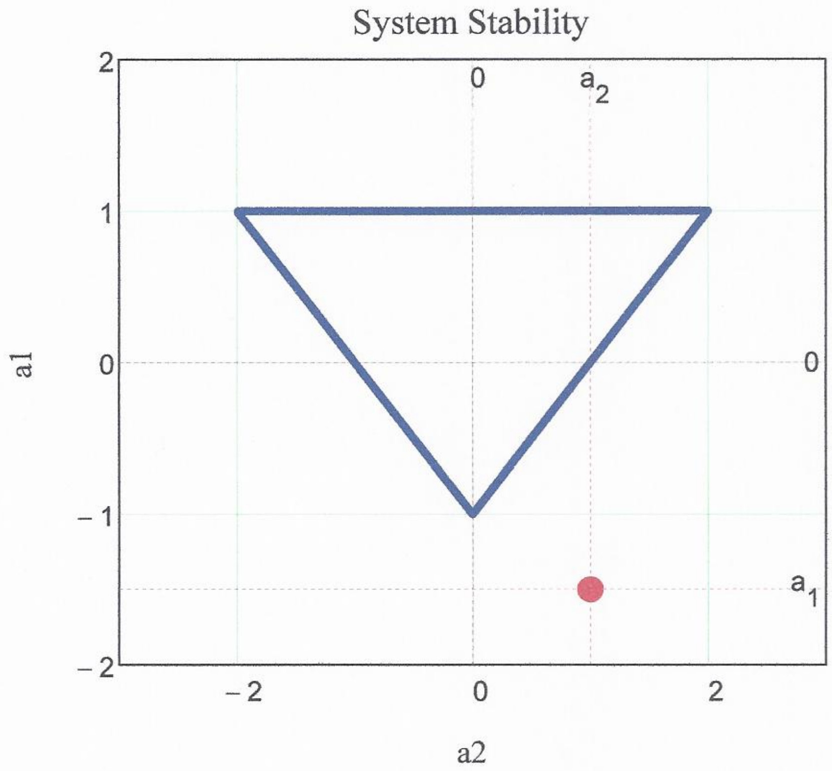


fig.(1.3.10)

CANONICAL SECOND ORDER SECTION

On the other hand, for a second order filter, having placed:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}, \qquad (1.3.8)$$

namely:

$$\frac{Y(z)}{W(z)} = b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}, \qquad (1.3.9)$$

$$\frac{W(z)}{X(z)} = \frac{1}{\left(1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}\right)}, \qquad (1.3.10)$$

or

$$W(z) = \frac{X(z)}{1 + \sum_{p=1}^{2} \left(a_p \cdot z^{-p}\right)} \qquad (1.3.11)$$

and

$$Y(z) = b_0 \cdot W(z) + b_1 \cdot z^{-1} \cdot W(z) + b_2 \cdot z^{-2} \cdot W(z), \qquad (1.3.12)$$

$$X(z) = W(z) + a_1 \cdot z^{-1} \cdot W(z) + a_2 \cdot z^{-2} \cdot W(z), \qquad (1.3.13)$$

The corresponding set of difference equations is:

state at time n $\quad w(n) = x(n) - a_1 \cdot w(n-1) - a_2 \cdot w(n-2), \qquad (1.3.14)$

output signal $\quad y(n) = b_0 \cdot w(n) + b_1 \cdot w(n-1) + b_2 \cdot w(n-2), \qquad (1.3.15)$

whose block diagram is given in the next drawing:

$$y_n = b_0 w_n + b_1 w_{n-1} + b_2 w_{n-2}$$

$$w_n = \quad x_n - a_1 w_{n-1} - a_2 w_{n-2}$$

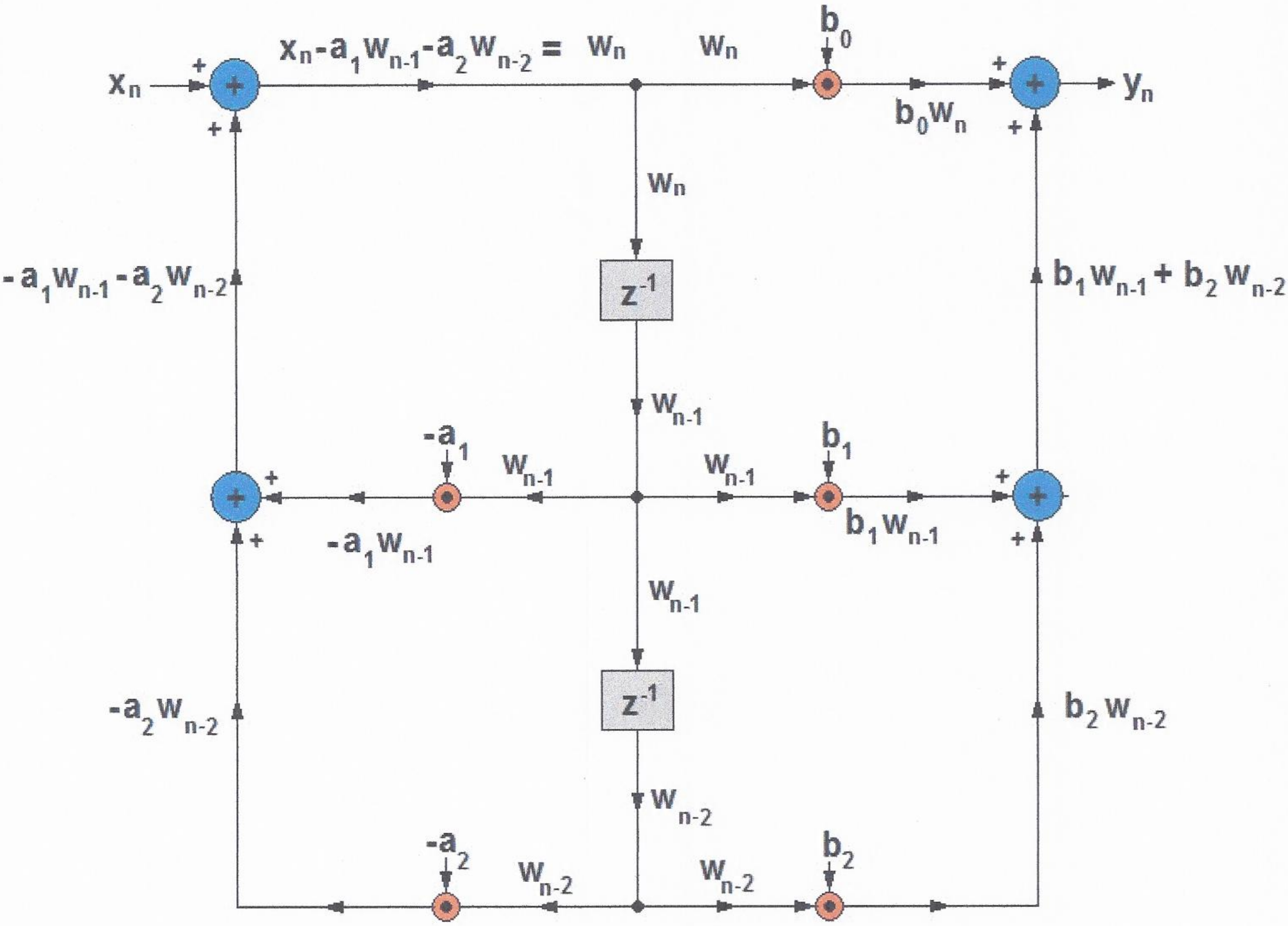


fig.(1.3.11)

DSP registers:

b_0
b_1
b_2
a_1
a_2
w_n
w_{n-1}
w_{n-2}
y_n
x_n

fig.(1.3.12)

Practical second order building block for IIR structure.
CANONICAL SECOND ORDER SECTION

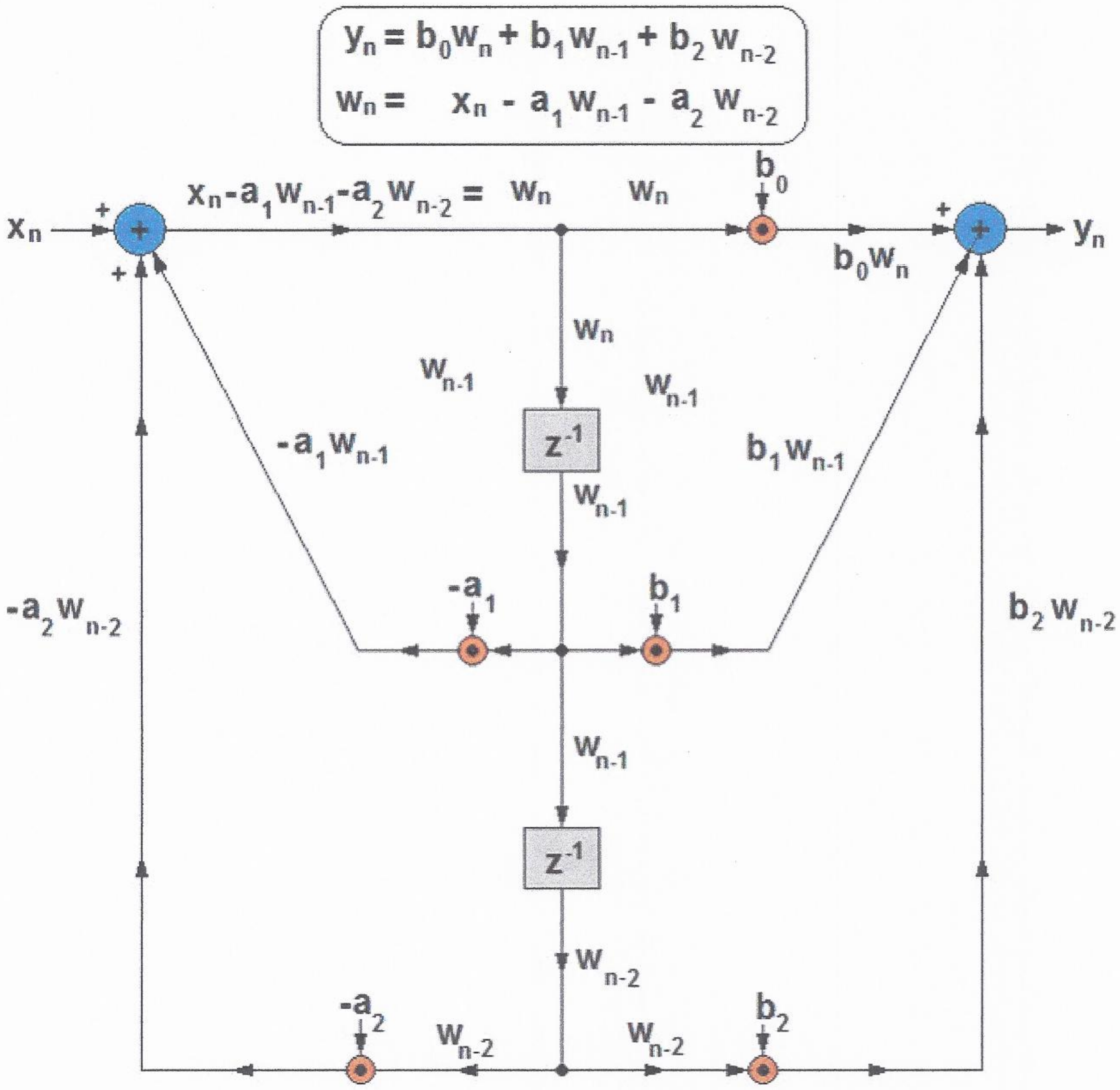
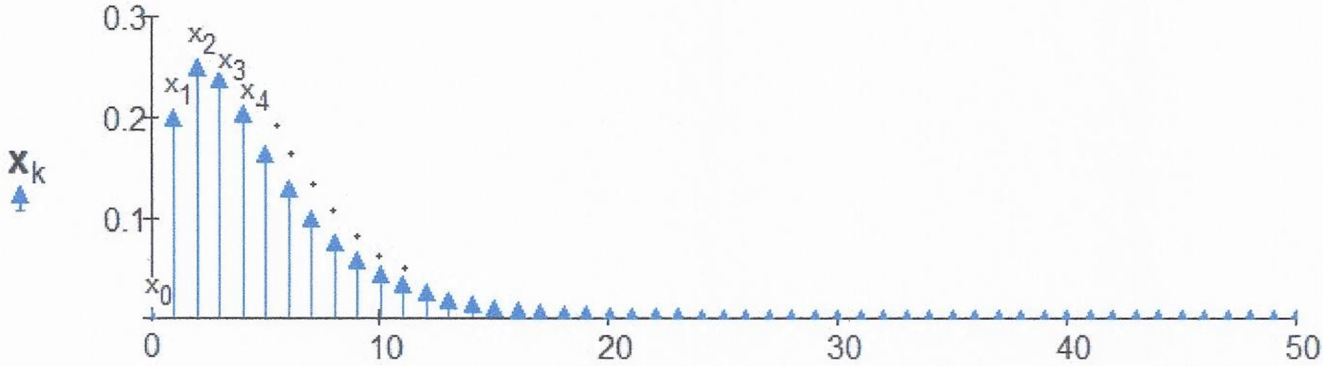


fig.(1.3.13)



$x_0\ \ x_1\ \ x_2\ x_3\ \ x_4\ \cdots$

↓ ↓ ↓ ↓ ↓

$w_0\ w_1\ w_2\ w_3\ w_4\ \cdots$

↓ ↓ ↓ ↓ ↓

$x_0\ x_1\ \neq 0\ \neq 0\ \neq 0$

$$w_2 = x_2 - a_1 w_1 - a_2 w_0$$

$$w_3 = x_3 - a_1 w_2 - a_2 w_1$$

$$w_4 = x_4 - a_1 w_3 - a_2 w_2$$

.
.
.

↓ ↓ ↓ ↓ ↓

$y_0\ \ y_1\ \ y_2\ y_3\ \ y_4\ \cdots$

i. c. $\neq 0\ \neq 0\ \neq 0$

$$y_2 = b_0 w_2 + b_1 w_1 + b_2 w_0$$

$$y_3 = b_0 w_3 + b_1 w_2 + b_2 w_1$$

$$y_4 = b_0 w_4 + b_1 w_3 + b_2 w_2$$

.
.
.

fig.(1.3.14)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$y_n = b_0 w_n + b_1 w_{n-1} + b_2 w_{n-2}$$
$$w_n = \quad - a_1 w_{n-1} - a_2 w_{n-2}$$
$$x_n = 0$$

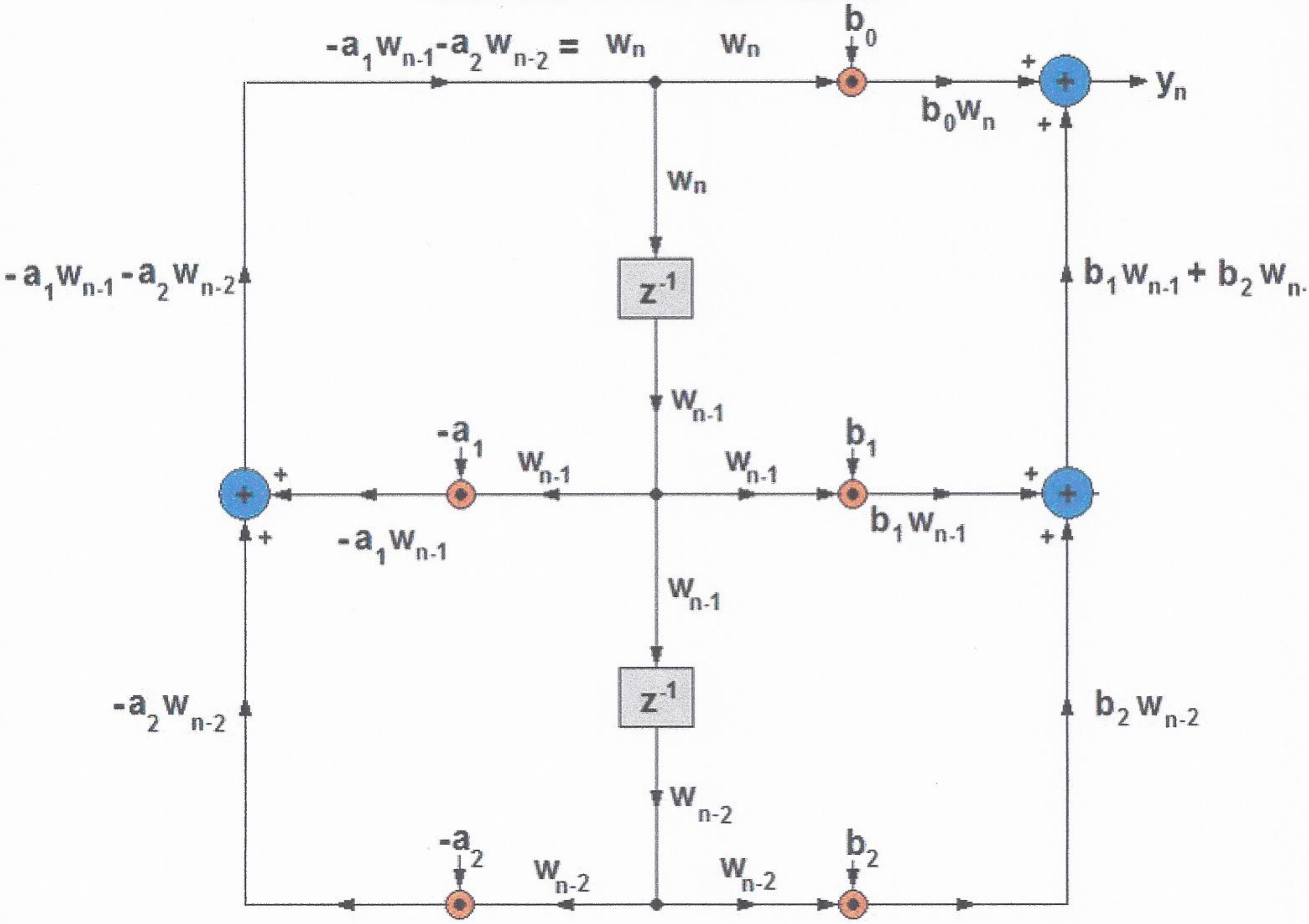


fig.(1.3.15)

Example $a_1 := 0.5$ $a_2 := 0.9$ $b_0 := 15$ $b_1 := -2.25$ $b_2 := 0$

$$H10(z) := \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} \qquad (1.3.16)$$

$n := 2 .. M1$ $k1 := 0 .. M1$ $x_n := 0$

$$x_{k1} := u(k1) \qquad (1.3.17)$$

Unitary pulse:

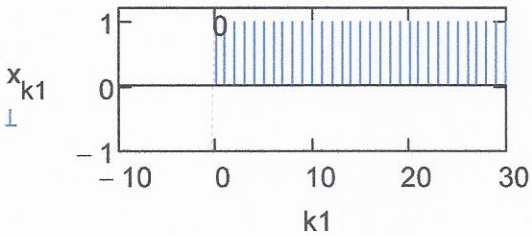


fig.(1.3.16)

$w_0 := x_0$ $w_1 := x_1$

$$w_n := x_n - a_1 \cdot w_{n-1} - a_2 \cdot w_{n-2} \qquad (1.3.18)$$

$$y1_n := b_0 \cdot w_n + b_1 \cdot w_{n-1} + b_2 \cdot w_{n-2} \qquad (1.3.19)$$

System Output

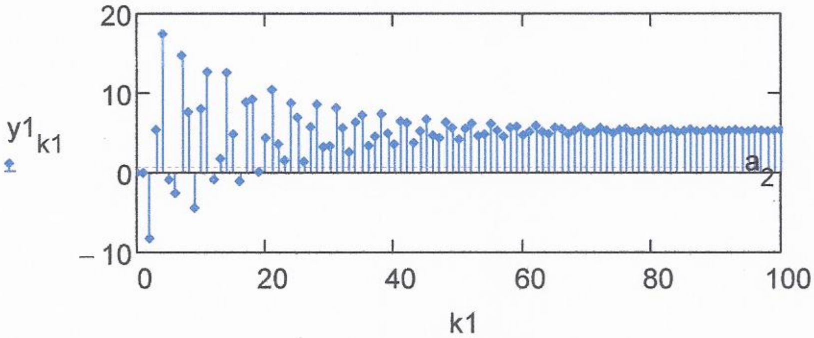


fig.(1.3.17)

Searching the poles of H(z):

$$v := \text{denom}(H10(z)) \text{ coeffs}, z \rightarrow \begin{pmatrix} 0 \\ 36.0 \\ 20.0 \\ 40.0 \end{pmatrix} \qquad (1.3.20)$$

$p := \text{polyroots}(v)$ $\text{rows}(p) = 3$

$$p^T = (-0.25 + 0.915j \quad -0.25 - 0.915j \quad 0) \qquad (1.3.21)$$

for the previous example we have: Check_stability(a) = "Stable"

$x := -3, -3 + 0.01 .. 3$

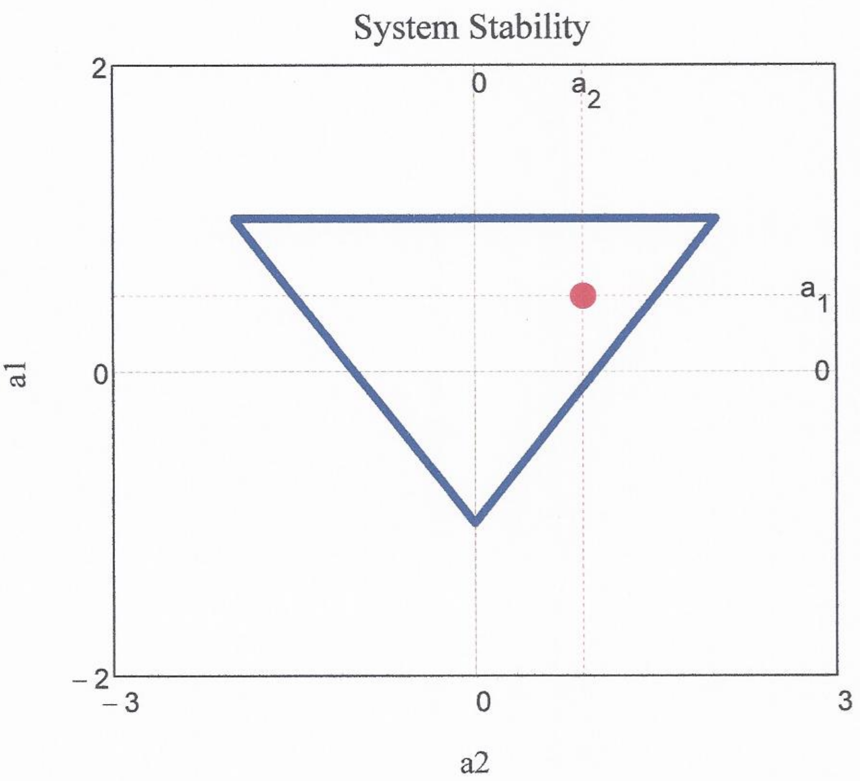


fig.(1.3.18)

$a_1 = 0.5 \qquad a_2 = 0.9$

Cascaded Block Diagram of a fourth order system.

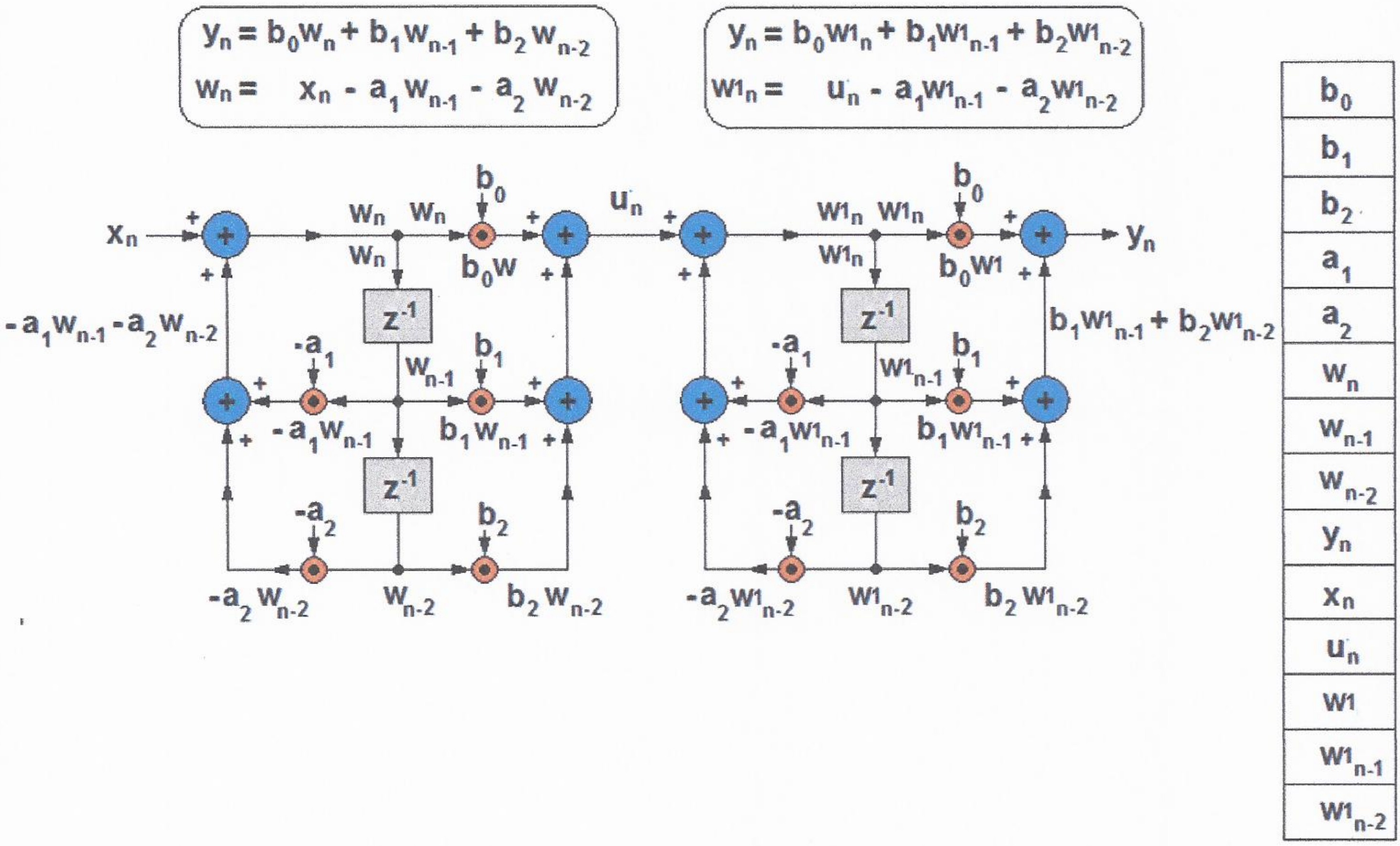
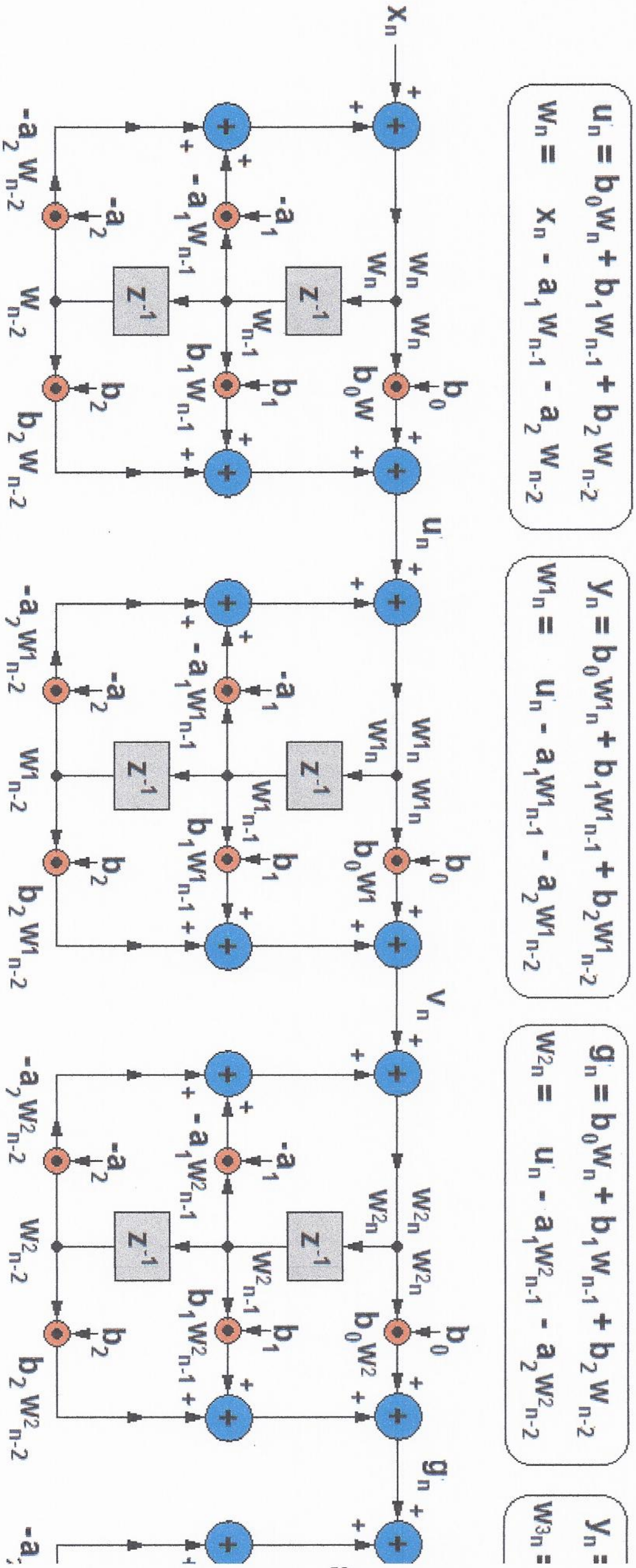


fig.(1.3.19)

Cascaded Block Diagram of a eighth order system.



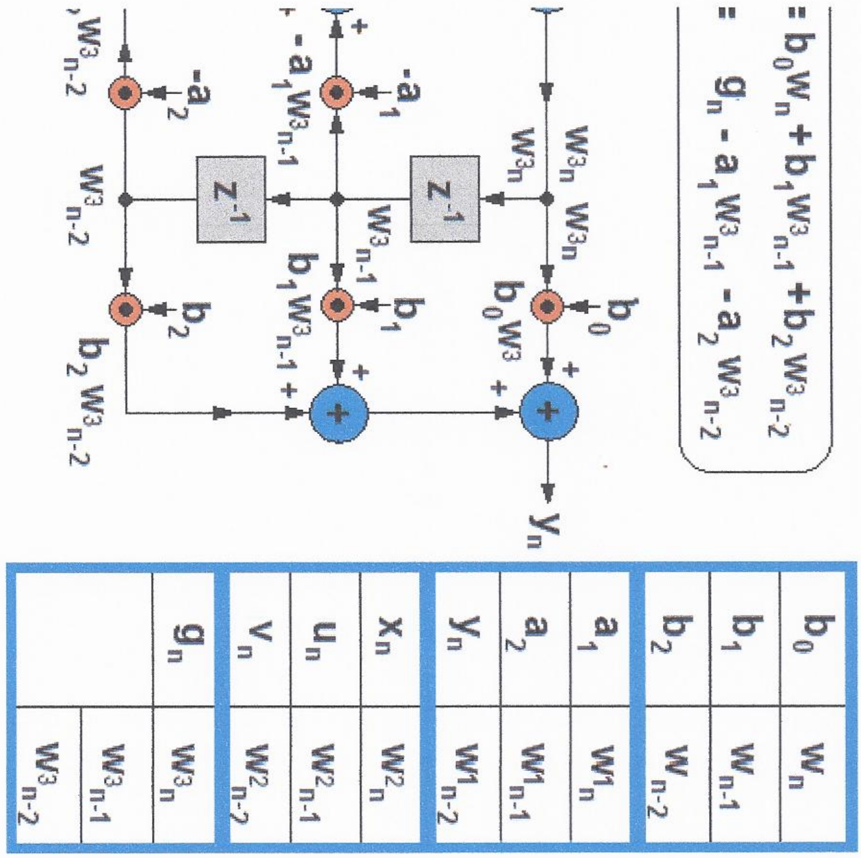


fig.(1.3.20)

Generally we can implement a digital filter of N° order using the cascade structure of second order subsystems. For this worth the following expression.

(for example: $N := 7$) ,

$$H(z) = B \cdot \prod_{k=1}^{L} \frac{1 + \beta_{1,k} \cdot z^{-1} + \beta_{2,k} \cdot z^{-2}}{1 - \alpha_{1,k} \cdot z^{-1} - \alpha_{2,k} \cdot z^{-2}}, \qquad (1.3.22)$$

where: $L := \text{floor}\left(\frac{N+1}{2}\right)$, $L = 4$ is the number of cascaded blocks.

(example: $l := 0..2$, $m := 0..L$, create the arrays $\beta_{l,m} := 0.0$, $\alpha_{l,m} := 0$. $B = \text{cost}$)

If N is odd the latter equation contains a first order term. The output signal is a function of the states $w_i(n-1)$, $w_i(n-2)$, i=1,...,L and of the input signal x(n), as follows:

$$y_n := w_{L,n} + \beta_{1,L} \cdot w_{L,n-1} + \beta_{2,L} \cdot w_{L,n-2} , \qquad (1.3.23)$$

$$w_{L,n} := w_{L-1,n} + \beta_{1,L-1} \cdot w_{L-1,n-1} + \beta_{2,L-1} \cdot w_{L-1,n-2} + \alpha_{1,L} \cdot w_{L,n-1} + \alpha_{2,L} \cdot w_{L,n-2} ,$$

.

.

.

$$w_{2,n} := w_{1,n} + \beta_{1,1} \cdot w_{1,n-1} + \beta_{2,1} \cdot w_{1,n-2} + \alpha_{1,2} \cdot w_{2,n-1} + \alpha_{2,2} \cdot w_{2,n-2} ,$$

$$w_{1,n} := B \cdot x_n + \alpha_{1,1} \cdot w_{1,n-1} + \alpha_{2,1} \cdot w_{1,n-2} .$$

Special cases.

Any sampled monochromatic signal switched on at n=0, of amplitude A, phase □$_1$, angular frequency ω_1 and sampling frequency f_s , at sampled time n=0 is given by:

$$x_n = A\cdot e^{j\cdot(\varphi_0\cdot n+\phi_1)},\ \text{for}\ n \ge 0, \qquad (1.3.24)$$

$$x_n = 0 \qquad \text{for}\ n<0.$$

normalized angular frequency: $\varphi_0 = \dfrac{\omega_1}{f_s}$, (1.3.25)

It follows that for two complex samples, namely:

$$x_0 = A\cdot e^{j\cdot\phi_1},$$

$$x_1 = A\cdot e^{j\cdot(\varphi_0+\phi_1)},$$

we can obtain:

normalized angular frequency $\varphi_0 = -j\cdot\ln\left(\dfrac{x_1}{x_0}\right)$,

signal amplitude $A_x = |x_0| = |x_1|$,

phase at sampled time n=0 $\phi_1 = -j\cdot\ln\left(\dfrac{x_0}{A_x}\right)$.

Example: $f := 1\cdot kHz$, $T := \dfrac{1}{f}$, $\omega_1 := 2\cdot\pi\cdot f$, $\phi_1 := \dfrac{\pi}{5}$, $A_x := 10$, $f_s := 12\cdot f$, $T_s := \dfrac{1}{f_s}$, $\varphi_0 := \dfrac{\omega_1}{f_s}$,

$$x(k) := A_x\cdot e^{j\cdot(\varphi_0\cdot k+\phi_1)}, \qquad (1.3.26)$$

$$\varphi_0 := -j\cdot\ln\left(\frac{x(1)}{x(0)}\right),\quad \varphi_0 = 0.524\ ,$$

'

$$\phi_1 := -j\cdot\ln\left(\frac{x(0)}{A_x}\right),\quad \phi_1 = 0.628\ ,$$

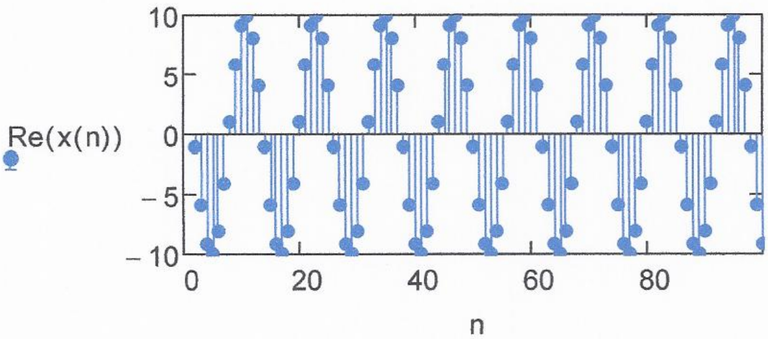


fig.(1.3.21)

$$z = e^{j\cdot\left(\omega_1\cdot\frac{1}{f_s}\right)}$$

$$H(z) = \frac{X(z)}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} = \frac{X(z)}{1 + \sum_{p=1}^{2} \left(a_p \cdot z^{-p}\right)} \qquad (1.3.27)$$

$$H(z) = \frac{X(z)}{1 + \sum_{p=1}^{2} \left[a_p \cdot e^{-j \cdot \left(\omega_1 \cdot \frac{p}{f_s}\right)}\right]} \qquad (1.3.28)$$

$$h(k) := \frac{1}{\left[1 + \sum_{p=1}^{2} \left[a_p \cdot e^{-j \cdot \left(\omega_1 \cdot \frac{p}{f_s}\right)}\right]\right]} \cdot \left[A_x \cdot e^{j \cdot \left(\omega_1 \cdot \frac{k}{f_s} + \phi_1\right)}\right] \qquad (1.3.29)$$

$$x(0) = 8.09 + 5.878j \qquad A_x \cdot e^{j \cdot \phi_1} = 8.09 + 5.878j \qquad (1.3.30)$$

$$x(1) = 4.067 + 9.135j \qquad x(0) \cdot e^{j \cdot \frac{\omega_1}{f_s}} = 4.067 + 9.135j \qquad (1.3.31)$$

$$h(k) := \frac{1}{\left[1 + \sum_{p=1}^{2} \left[a_p \cdot \left(\frac{x(1)}{x(0)}\right)^{-p}\right]\right]} \cdot x(0) \cdot \left(\frac{x(1)}{x(0)}\right)^{k} \qquad (1.3.32)$$

Thus, known the first two sample of the monochromatic input, we can determine the memory of the filter so that the filter is in its settled state.

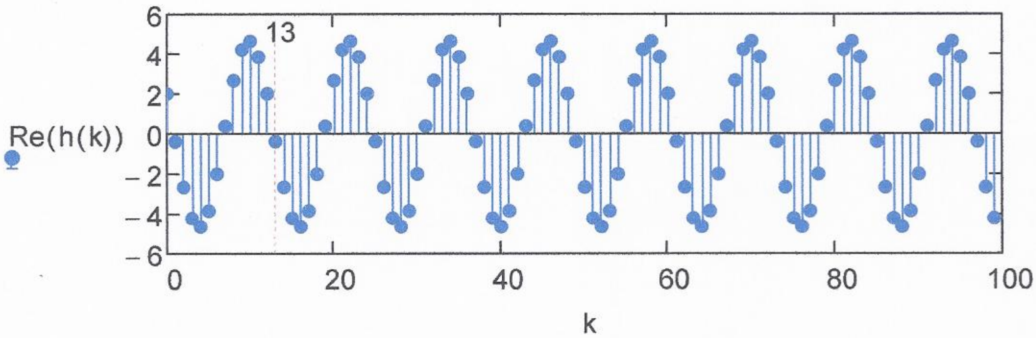


fig.(1.3.22)

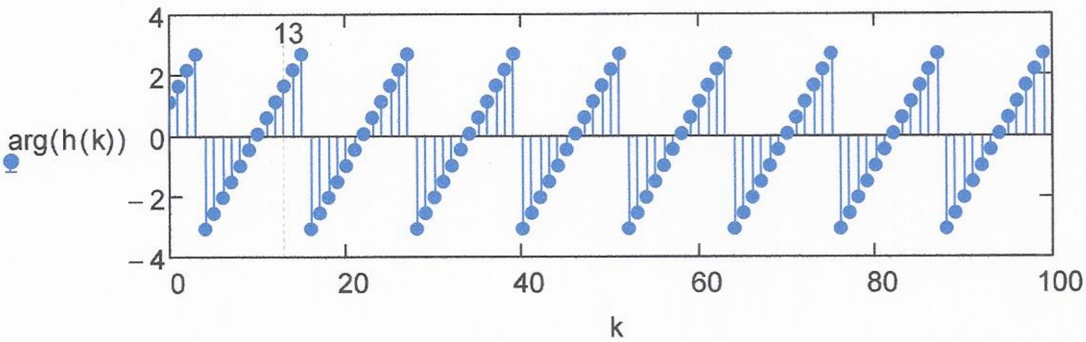


fig.(1.3.23)

For a monochromatic input signal and a cascaded structure worth the relation for state of the λ stage

$$\lambda := 2..L \ , \quad \nu := 0..-2 \ ,$$

$$H_{\nu,\lambda} := \frac{1+\beta_{1,(\lambda-1)}\cdot\left(\frac{x(1)}{x(0)}\right)^{-1}+\beta_{2,(\lambda-1)}\cdot\left(\frac{x(1)}{x(0)}\right)^{-2}}{1-\alpha_{1,\lambda}\cdot\left(\frac{x(1)}{x(0)}\right)^{-1}-\alpha_{2,\lambda}\cdot\left(\frac{x(1)}{x(0)}\right)^{-2}}\cdot W_{0,\lambda-1}\left(\frac{x(1)}{x(0)}\right)^{\nu} \ , \qquad (1.3.33)$$

$$H_{n,1} := \frac{B}{1-\alpha_{1,\lambda}\cdot\left(\frac{x(1)}{x(0)}\right)^{-1}-\alpha_{2,\lambda}\cdot\left(\frac{x(1)}{x(0)}\right)^{-2}}\cdot x(0)\cdot\left(\frac{x(1)}{x(0)}\right)^{n} \qquad (1.3.34)$$

Knowing H_1 we can pre initialize the memory of an IIR filter in the cascade structure to shorten the filter settling time.

§1.4) Consequences due the quantization error in IIR digital filters

In a DSP a finite number of bits is used to represent constants and variables and to perform arithmetic operations. Typical word lengths in modern DSP processors are 16 or 32 bit . The use of finite word lengtl produces errors which influences the performances of the DSP.

The causes of errors in DSP are:

1) ADC quantization errors,
2) Difference equation coefficients quantization errors,
3) Overflow errors due to operation results which are greater than word length,
4) Roundoff errors.

§1.5) PERIODIC SEQUENCES. FOURIER SERIES.

A periodic sequence with period N, is so defined:

$$x_p(n) = x_p(n + k\cdot N)\ , \quad k=0,1,2....N\ . \tag{1.5.1}$$

Its Fourier series is:

$$x_p(n) = \frac{1}{N}\cdot\sum_{k=0}^{N-1}\left(X_p(k)\cdot e^{j\cdot\frac{2\cdot\pi}{N}\cdot n\cdot k}\right). \tag{1.5.2}$$

$$X_p(k) = \sum_{n=0}^{N-1}\left(x_p(n)\cdot e^{-j\cdot\frac{2\cdot\pi}{N}\cdot n\cdot k}\right). \tag{1.5.3}$$

placing:

$$W_N = e^{-j\cdot\frac{2\cdot\pi}{N}}\ , \tag{1.5.4}$$

we can write

$$X_p(k) = \sum_{n=0}^{N-1}\left(x_p(n)\cdot W_N^{\,n\cdot k}\right), \tag{1.5.5}$$

$$x_p(n) = \frac{1}{N}\cdot\sum_{k=0}^{N-1}\left(X_p(k)\cdot W_N^{\,-n\cdot k}\right). \tag{1.5.6}$$

Indicating in a period: $x(n) = x_p(n) \quad \text{for} \quad 0 \le n \le N-1$

$x(n) = 0 \quad \text{otherwise,}$

we can deduce its z transform:

$$X(z) = \sum_{n=0}^{N-1}\left(x(n)\cdot z^{-n}\right)\ , \tag{1.5.7}$$

$$X_p(k) = X\left(W_N^{\,-k}\right). \tag{1.5.8}$$

Example:

$\nu := M1 - 1$ period: $N := 10$ Amplitude: $V_m := 1$

$$x1(n) := \frac{V_m}{2}\cdot\left[\sum_{k=0}^{\nu}\left[u(n - k\cdot N) - 2\cdot u\left[n - \left(\frac{2\cdot k + 1}{2}\cdot N\right)\right] + u[n - (k + 1)\cdot N]\right] + u(n)\right] \tag{1.5.9}$$

Unit Pulses Sequence.

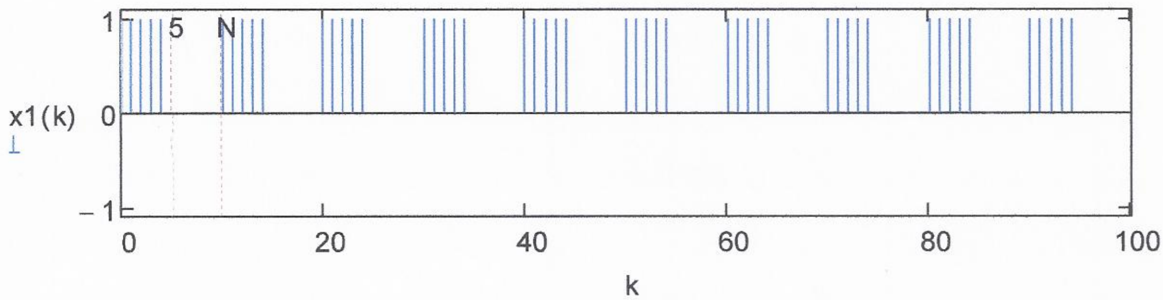


fig.(1.5.1)

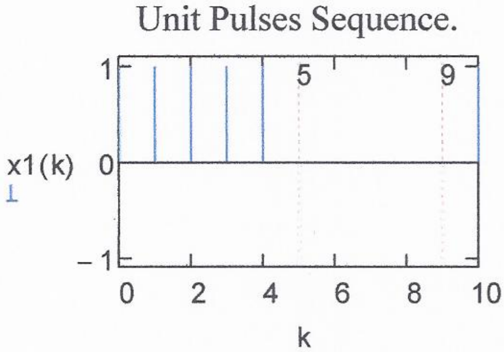


fig.(1.5.2)

$$X_p(k) = \sum_{n=0}^{4} W_{10}^{n \cdot k} = \sum_{n=0}^{4} e^{-j \cdot \frac{2 \cdot \pi}{10} \cdot n \cdot k} = e^{-j \cdot \frac{4 \cdot \pi}{10} \cdot k} \cdot \frac{\sin\left(k \cdot \frac{\pi}{2}\right)}{\sin\left(k \cdot \frac{\pi}{10}\right)} \qquad (1.5.10)$$

$$\lim_{k \to 30} \frac{\sin\left(k \cdot \frac{\pi}{2}\right)}{\sin\left(k \cdot \frac{\pi}{10}\right)} \to 5$$

Function Xp(k)

$|X_p(30)| = 5$ $\qquad |X_p(70)| = 5$

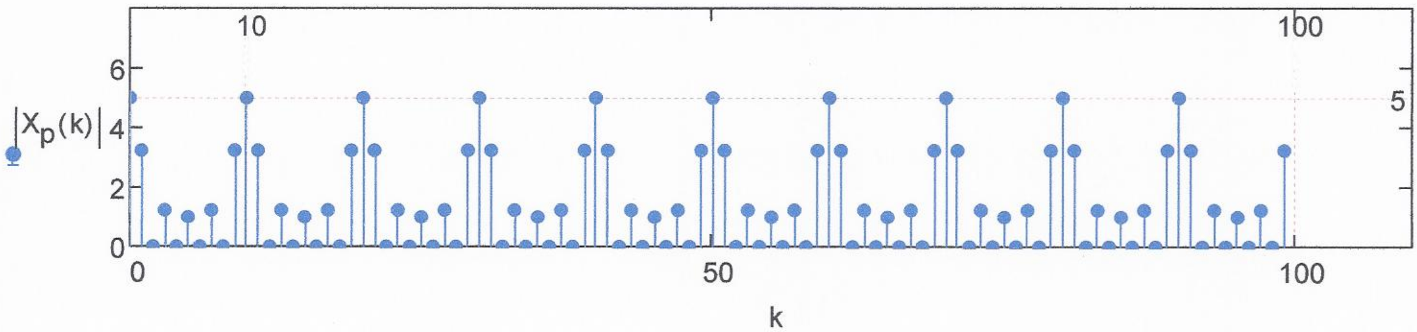


fig.(1.5.3)

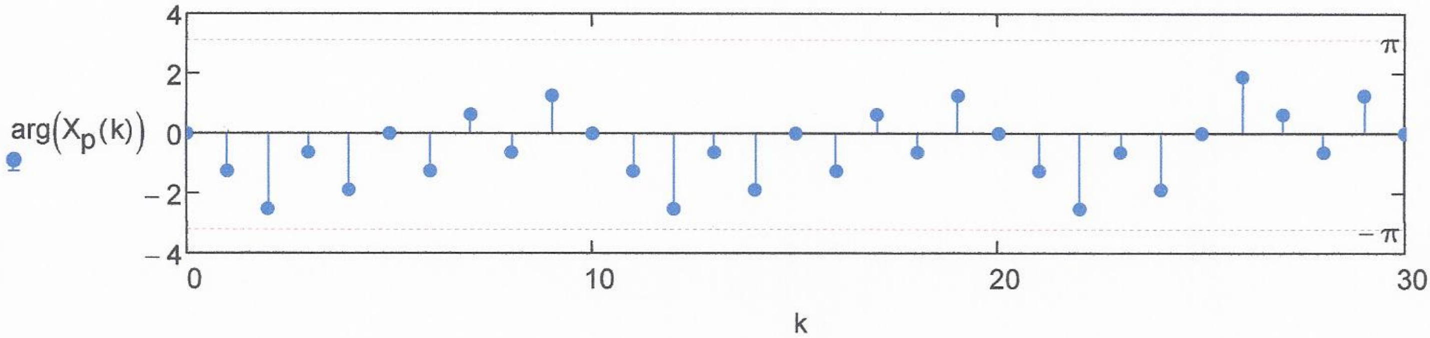


fig.(1.5.4)

$$X(z) = \sum_{n=0}^{N-1} \left(x_p(n) \cdot z^{-n} \right) \qquad (1.5.11)$$

$$X(\omega) := e^{-j \cdot 2 \cdot \omega} \cdot \frac{\sin\left(\frac{5}{2} \cdot \omega\right)}{\sin\left(\frac{\omega}{2}\right)} \quad \text{for} \quad z = e^{j \cdot \omega} \qquad (1.5.12)$$

Signal Spectrum

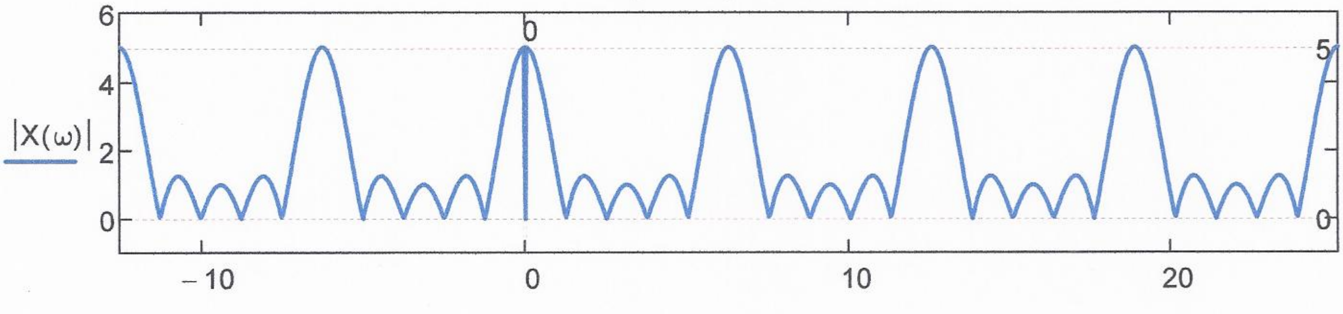


fig.(1.5.5)

Phase Spectrum

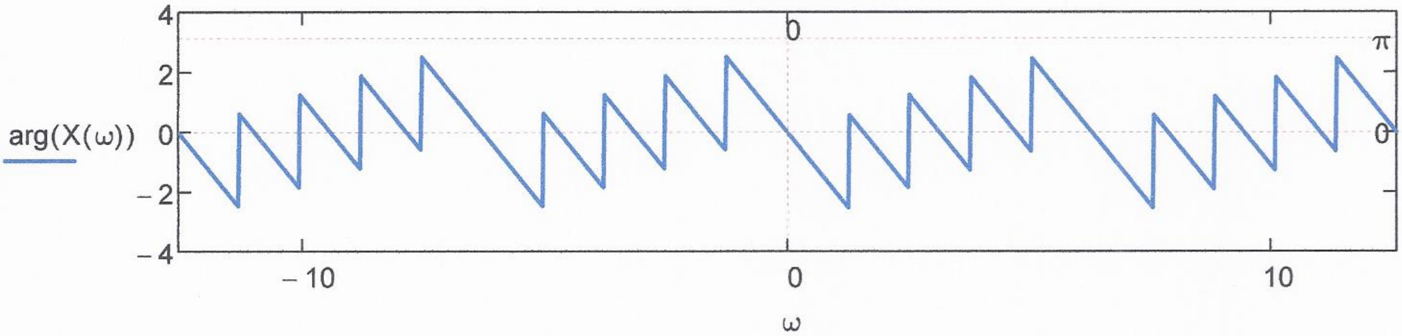


fig.(1.5.6)

FFT application: Frequency response estimation .

The frequency response of a discrete-time system is simply the Fourier transform of its impulse response Given the two vectors of the coefficients of the numerator (N) and denominator (M) a and b, we create th FFT of a and b namely A(k) and B(k). The ratio of the two FFTs gives the frequency response.

$$H\left(e^{j \cdot \omega_k \cdot T}\right) = \frac{A(k)}{B(k)}, \quad k = 0 .. \frac{N}{2} \qquad (1.5.13)$$

Example:

$$H(z) = \frac{z+1}{z-0.7071} = \frac{1+z^{-1}}{1-0.7071 \cdot z^{-1}} \qquad (1.5.14)$$

$k := 0 .. 2 \cdot M1$

$a_k := 0 \quad b_k := 0$

$a_0 := 1 \quad a_1 := -0.7071 \quad a_2 := 0. \qquad b_0 := 1.0 \quad b_1 := 1 \qquad b_2 := 0$

$$H(z) := \frac{b_0 + b_1 \cdot z^{-1}}{1 + a_1 \cdot z^{-1}} \qquad (1.5.15)$$

$a^T =$

1	-0.707	0	0	0	0	0	0	0	...

$b^T =$

1	1	0	0	0	0	0	0	0	0	0	0	0	0	...

$A_k := 0 \qquad B_k := 0$

$$A := \mathrm{CFFT}(a) \tag{1.5.16}$$

$$B := \mathrm{CFFT}(b)$$

$$h_k := \frac{B_k}{A_k}$$

$h^T =$

6.828	6.772-0.617j	6.609-1.204j	6.353-1.737j	6.026-2.199j	5.651-2.579j	...

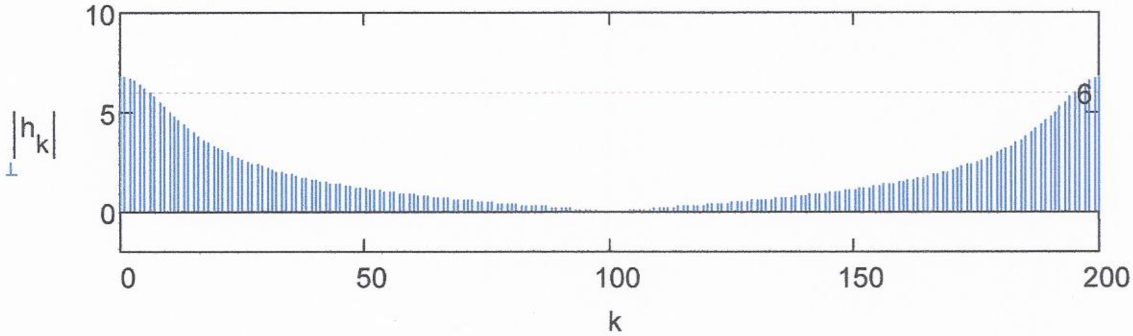


fig.(1.5.7)

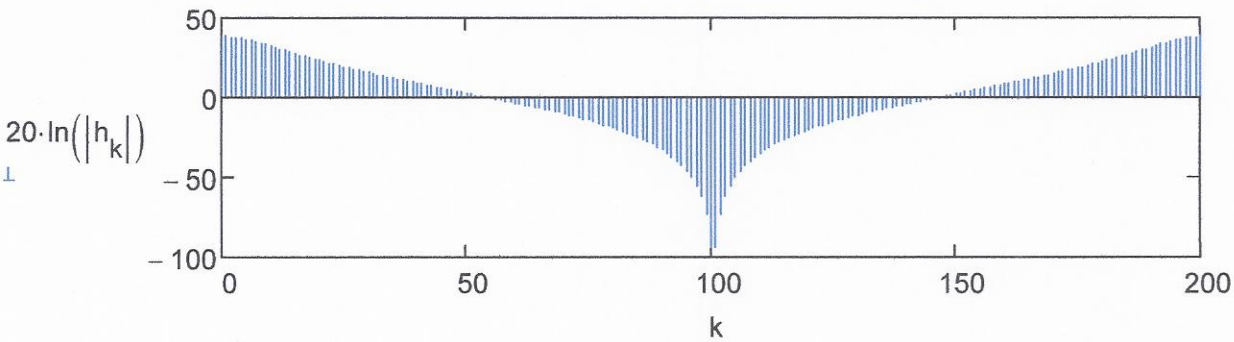


fig.(1.5.8)

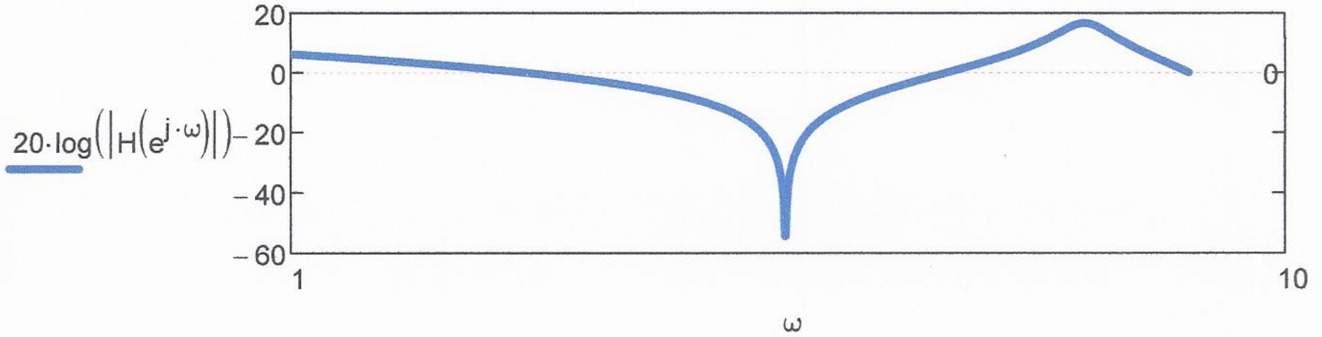


fig.(1.5.9)

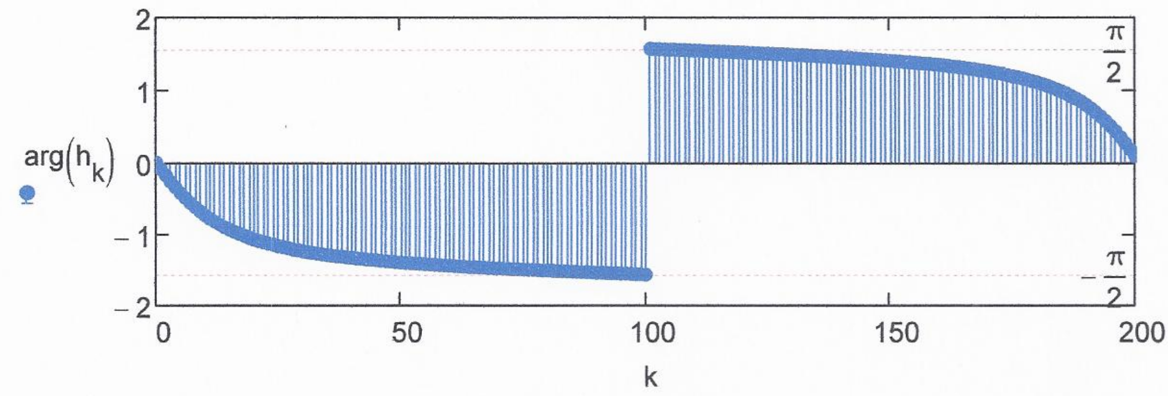


fig.(1.5.10)

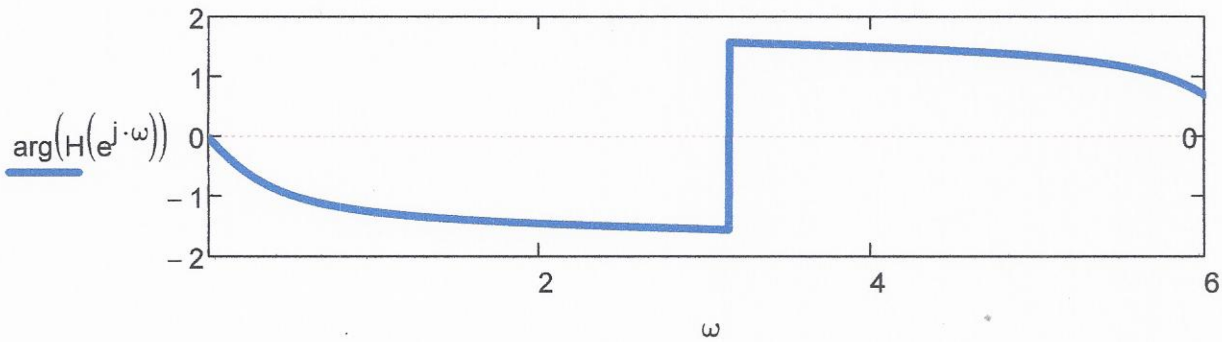


fig.(1.5.11)