4
Equivalent Digital High Pass Filter (I°order)

(All formulas and graphs)

Introduction.

This worksheet, developed using a common operational amplifier in an inverting configuration, begins with a brief summary of the main results of the circuit analysis, enriched with graphics and examples. Seven signals from an external file (Test Signals.xmcd), among the most common, are generated and used as input of the amplifier or, once sampled, as input to the digital filter. The many algorithms to implement the corresponding digital filter are derived applying the z-transform. Two approximations are applied to each result to derive the corresponding difference equations. Thus one will see that, as the analog filter is effective, just is the digital one with the used approximations. After reviewing this worksheet, the reader just has to choose the algorithm and implement the firmwere for the DSP.

Content:

Definitions and necessary constants.
Refer to the file "Test Signals.xmcd"

To test the effectiveness of the filter, several test signals are created and defined in "Test Signals.xmcd" and are used in the whole worksheet.

Defined in "Test Signals.xmcd" $T_0 = 5 \cdot \mu s$

Defined in "Test Signals.xmcd" $\tau = 0.796 \cdot \mu s$

Defined in "Test Signals.xmcd" Cutoff frequency of the filte $f_0 = 200 \cdot kHz$ $\omega_0 = 1.257 \cdot \frac{Mrads}{sec}$

Riferimento:C:\Nuova cartella\Test Signals.xmcd(R)

Chosen test signal period, $T_{test} := 2 \cdot T1_{test}$ ie a multiple of the analog filter time constant defined in "Test Signals.xmcd" ($T1_{test} = 2 \cdot \pi \cdot \tau$).

$$T_{test} = 40 \cdot \mu s$$

Then the signal frequency $f_{test} := \frac{1}{T_{test}}$, $f_{test} = 25 \cdot kHz$: $\omega_0 = 1.257 \cdot \frac{Mrads}{sec}$. . $\omega_{test} := 2 \cdot \pi \cdot f_{test}$,

$\omega_{test} = 0.157 \cdot \frac{Mrads}{sec}$ is lesser than the cutoff frequency of the filter , $\frac{\omega_0}{\omega_{test}} = 8$,

As a result, the signal at the filter output should be strongly attenuated.

Amplifier Gain: $A_0 \equiv -1.5$

sampling frequency: $f_{smp} := 20 \cdot f_0$, $f_{smp} = 4 \cdot MHz$,

sampling period: $T_{smp} := \frac{1}{f_{smp}}$, $T_{smp} = 0.25 \cdot \mu s$,

generic pulse delay time: $\tau_0 := 4 \cdot T_{smp}$

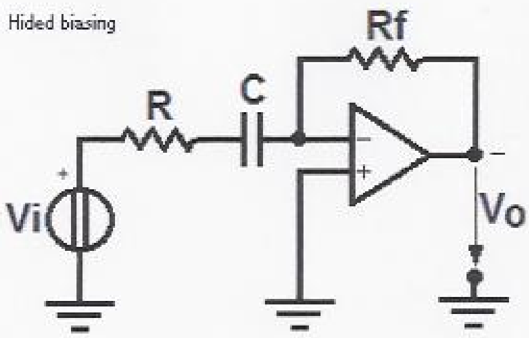
sampling angular frequency: $\omega_{smp} := 2 \cdot \pi \cdot f_{smp}$ · $\omega_{smp} = 25.133 \cdot \frac{Mrads}{sec}$,

sampling time step: $n_k := \frac{k}{f_{smp}}$,

The Bode diagrams will have an extension defined by a multiple $U \equiv 100$, of ω_s, freely chosen.

4.1
Analog HIGH PASS Filter (I° order):

Consider the simple analog high pass active filter (derivator) below depicted:



Hereafter are reported the main results of the analysis: the transfer function, the graph of the impulse response, the Bode plots.

The transfer function (ideal Op. Amp.) is: $W_{hp}(s) = \dfrac{-Rf}{R + \dfrac{1}{s \cdot C}} = -\dfrac{Rf}{R} \cdot \dfrac{s}{s + \dfrac{1}{R \cdot C}}$.

Placing:

$$\omega_0 = 2 \cdot \zeta \cdot Q = \frac{1}{R \cdot C} = \frac{1}{\tau} \text{ and } A_0 = -\frac{R_f}{R}.$$

$$W_{hp}(s) = A_0 \cdot \frac{s}{s + \omega_0}$$

The angular frequency by which the voltage gain is 0dB, is:

$$\omega_{0dB} = \frac{1}{C \cdot \sqrt{Rf^2 - R^2}} = \frac{\omega_0}{\sqrt{(A_0)^2 - 1}},$$

so:

$$\omega_{0dB} := \frac{\omega_0}{\sqrt{(A_0)^2 - 1}}$$

Finally, the transfer function can be written

$$W_{hp}(s) := \frac{A_0 \cdot s}{s + \omega_0}, \quad A_0 = -1.5 ,$$

or

$$W_{hp}(s) = \frac{A_0 \cdot s}{\omega_0 \cdot \left(\dfrac{s}{\omega_0} + 1\right)}.$$

From the definition of the transfer function we derive that:

$$V_o(s) = W_{hp}(s)\cdot V_i(s),$$

To this product corresponds, antitransforming, a convolution product in the time domain. So that the exact system output is:

$$: \qquad v_o(t) = A_0\cdot\int_0^t v_i(\sigma)\cdot\left(\Delta(t-\sigma) - \frac{e^{-\frac{t-\sigma}{\tau}}}{\tau}\right) d\sigma = A_0\cdot v_i(t) - \frac{A_0}{\tau}\cdot\int_0^t v_i(\sigma)\cdot e^{-\frac{t-\sigma}{\tau}}\, d\sigma,$$

Hence the time domain filter's output is:

$$\boxed{v_o(t) = A_0\cdot\left(v_i(t) - e^{-\frac{t}{\tau}}\cdot\int_0^{\frac{t}{\tau}} v_i(\xi\cdot\tau)\cdot e^{\xi}\, d\xi\right)}\ .$$

Example: $v_i(t) = \Delta(t)$,

$$v_o(t) = A_0\cdot\left(\Delta(t) - \frac{e^{-\frac{t}{\tau}}}{\tau}\cdot\int_0^t \Delta(\sigma)\cdot e^{\frac{\sigma}{\tau}}\, d\sigma\right) = A_0\cdot\left(\Delta(t) - \frac{1}{\tau}\cdot e^{-\frac{t}{\tau}}\cdot\Phi(t)\right)$$

Approximation.

Given the transfer function:

$$W_{hp}(s) = \frac{A_0\cdot s}{\omega_0\cdot\left(\frac{s}{\omega_0}+1\right)},$$

if $\left(\left|\frac{s}{\omega_0}\right|\right) \ll 1$, namely for time armonic signals, $s = j\cdot\omega$, $\omega \ll \omega_0$, or $T \gg (2\cdot\pi\cdot\tau)$, the fraction can be developed in a McLaurin series after having placed $x = \frac{s}{\omega_0}$, so that :

$\left(\frac{1}{1+x}\right) (\approx) \left(1 - x + x^2 - x^3 - x^5\right)$ and, in first approximation:

$$W_{hp}(s) = \frac{A_0\cdot s}{\omega_0}, \qquad \frac{A_0}{\omega_0} = -\frac{Rf}{R}\cdot\tau = A_0\cdot\tau = -Rf\cdot C = -\tau_f$$

$$\tau_f := -\frac{A_0}{\omega_0}$$

from which the input-output bond is:

$$V_o(s) = -\tau_f\cdot s\cdot V_i(s),$$

where $V_i(s)$ is the L. T. of the input signal.

Antitransforming and considering zero initial conditions, the approximated temporal trend of the output signal is:

$$v_o(t) = -\tau_f\cdot\frac{\partial}{\partial t}v_i(t) \qquad \text{for } \omega \ll \omega_0 \text{ or } T \gg (2\cdot\pi\cdot\tau).$$

Impulse response:

Defined in "Test Signals.xmcd" $\varepsilon_t = 0.1 \cdot ns$

$$t_\delta := -10\cdot\varepsilon_t, -10\cdot\varepsilon_t + \frac{\varepsilon_t}{100} .. 2000\cdot\varepsilon_t$$

$$A_0 = -1.5 \qquad A_0 := A_0 \qquad s := s \quad a := a \qquad \omega_0 := \omega_0$$

Dirac pulse response: $w_{hp}(t) := \frac{A_0 \cdot s}{s + \omega_0}$ invlaplace, s, t $\rightarrow A_0 \cdot \left(\Delta(t) - \omega_0 \cdot e^{-t\cdot\omega_0}\right)$

$$w_{hp}(t) := A_0 \cdot \left(\Delta_\varepsilon(t, \varepsilon_t) - \omega_0 \cdot e^{-\omega_0 \cdot t}\right)$$

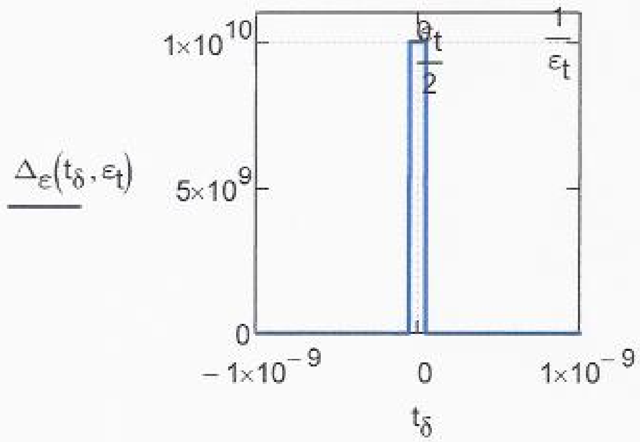
Graph of the impulse response. (Consider a negative Dirac pulse of area $|A_0|$ at the origin).

Geometric tangent to the curve at the point (0, w(0)):

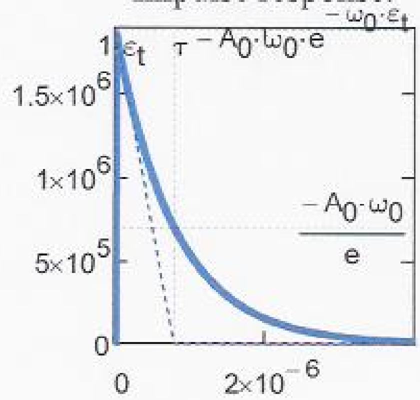
$$f_a(t) := A_0 \cdot \omega_0 \cdot (\omega_0 \cdot t - 1) \cdot (\Phi(t) - \Phi(t - \tau))$$

$$t := 0\cdot\tau, \frac{20\cdot\tau}{1000} .. 20\cdot\tau \qquad A_0 = -1.5$$

Finite Pulse



Impulse response.



Bode plots (HIGH PASS (I° order)):

For time armonic signals the complex variable is purely immaginary : $s = j\cdot\omega$

$$\omega_0 = 1.257\cdot\frac{\text{Mrads}}{\text{sec}}$$

Transfer function: $$\frac{A_0\cdot j\cdot\omega}{j\cdot\omega+\omega_0} = \frac{|A_0|\cdot\omega}{\sqrt{\omega^2+\omega_0^2}}\cdot e^{j\cdot\left(-\frac{\pi}{2}-\text{atan}\left(\frac{\omega}{\omega_0}\right)\right)}$$

Magnitude in dB $$W_{hpdB}(\omega) := 20\cdot\log\left(|A_0|\cdot\omega\cdot\text{sec}\right) - 20\cdot\log\left(\sqrt{\omega^2+\omega_0^2}\cdot\text{sec}\right)$$

Phase: $$\varphi_{hp}(\omega) := -\frac{\pi}{2} - \text{atan}\left(\frac{\omega}{\omega_0}\right)$$

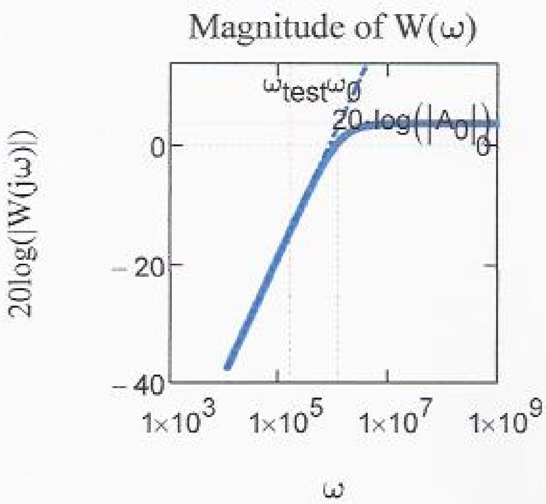
$$\lim_{\omega\to\infty} W_{hpdB}(\omega) = 20\cdot\log(A_0)$$

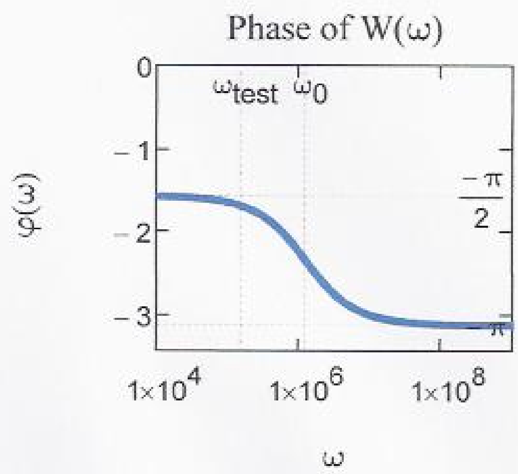
$$\lim_{\omega\to 0} W_{hpdB}(\omega) = 20\cdot\log\left(|A_0|\cdot\omega\right) - 20\cdot\log(\omega_0)$$

Asymptote: $$Asy_{dB}(\omega) := 20\cdot\log\left(|A_0|\cdot\omega\cdot\text{sec}\right) - 20\cdot\log\left(\omega_0\cdot\text{sec}\right)$$

$$\omega := \frac{\omega_{0dB}}{U}, \frac{\omega_{0dB}}{U} + \frac{\omega_0\cdot U - \frac{\omega_{0dB}}{U}}{U^2} .. 10\cdot U\cdot\omega_0$$

$\omega_0 = 1.257\cdot\frac{\text{Mrads}}{\text{sec}}$ $\omega_{0dB} = 1.124\cdot\frac{\text{Mrads}}{\text{sec}}$ $\frac{\omega_{0dB}}{2\cdot\pi} = 178.885\cdot\text{kHz}$





$\omega_0 = 1.257\cdot\frac{\text{Mrads}}{\text{sec}}$. . . , $\omega_{test} = 157.08\cdot\frac{\text{krads}}{\text{sec}}$, $\frac{\omega_0}{\omega_{test}} = 8$

Attenuation:

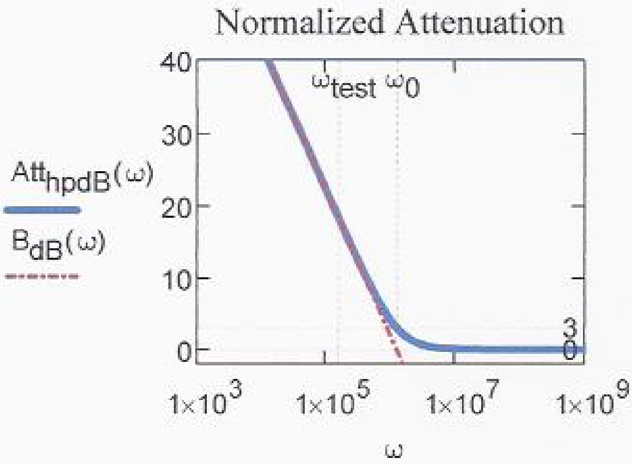
$$Att_{hpdB}(\omega) := -20\cdot\log(\omega\cdot sec) + 20\cdot\log\left(\sqrt{\omega^2+\omega_0^{\,2}}\cdot sec\right)$$

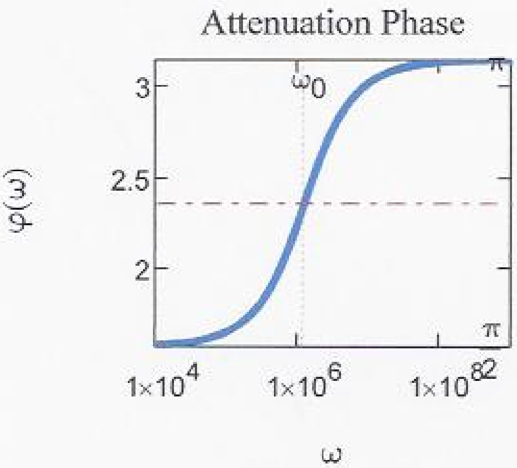
$$\varphi_{atthp}(\omega) := \frac{\pi}{2} + atan\left(\frac{\omega}{\omega_0}\right)$$

$$\lim_{\omega\to\infty} Att_{hpdB}(\omega) = 0$$

$$\lim_{\omega\to 0} Att_{hpdB}(\omega) = -20\cdot\log(\omega) + 20\cdot\log(\omega_0)$$

$$B_{dB}(\omega) := -20\cdot\log(\omega\cdot sec) + 20\cdot\log(\omega_0\cdot sec)$$





Attenuation:

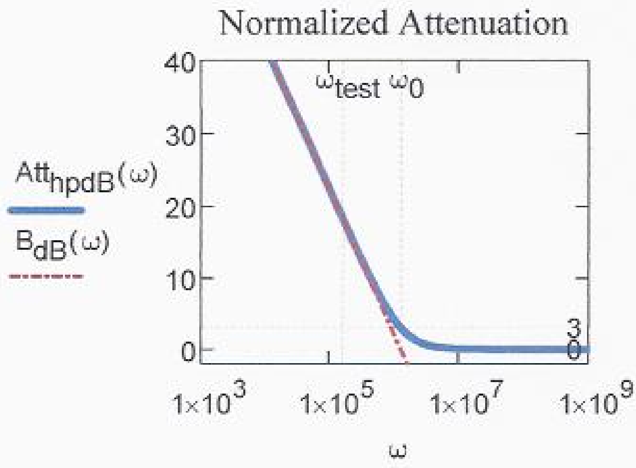
$$Att_{hpdB}(\omega) := -20 \cdot \log(\omega \cdot sec) + 20 \cdot \log\left(\sqrt{\omega^2 + \omega_0^2} \cdot sec\right)$$

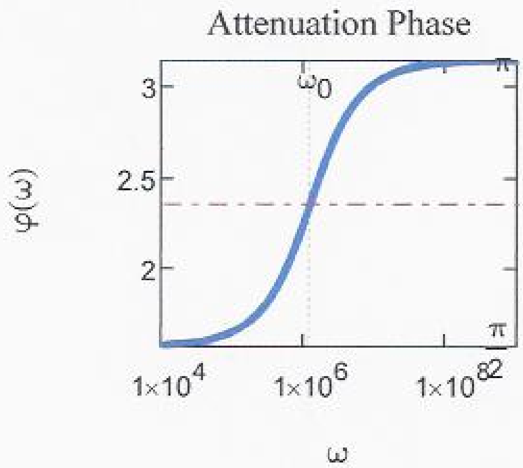
$$\varphi_{atthp}(\omega) := \frac{\pi}{2} + atan\left(\frac{\omega}{\omega_0}\right)$$

$$\lim_{\omega \to \infty} Att_{hpdB}(\omega) = 0$$

$$\lim_{\omega \to 0} Att_{hpdB}(\omega) = -20 \cdot \log(\omega) + 20 \cdot \log(\omega_0)$$

$$B_{dB}(\omega) := -20 \cdot \log(\omega \cdot sec) + 20 \cdot \log(\omega_0 \cdot sec)$$





4.2
ANALOG FILTER OUTPUT ANALYSIS

Chosen period of the test signal, $T_{test} = 40 \cdot \mu s$. At the corresponding frequency, the voltage gain of the filter is $20 \cdot \log\left(\left|W_{hp}(j \cdot \omega_{test})\right|\right) = -14.607 \cdot dB$.

ANALOG FILTER OUTPUT ANALYSIS

4.2.1) The ramp response: Ramp slope: $V_i = 5 \times 10^3 \cdot mV$

Ramp response: $$V_o(s) = \frac{A_0 \cdot s}{s + \omega_0} \cdot \frac{V_i}{s^2},$$

$A_0 := A_0,$

ramp response: $$Y_{rr1}(t) := \frac{A_0 \cdot s}{s + \omega_0} \cdot \frac{V_i}{s^2} \text{ invlaplace}, s, t \rightarrow -\frac{A_0 \cdot V_i \cdot \left(e^{-t \cdot \omega_0} - 1\right)}{\omega_0}$$

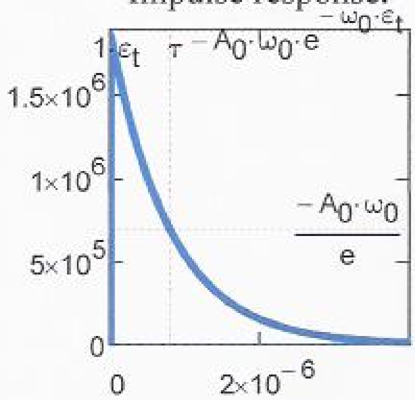
$$Y_{rr}(t) := -\frac{A_0 \cdot V_i \cdot \left(e^{-t \cdot \omega_0} - 1\right)}{\omega_0} \cdot \Phi(t)$$

$$y_{as}(t) := \begin{cases} A_0 \cdot V_i \cdot t & \text{if } t \le \tau \\ (\text{break}) & \text{otherwise} \end{cases}$$

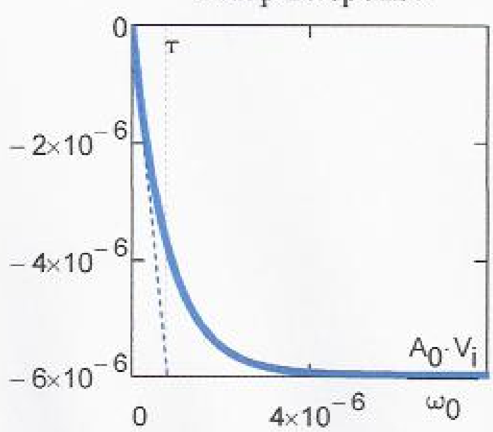
$$\frac{A_0 \cdot V_i}{\omega_0} = -5.968 \times 10^{-6} \cdot \text{volt} \cdot \text{sec} \qquad t := 0 \cdot \tau, \frac{20 \cdot \tau}{1000} .. 20 \cdot \tau$$

$$\frac{A_0 \cdot V_i}{\omega_0} = -\tau_f \cdot V_i$$

Impulse response.



Ramp Response



4.2.2) Window function response: $V_w(t) := v_w(t, \tau_0, T_{smp}, \delta1)$

$$v2_o(t) := A_0 \cdot \left(V_w(t) - e^{-\frac{t}{\tau}} \cdot \int_0^{\frac{t}{\tau}} V_w(\xi \cdot \tau) \cdot e^{\xi} \, d\xi \right)$$

calculation result:

$$v2_{oc}(t) := A_0 \cdot \left[V_i \cdot \left(\Phi(t - \tau_0) - \Phi(t - \delta1 \cdot T_{smp} - \tau_0) \right) - e^{-\frac{t}{\tau}} \cdot \left[V_i \cdot e^{\frac{\tau_0}{\tau}} \cdot \left(e^{\frac{T_{smp} \cdot \delta1}{\tau}} - 1 \right) \right] \right]$$

Approximated output:

$$v2_{oapp}(t) = -\tau_f \cdot \frac{\partial}{\partial t} v_i(t) = -\tau_f \cdot V_i \cdot \frac{\partial}{\partial t} \left(\Phi(t - \tau_0) - \Phi(t - \delta1 \cdot T_{smp} - \tau_0) \right)$$

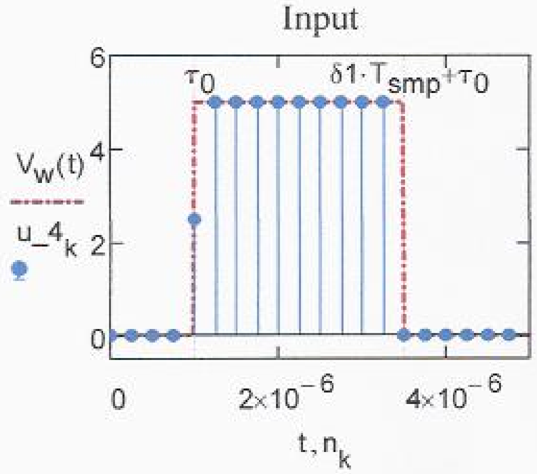
$$v2_{oapp}(t) := \left(\Delta_\varepsilon(t - \tau_0 - T_{smp} \cdot \delta1, \varepsilon_t) - \Delta_\varepsilon(t - \tau_0, \varepsilon_t) \right) \cdot \tau_f \cdot V_i$$

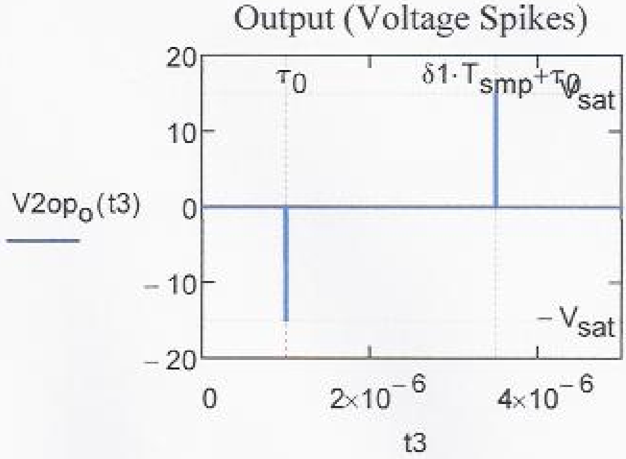
$u\_4_k := V_w(n_k)$ $\qquad \tau_f \cdot V_i = 5.968 \times 10^3 \cdot \text{volt} \cdot \text{ns}$

Graph of the window response considering the OpAmp saturation:

$$V2op_o(t) := \text{if}\left(-V_{sat} \le v2_{oapp}(t) \le V_{sat}, v2_{oapp}(t), \text{if}\left(v2_{oapp}(t) \le V_{sat}, -V_{sat}, V_{sat}\right)\right)$$

$A_0 = -1.5 \quad T_{test} = 0.04 \cdot \text{ms} \qquad t3 := 0, \frac{\tau_0 + 2 \cdot (\delta1 \cdot T_{smp})}{15000} .. \tau_0 + 2 \cdot (\delta1 \cdot T_{smp})$





$\delta1 \cdot T_{smp} = 2.5 \cdot \mu s$ <u>(Always Avoid spikes!)</u> $4 \cdot \delta1 \cdot T_{smp} = 10 \cdot \mu s$

ANALOG FILTER OUTPUT ANALYSIS

$\omega_{test} \ll \omega_0 \qquad \frac{\omega_0}{\omega_{test}} = 8$

4.2.3) Triangular wave respons: $\quad T_{test} = 40 \cdot \mu s \qquad \tau = 0.796 \cdot \mu s$

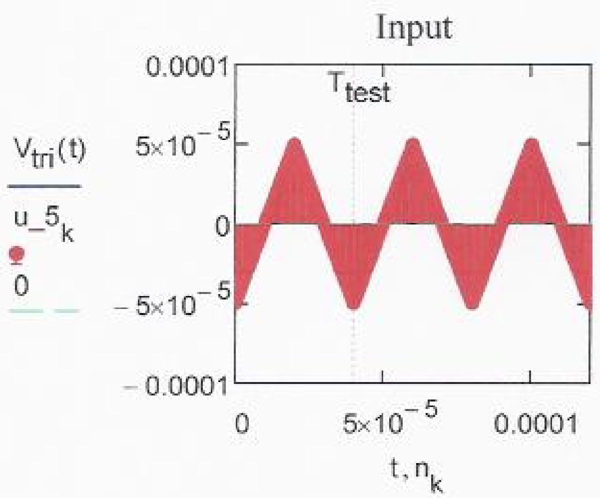
$V_{tri}(t) := v_{tri0}(t, T_{test}, V_i) \qquad N = 50$

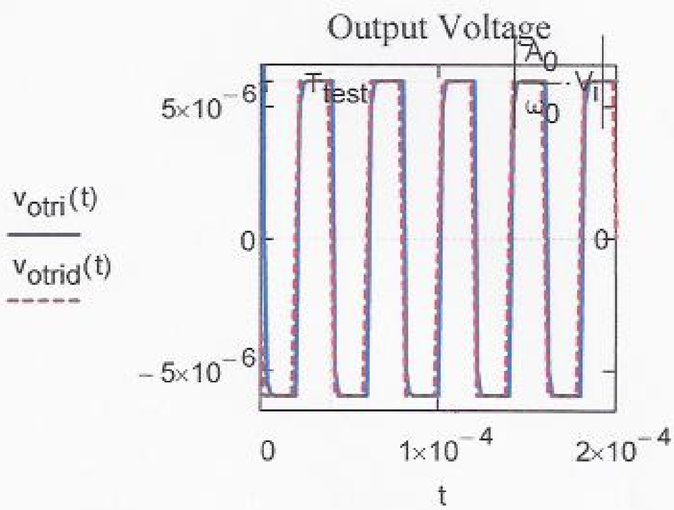
Exact output: $\quad v_{otri}(t) := A_0 \cdot \left(V_{tri}(t) - e^{-\frac{t}{\tau}} \cdot \int_0^{\frac{t}{\tau}} V_{tri}(\xi \cdot \tau) \cdot e^{\xi} \, d\xi \right) \qquad V_i = 5V$

Approximated output: $\quad v_{otrid}(t) := \frac{A_0}{\omega_0} \cdot \frac{\partial}{\partial t} V_{tri}(t)$ holds for $\omega \ll \omega_0$ or $T \gg (2 \cdot \pi \cdot \tau)$

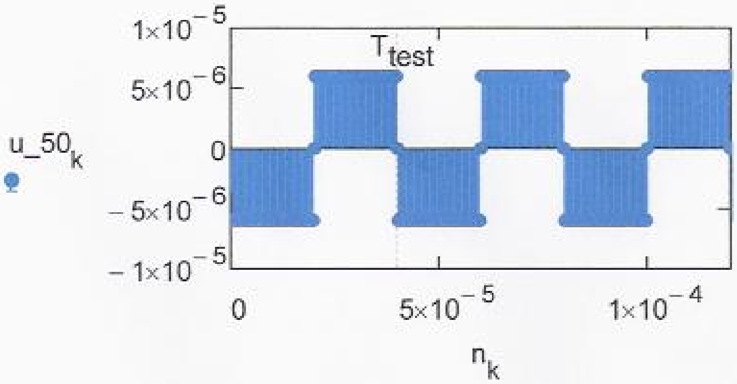
Input sampling: $u\_5_k := V_{tri}(n_k) \qquad A1 := 1.1 \cdot \left| \frac{A_0}{\omega_0} \cdot V_i \right|$

$T_{test}^{-1} = 25 \cdot kHz \qquad t := 0 \cdot T_{test}, 0 \cdot T_{test} + \frac{5 \cdot T_{test}}{100} .. 5 \cdot T_{test} \qquad f_{smp} = 4 \times 10^3 \cdot kHz$

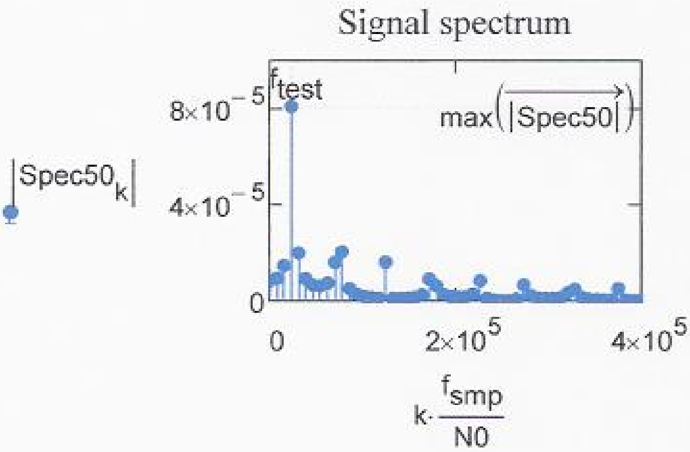


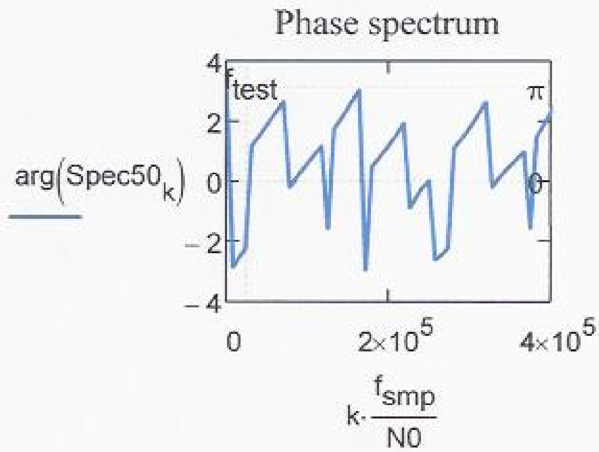


Output sampling: $u\_50_k := v_{otrid}(n_k)$



Spec50 := fft(u_50)





4.2.4) Sawtooth wave response.

$\delta t := 3 \cdot T_{test}$ $\quad v_{sw}(t) := v1_{sw}(t, \delta t, p)$ $\quad \delta t \cdot p \cdot V_i = 90.681 \cdot volt \cdot \mu s$

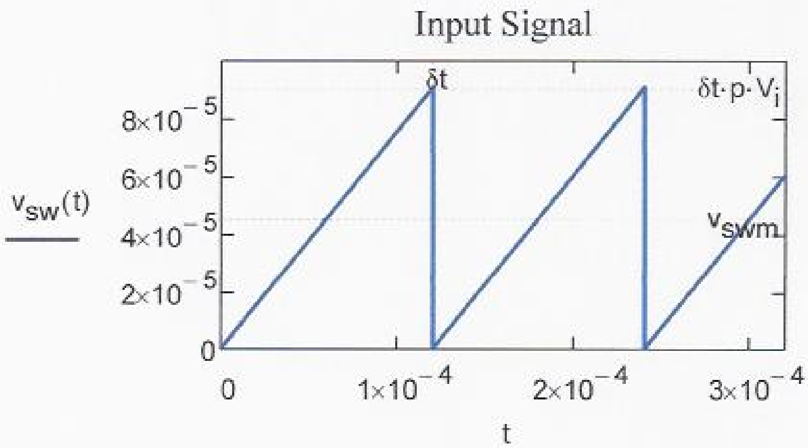
Exact output: $$v_{osw}(t) := A_0 \cdot \left(v_{sw}(t) - e^{-\frac{t}{\tau}} \cdot \int_0^{\frac{t}{\tau}} v_{sw}(\xi \cdot \tau) \cdot e^{\xi} \, d\xi \right)$$ $V_i = 5V$

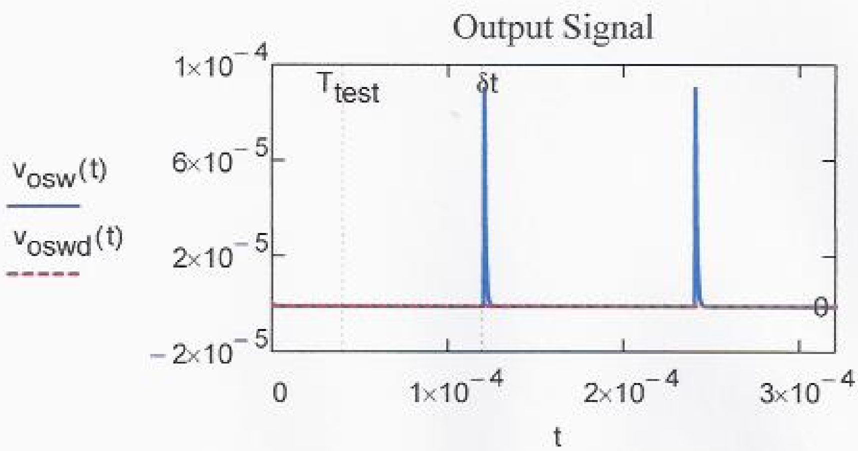
Approximated output: $v_{oswd}(t) := \frac{A_0}{\omega_0} \cdot \frac{\partial}{\partial t} v_{sw}(t)$ $\quad \tau_f = \frac{A_0}{\omega_0}$ $\quad T_{test} = 0.04 \cdot ms$

$p = 0.151$

$$t := T_{test} \cdot 0, T_{test} \cdot 0 + \frac{8 \cdot T_{test} - T_{test} \cdot 0}{1000} .. 8 \cdot T_{test}$$ $\delta t = 120 \cdot \mu s$

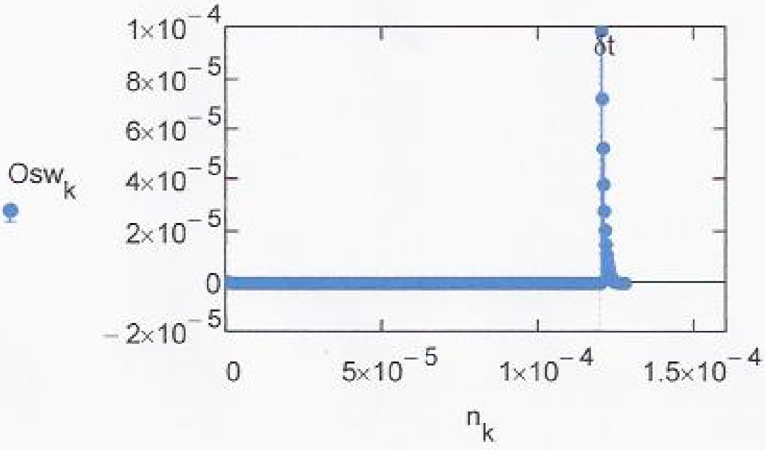
$\delta t \cdot p \cdot V_i = 90.681 \cdot volt \cdot \mu s$ $\quad v_{swm} := \frac{1}{\delta t} \cdot \int_0^{\delta t} v_{sw}(t) \, dt$ $\quad v_{swm} = 45.341 \cdot volt \cdot \mu s$





Output sampling: $Osw_k := v_{osw}(n_k)$

$T_{test} = 40 \cdot \mu s$ $T_{smp} = 0.25 \cdot \mu s$ $f_{test} = 25 \cdot kHz$ $f_{smp} = 4 \cdot MHz$

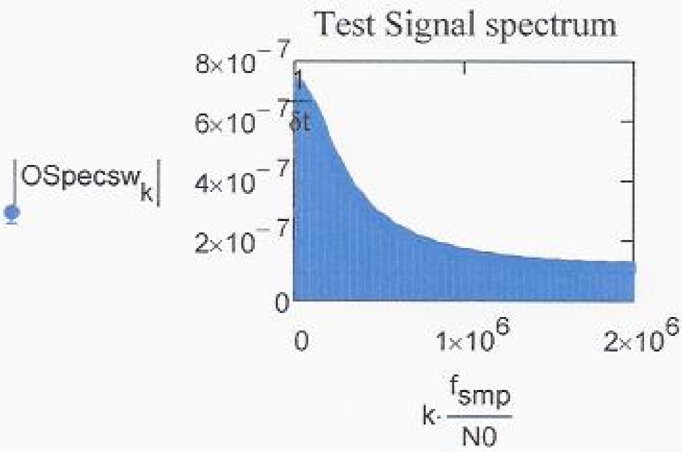


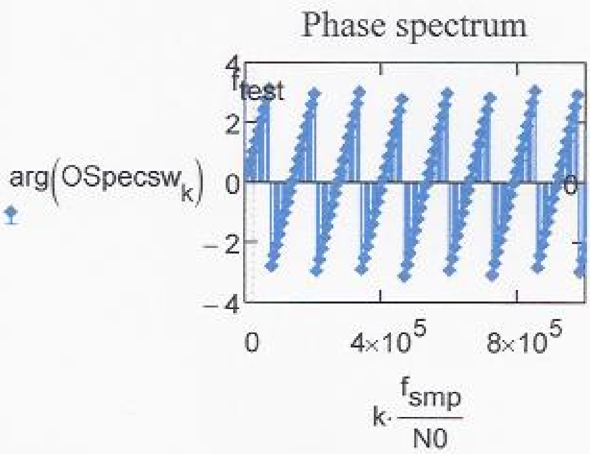
Fourier Transform of the test signal

$$f_{test} = 25 \cdot kHz \qquad \frac{f_{smp}}{f_{test}} = 160$$

$$OSpecsw := FFT(Osw)$$

$$\left|\max\left(\overrightarrow{OSpecsw}\right)\right| = 0.977 \cdot volt \cdot \mu s$$



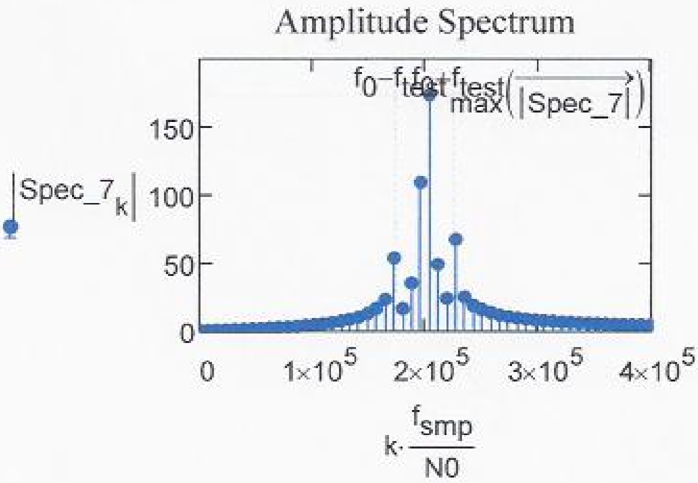


4.2.5) AM Signal response.

$V2_i(t) := v2_i(t, \omega_{test}, \omega_0)$ $m_{am} = 45 \cdot \%$

$$u\_7_k := \frac{v2_i(n_k, \omega_{test}, \omega_0)}{volt}$$

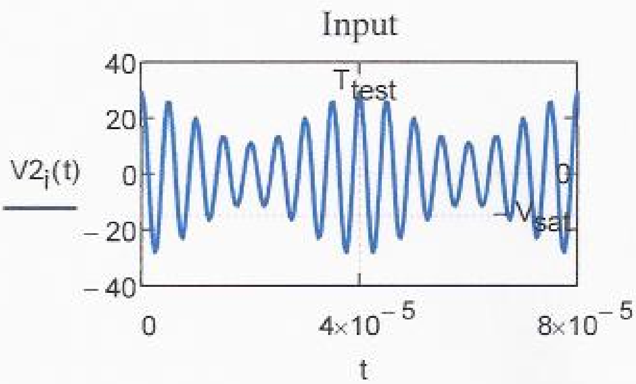
$$Spec\_7 := fft(u\_7)$$

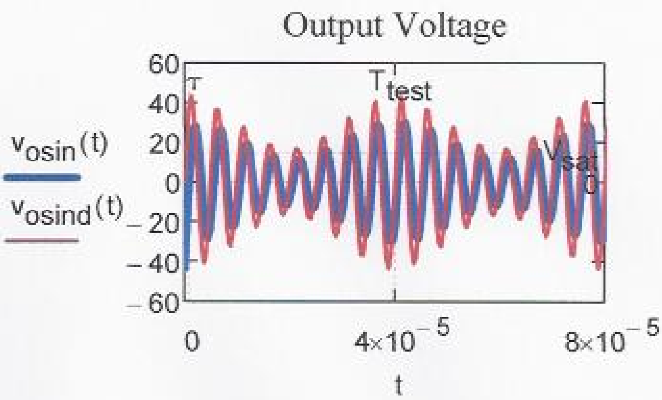


Exact output: $v_{osin}(t) := A_0 \cdot \left(V2_i(t) - e^{-\frac{t}{\tau}} \cdot \int_0^{\frac{t}{\tau}} V2_i(\xi \cdot \tau) \cdot e^{\xi} \, d\xi \right)$ $V_i = 5V$

Approximated output: $v_{osind}(t) := -\tau_f \cdot \frac{\partial}{\partial t} V2_i(t)$ holds for $\omega \ll \omega_0$ or $T \gg (2 \cdot \pi \cdot \tau)$

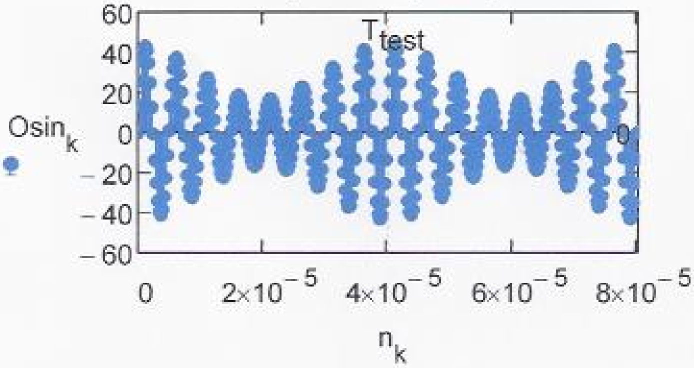
$$t := 0 \cdot T_{test}, 0 \cdot T_{test} + \frac{10 \cdot T_{test}}{1000} .. 10 \cdot T_{test} \qquad T_{test} = 40 \cdot \mu s$$





Output sampling: $Osin_k := v_{osind}(n_k)$

Sampled output signal

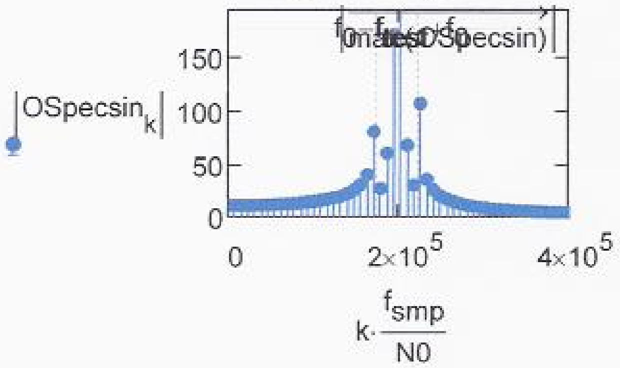


Fourier Transform of the test signal

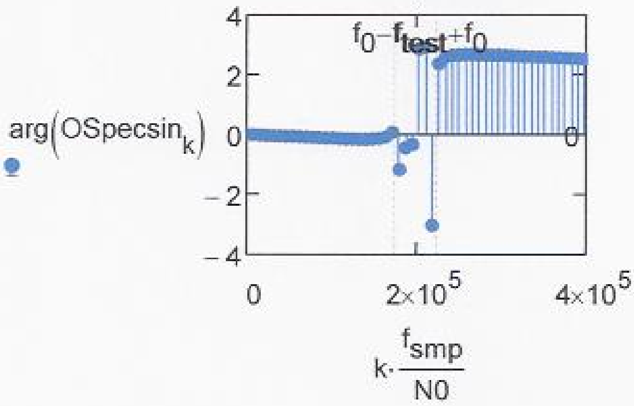
$f_{test} = 25 \cdot kHz$ $\frac{f_{smp}}{f_{test}} = 160$

OSpecsin := fft(Osin)

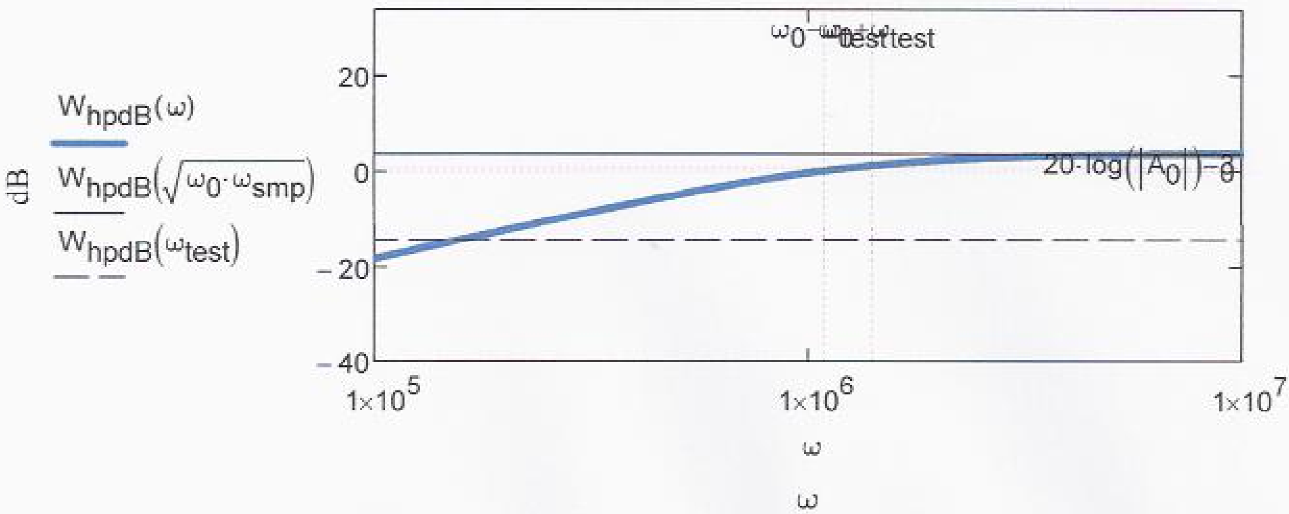
Magnitude Spectrum



Phase Spectrum



BODE Plots of H(z) compared with that of W(jω)



ANALOG FILTER OUTPUT ANALYSIS

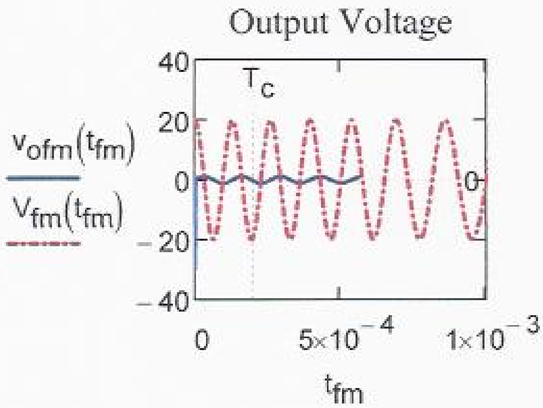
4.2.6) Frequency Modulated carrier response. $T_c := T2_c$ $T_m := T2_m$

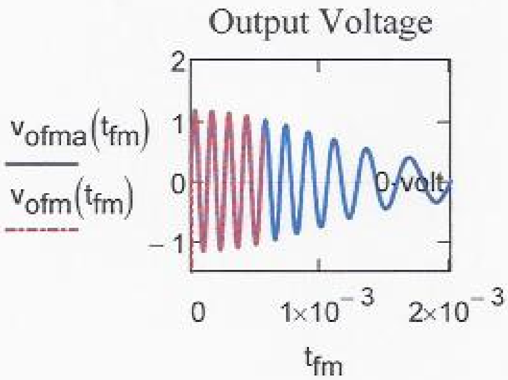
Defined in "Test Signals.xmcd" $f_c := f2_c$ $\omega_c := \omega 2_c$ $\omega_m := \omega 2_m$

$f_{sfm} := f1_{sfm}$ $V_{fm}(t) := v_{fm}(t, \omega_c, \omega_m, A_{fm}, m_f)$ $A_{fm} = 20V$

Exact output: $v_{ofm}(t) := A_0 \cdot \left(V_{fm}(t) - e^{-\frac{t}{\tau}} \cdot \int_0^{\frac{t}{\tau}} V_{fm}(\xi \cdot \tau) \cdot e^{\xi} \, d\xi \right)$ $V_i = 5V$

Approximated output: $v_{ofma}(t) := \frac{A_0}{\omega_0} \cdot \frac{\partial}{\partial t} V_{fm}(t)$ $T_{test} = 40 \cdot \mu s$





Output sampling: $Ofm_k := \frac{v_{ofm}(nfm_k)}{volt}$

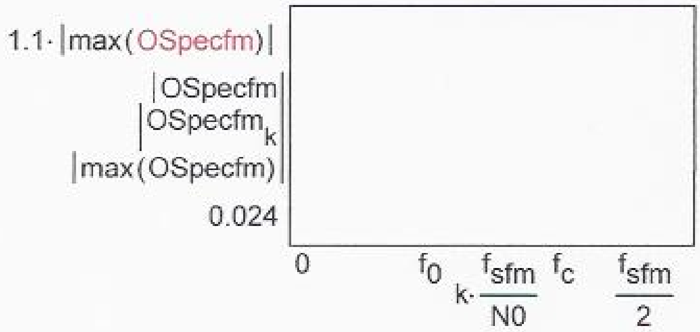
Sampled signal

Fourier Transform of the test signal

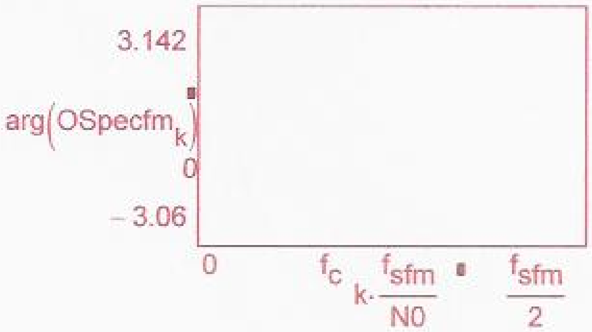
$$f_c = 5 \times 10^{-3} \cdot MH \quad \frac{f_{sfm}}{f_c} = 8$$

$$OSpecfm := fft(Ofm) \qquad m_f = 11 \qquad \omega_m = 1.571 \times 10^{-3} \cdot \frac{Mrads}{sec}$$

FM Test Signal spectrum



Phase spectrum



On the other hand if the carrier frequency is located in the passing band, the filter response is:

$$f2_{sfm} := f1_{sfm} \qquad V1_{fm}(t) := v_{fm}\left(t, 100\cdot\omega_0, 100\cdot\omega_m, A_{fm}, m_f\right) \qquad A_{fm} = 20V$$

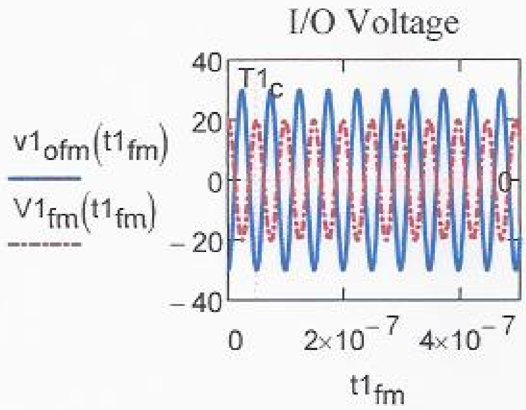
$$\omega2_0 := 100\cdot\omega_0$$

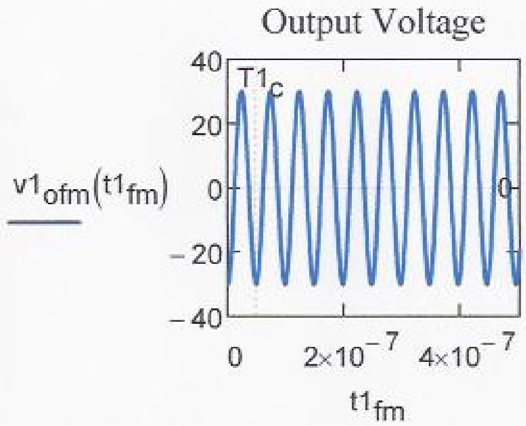
$$\text{Exact output:} \quad v1_{ofm}(t) := A_0 \cdot \left(V1_{fm}(t) - e^{-\frac{t}{\tau}} \cdot \int_0^{\frac{t}{\tau}} V1_{fm}(\xi\cdot\tau)\cdot e^{\xi}\, d\xi \right) \qquad V_i = 5V$$

$$\text{Approximated output:} \quad v1_{ofma}(t) := \frac{A_0}{\omega_0} \cdot \frac{\partial}{\partial t} V1_{fm}(t) \qquad T_{test} = 40\cdot\mu s$$

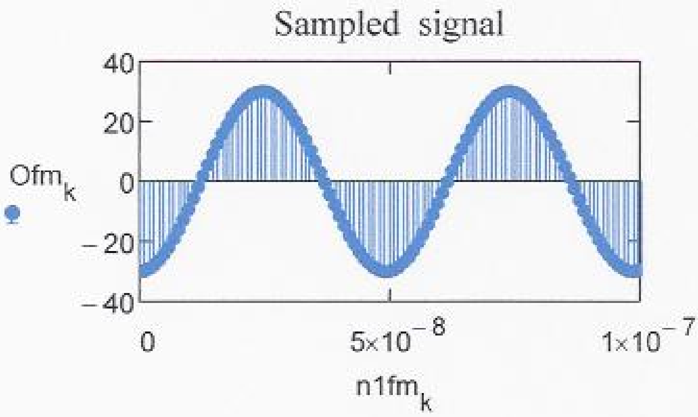
$$T1_c := \frac{2\cdot\pi}{100\cdot\omega_0} \qquad t1_{fm} := T2_m\cdot 0, T2_m\cdot 0 + \frac{\left|T2_m\cdot 0 - 10\cdot T1_c\right|}{1000} .. 10\cdot T1_c \qquad T1_c = 50\cdot ns$$

$$\frac{1}{f2_{sfm}} = 25\cdot\mu s$$



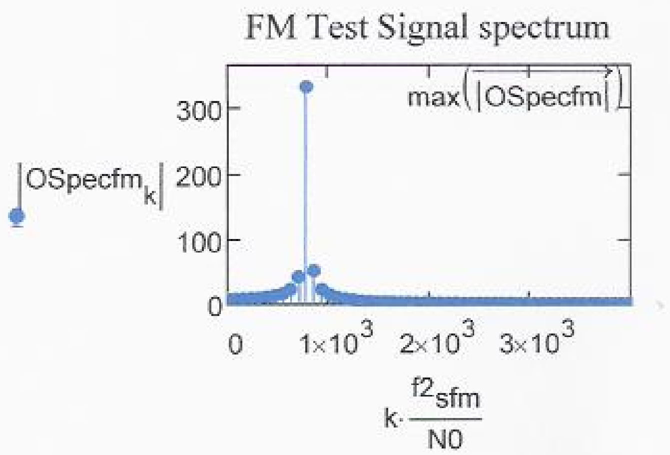


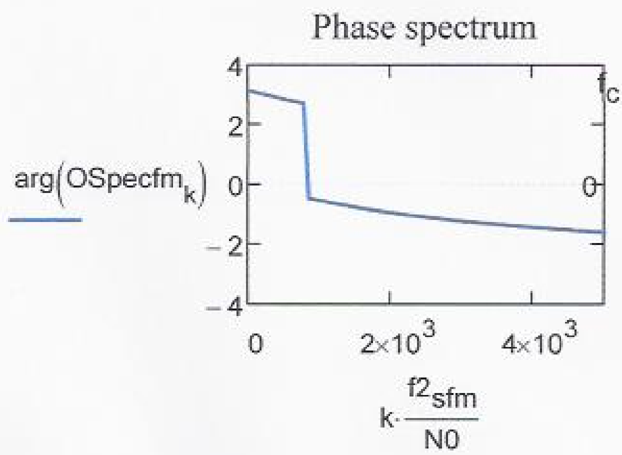
$$n1fm_k := \frac{k}{N0} \cdot T1_c \cdot 10 \qquad \text{Output sampling: } Ofm_k := \frac{v1_{ofm}(n1fm_k)}{volt}$$



Fourier Transform of the test signal $f1_c = 0.1 \cdot MHz$ $\qquad \frac{f1_{sfm}}{f1_c} = 0.4 \qquad m_f = 11$

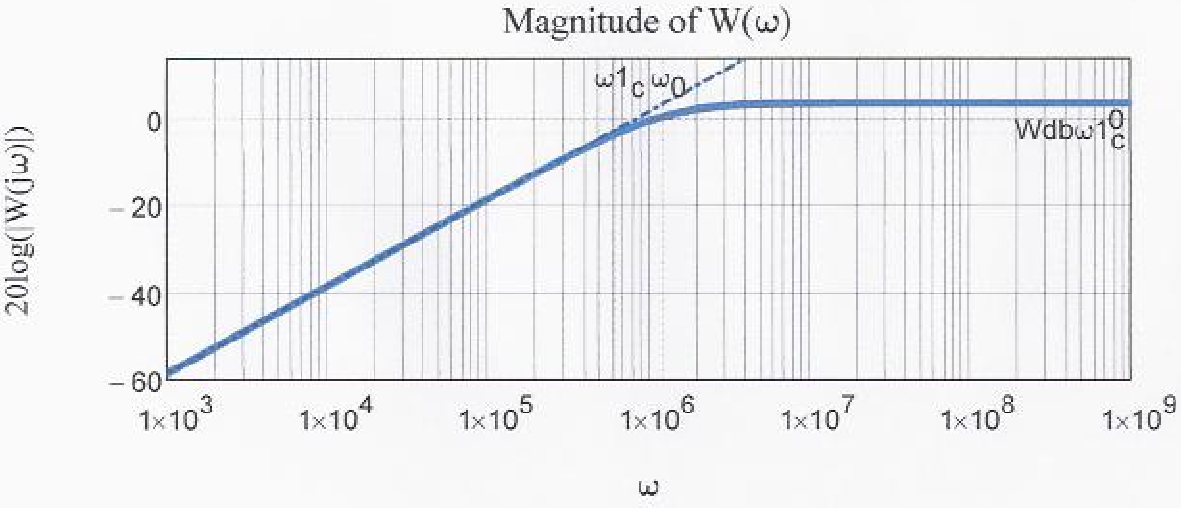
$$OSpecfm := fft(Ofm) \qquad \omega_m = 1.571 \times 10^{-3} \cdot \frac{Mrads}{sec}$$





$$\omega := \frac{\omega_0}{20 \cdot U}, \frac{\omega_0}{20 \cdot U} + \frac{\omega_0 \cdot U - \frac{\omega_0}{20 \cdot U}}{U^2} .. 10 \cdot U \cdot \omega_0$$

$$Wdb\omega1_c := 20\cdot\log(|W_{hp}(j\cdot\omega1_c)|) \qquad Wdb\omega1_c = -3.468\cdot dB$$



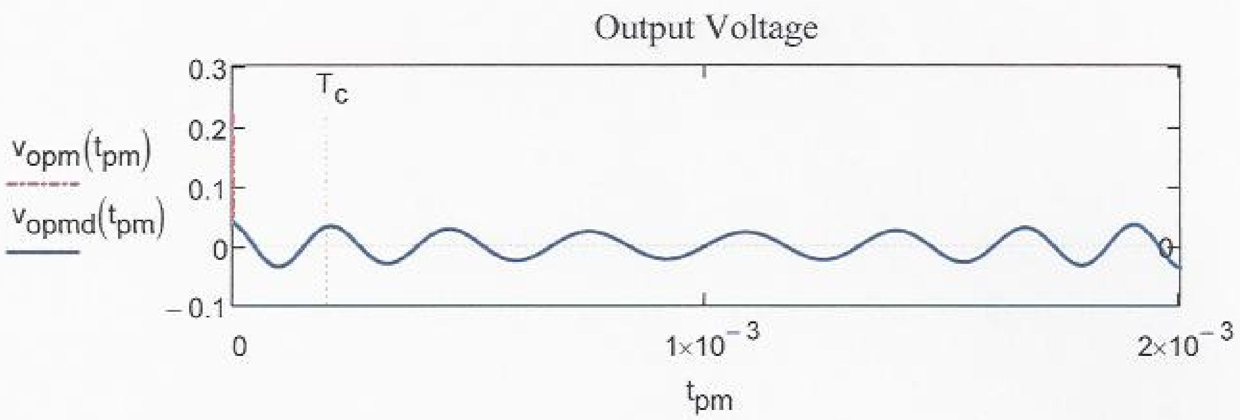
4.2.7) Phase Modulated carrier response.

$f_{spm} := f1_{spm}$ $\qquad V_{pm}(t) := v_{pm}(t, \omega_c, \omega_m, A, m_p)$

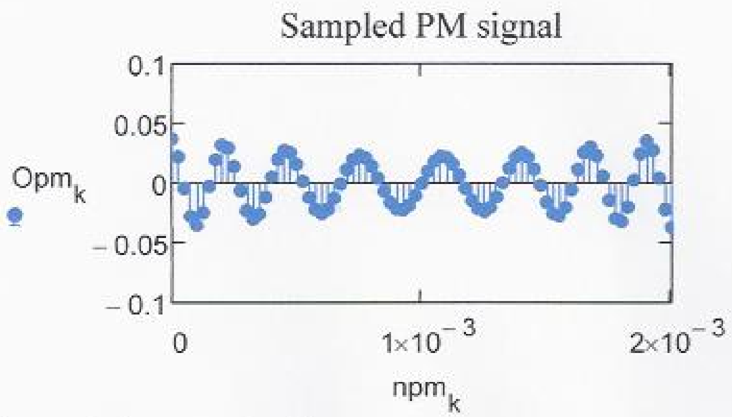
Exact output: $$v_{opm}(t) := A_0 \cdot \left(V_{pm}(t) - e^{-\frac{t}{\tau}} \cdot \int_0^{\frac{t}{\tau}} V_{pm}(\xi \cdot \tau) \cdot e^{\xi} \, d\xi \right) \quad V_i = 5V$$

Approximated output: $$v_{opmd}(t) := \frac{A_0}{\omega_0} \cdot \frac{\partial}{\partial t} V_{pm}(t)$$ $T_{test} = 40 \cdot \mu s$

$\tau = 0.796 \cdot \mu s$



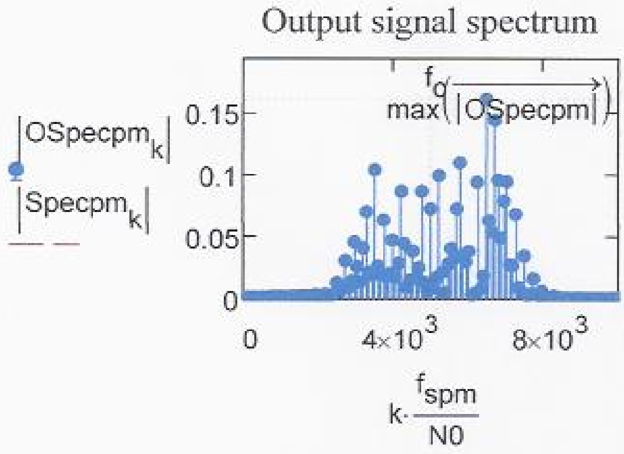
$$Opm_k := \frac{v_{opmd}(npm_k)}{volt}$$

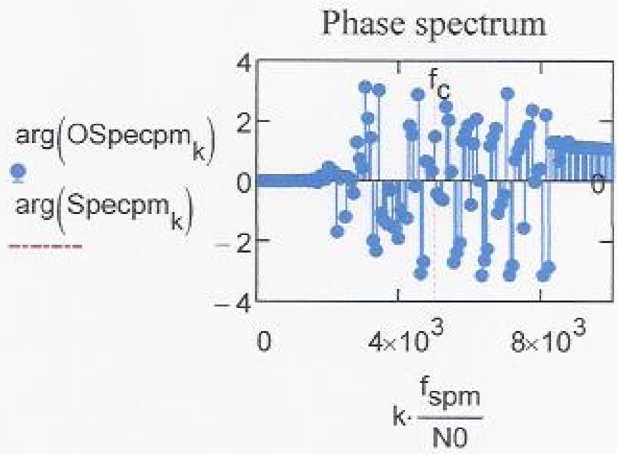


Fourier Transform of the test signal

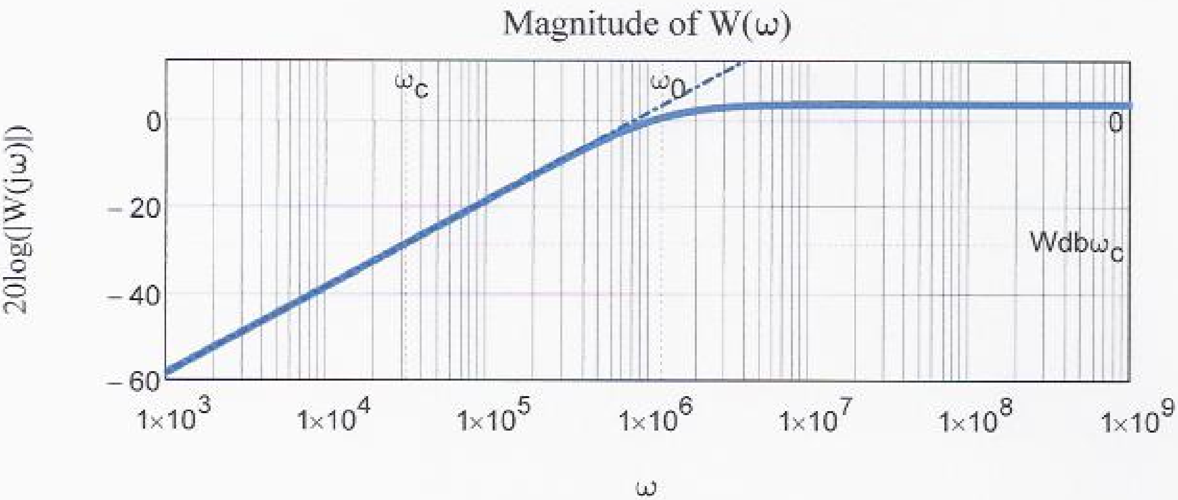
$f_c = 5 \times 10^{-3} \cdot MHz$ $\qquad \frac{f_{spm}}{f_c} = 8$

$OSpecpm := fft(Opm)$ $\qquad m_p = 8$ $\qquad \omega_m = 1.571 \times 10^{-3} \cdot \frac{Mrads}{sec}$





$$Wdb\omega_c := 20\cdot\log\left(\left|W_{hp}(j\cdot\omega_c)\right|\right) \qquad Wdb\omega_c = -28.522\cdot dB$$



4.3
Equivalent Digital High Pass Filter (I°order)

4.3.1) Z-transfer function of the I° Order High Pass Digital Filter.

Given the transfer function: $W_{hp}(s) = \frac{A_0 \cdot s}{s + \omega_0}$, the corresponding z-transform can be calculated

with the change of variable: $s = \frac{1 - z^{-1}}{T_{smp}}$,

$$\omega_{smp} = 25.133 \cdot \frac{Mrads}{sec} \qquad \omega_0 = 1.257 \cdot \frac{Mrads}{sec}$$

$$A_0 := A_0 \quad T_{smp} := T_{smp} \quad \omega_{smp} := \omega_{smp}$$

$$H1_{hp}(z) := \frac{A_0 \cdot s}{s + \omega_0} \left| \begin{array}{l} \text{substitute}, s = \frac{1}{T_{smp}} \cdot \left(1 - z^{-1}\right) \\ \text{collect}, z \\ \text{collect}, A_0 \end{array} \right. \rightarrow \frac{z - 1}{z \cdot \left(T_{smp} \cdot \omega_0 + 1\right) - 1} \cdot A_0$$

and after some algebraic manipulation and the definition of the following parameters:

$$\frac{z - 1}{z \cdot \left(T_{smp} \cdot \omega_0 + 1\right) - 1} = \frac{\left(1 - z^{-1}\right) \cdot A_0}{1 + T_{smp} \cdot \omega_0 - z^{-1}} = \frac{A_0}{\left(1 + T_{smp} \cdot \omega_0\right)} \cdot \frac{\left(1 - z^{-1}\right)}{\left[1 - \frac{z^{-1}}{\left(1 + T_{smp} \cdot \omega_0\right)}\right]}$$

Coefficients: $u0 := \frac{A_0}{\left(1 + T_{smp} \cdot \omega_0\right)}$ $\qquad v0 := \frac{1}{\left(1 + \omega_0 \cdot T_{smp}\right)}$

$u0 = -1.141414165$ $\quad v0 = 0.760942776$

we get the following result for the t. f. as a function of z^{-1}:

$$H_{hp}(z) := u0 \cdot \frac{1 - z^{-1}}{1 - z^{-1} \cdot v0}$$

$$\omega_0 = 1.257 \cdot \frac{Mrads}{sec} \qquad \omega_{test} = 0.157 \cdot \frac{Mrads}{sec}$$

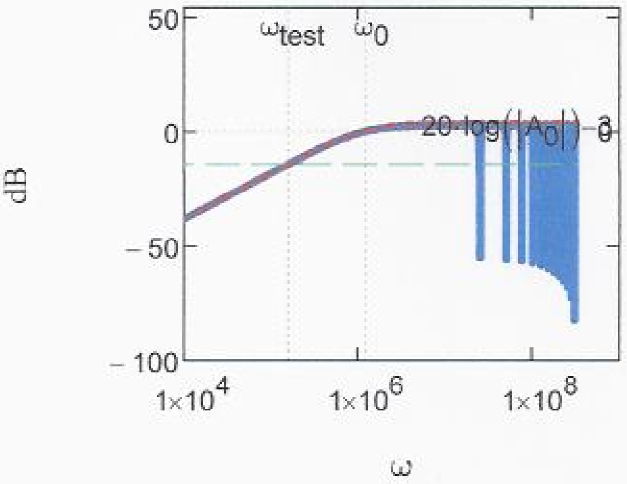
$$20 \cdot \log\left(\left|H_{hp}\left(e^{j \cdot \omega_{test} \cdot T_{smp}}\right)\right|\right) = -14.629 \cdot dB$$

$$HdB1(\omega) := 20\cdot\log\left(\left|H_{hp}\left(e^{j\cdot\omega\cdot T_{smp}}\right)\right|\right) \qquad \varphi1(\omega) := \arg\left(H_{hp}\left(e^{j\cdot\omega\cdot T_{smp}}\right)\right)$$

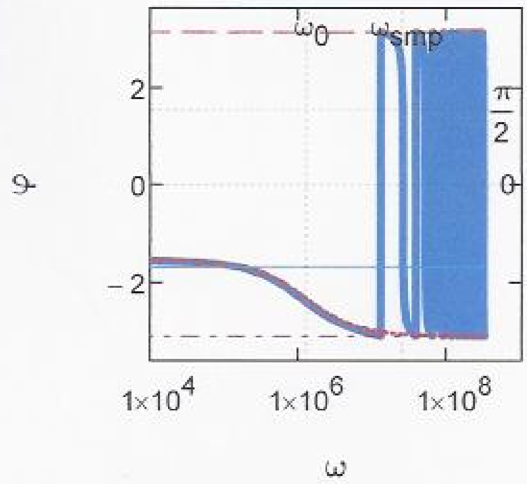
$$HdBc := 20\cdot\log\left(\left|H_{hp}\left(e^{j\cdot\omega_{test}\cdot T_{smp}}\right)\right|\right) \qquad \varphi1c := \arg\left(H_{hp}\left(e^{j\cdot\frac{2\cdot\pi}{T_{test}}\cdot T_{smp}}\right)\right)$$

$$\omega := \frac{\omega_0}{8\cdot U}, \frac{\omega_0}{8\cdot U} + \frac{\omega_{smp}\cdot\frac{U}{8} - \frac{\omega_0}{8\cdot U}}{4\cdot U^2} .. \frac{U}{8}\cdot\omega_{smp} \qquad \frac{\omega_{smp}}{\omega_{test}} = 160$$

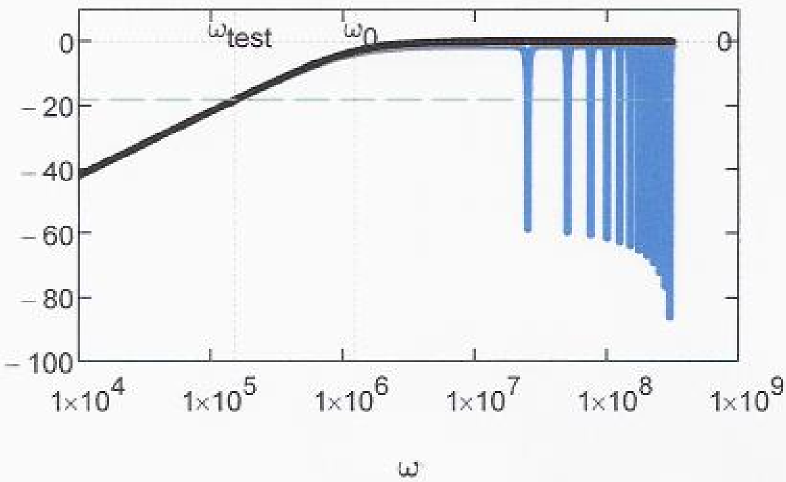
Magnitude of H(z)



Phase of H(z)



Normalized Magnitude of H(z) and W(jω)



4.3.2) Difference equations (HIGH PASS FILTER (I°order)). Canonical form.

Given the z transfer function $H_{hp}(z) = u0\cdot\frac{1-z^{-1}}{1-z^{-1}\cdot v0}$, it can be decomposed so that:

$$H_{hp}(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W\_(z)}\cdot\frac{W\_(z)}{X(z)}$$

$$\frac{Y(z)}{W\_(z)}\cdot\frac{W\_(z)}{X(z)} = u0\cdot\frac{1-z^{-1}}{1-z^{-1}\cdot v0}$$

$$\frac{Y(z)}{W\_(z)} = u0\cdot\left(1-z^{-1}\right)$$

$$Y(z) = u0\cdot W\_(z) - u0\cdot z^{-1}\cdot W\_(z)$$

$$y(n) = u0\cdot w(n) - u0\cdot w(n-1)$$

$$\frac{W\_(z)}{X(z)} = \frac{1}{1-z^{-1}\cdot v0}$$

$$X(z) = \left(1-z^{-1}\cdot v0\right)\cdot W\_(z) = W\_(z) - z^{-1}\cdot v0\cdot W\_(z)$$

$$x(n) = w(n) - v0\cdot w(n-1)$$

Ultimately the corresponding set of difference equations is:

1) $w(n) = x(n) + v0\cdot w(n-1)$

2) $y(n) = u0\cdot(w(n) - w(n-1))$

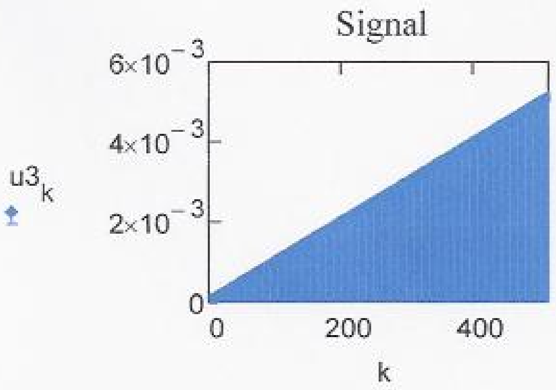
$z := z \quad u0 := u0 \quad v0 := v0$

Z T. Initial value theorem: $\lim_{z\to\infty}\left(u0\cdot\frac{1-z^{-1}}{1-z^{-1}\cdot v0}\right) \to u0$ $\qquad u0 = -1.141$

Z T. Final value theorem: $\lim_{z\to 0}\left(u0\cdot\frac{1-z^{-1}}{1-z^{-1}\cdot v0}\right) \to \begin{cases}\frac{u0}{v0} & \text{if } v0 \neq 0\\ \text{undefined} & \text{otherwise}\end{cases}$ $\qquad \frac{u0}{v0} = -1.5$

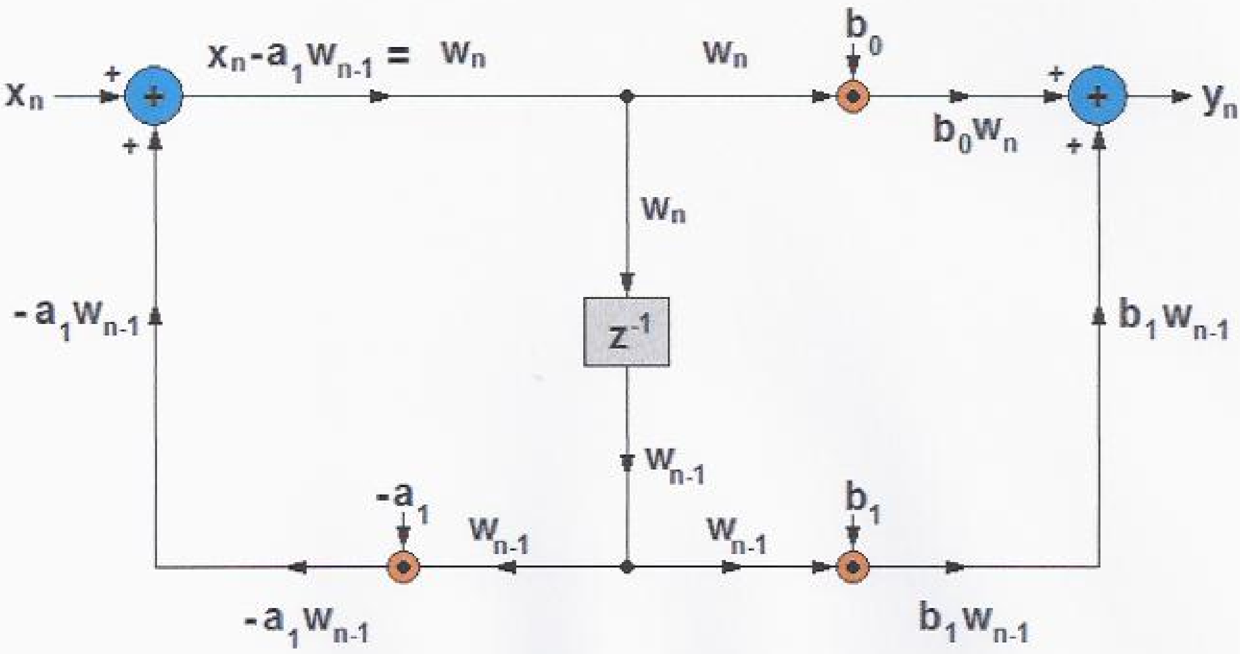
4.3.2.1) Sequence of the Voltage ramp response.

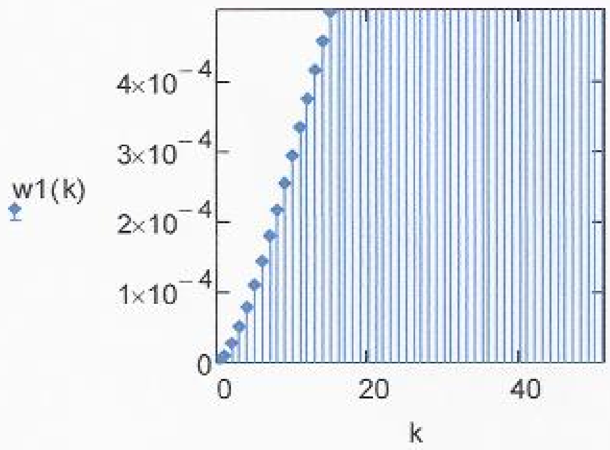
$$x1_i(n) := u3_n$$

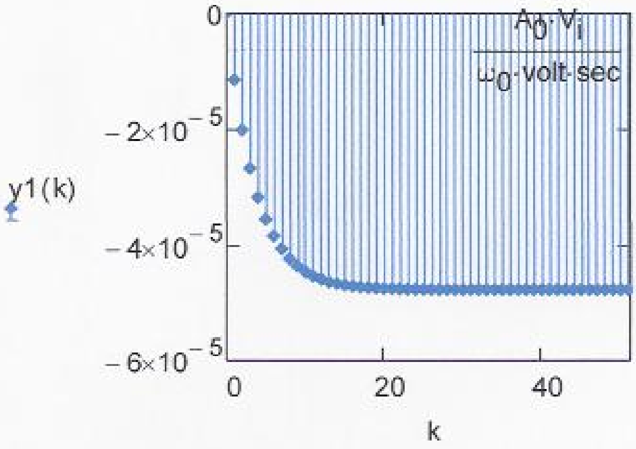


1) $$w1(n) := \begin{cases} x1_i(n) + v0 \cdot w1(n-1) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

2) $$y1(n) := \begin{cases} u0 \cdot (w1(n) - w1(n-1)) & \text{if } n > 0 \\ (0) & \text{otherwise} \end{cases}$$



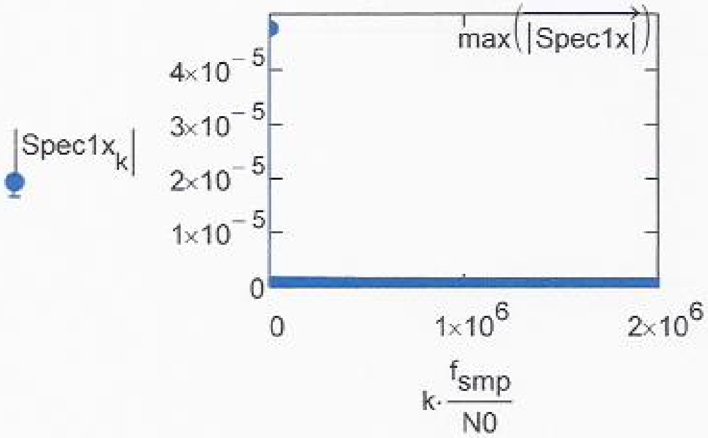




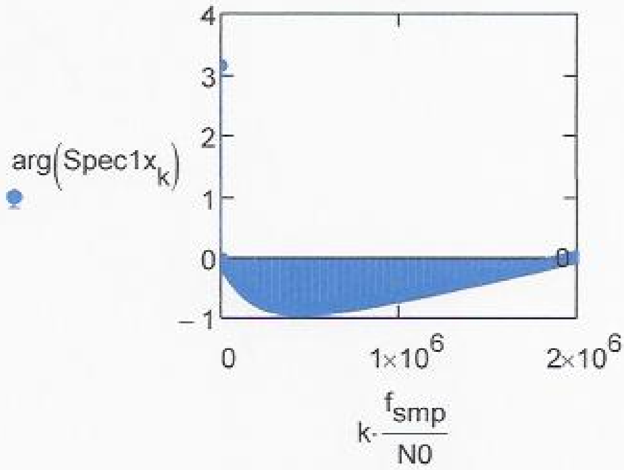
Sampled signal: $v1x_k := y1(k)$

$Spec1x := FFT(v1x)$

Amplitude Spectrum



Phase spectrum

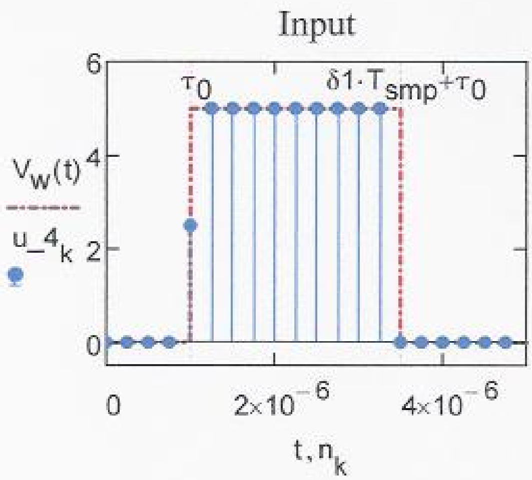


4.3.2.2) Sequence of the Voltage window response.

Digital first order High pass filter difference equations:

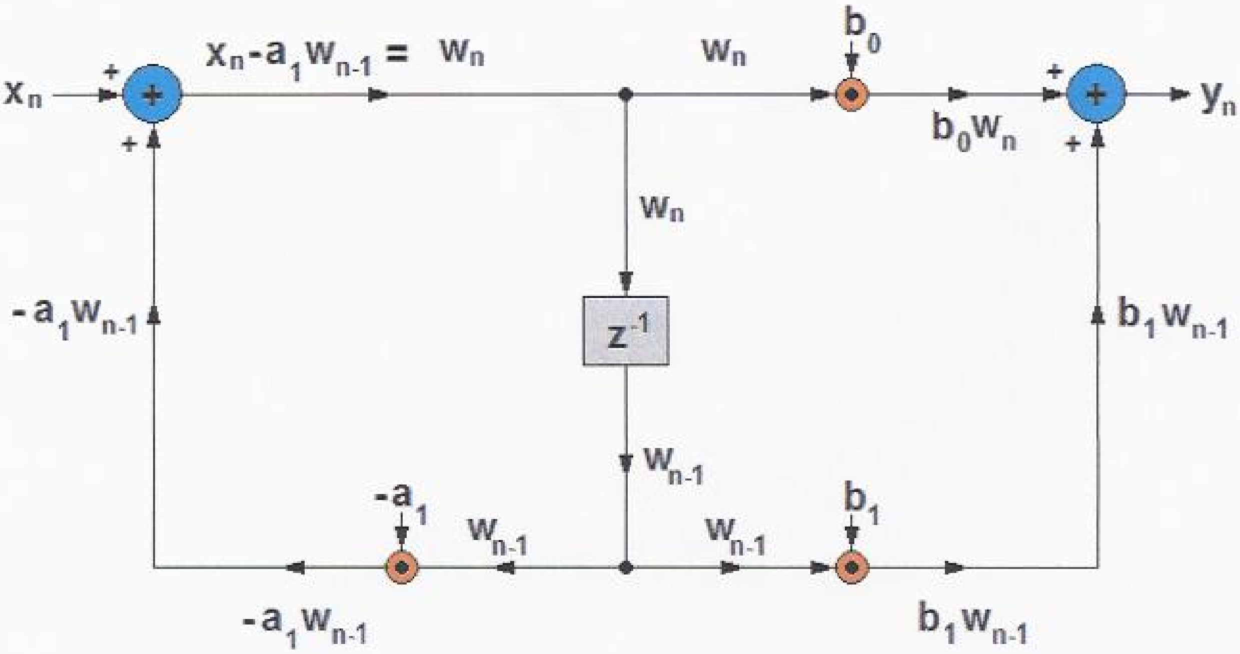
$$x2_i(n) := u\_4_n$$

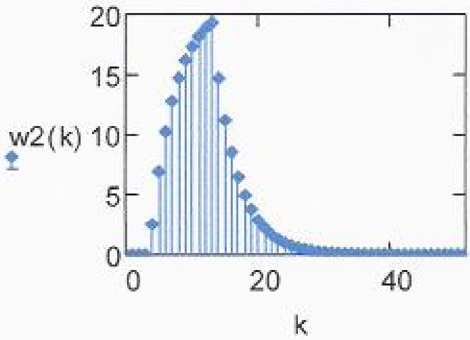
$$t := 0, \frac{\tau_0 + 2\cdot(\delta 1\cdot T_{smp})}{10000} .. \tau_0 + 2\cdot(\delta 1\cdot T_{smp})$$

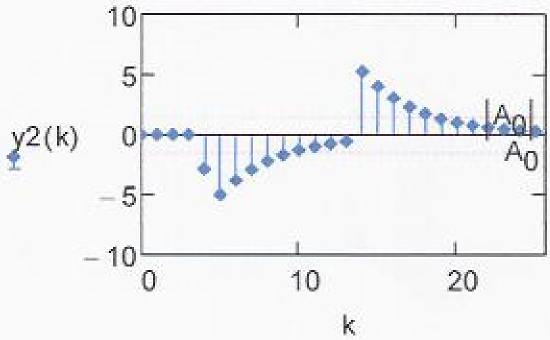


1) $w2(n) := \begin{cases} x2_i(n) + v0\cdot w2(n-1) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$

2) $y2(n) := \begin{cases} u0\cdot(w2(n) - w2(n-1)) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$

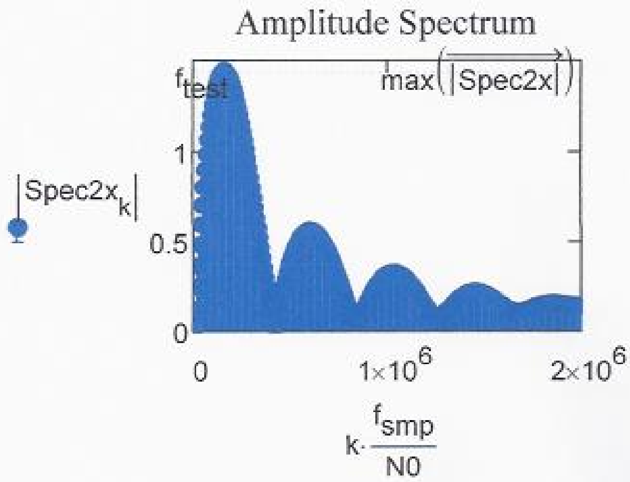


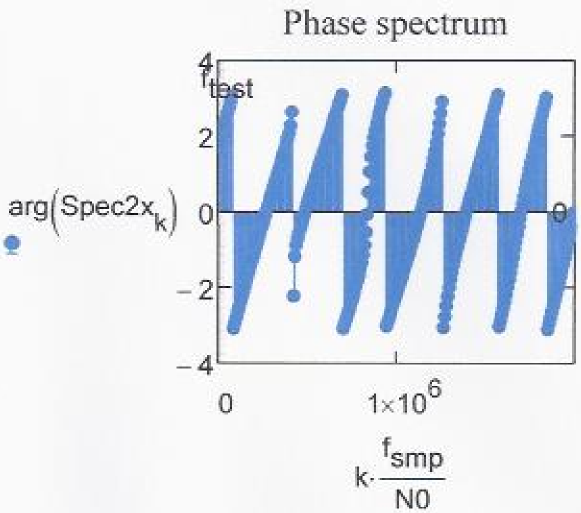


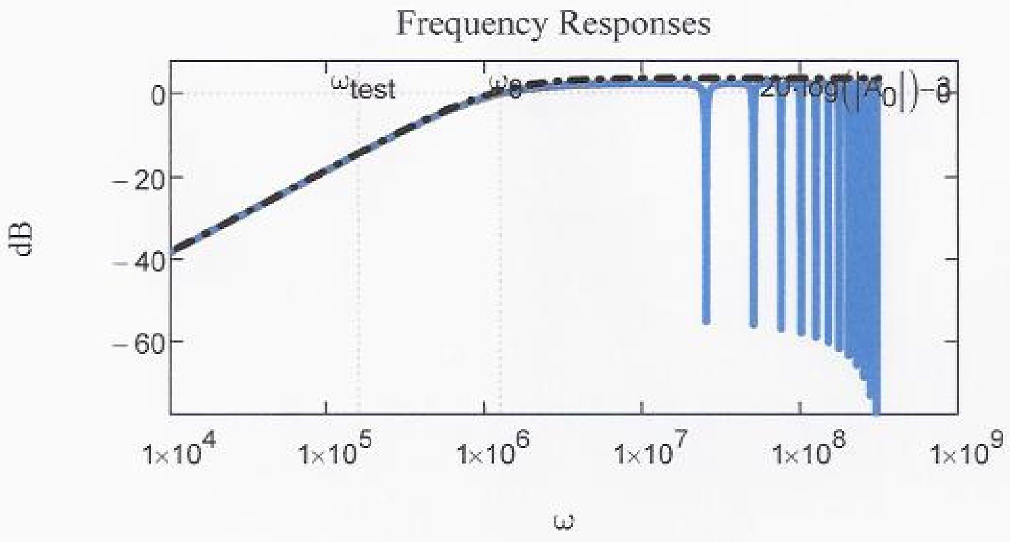


Sampled signal: $v2x_k := y2(k)$

$Spec2x := fft(v2x)$





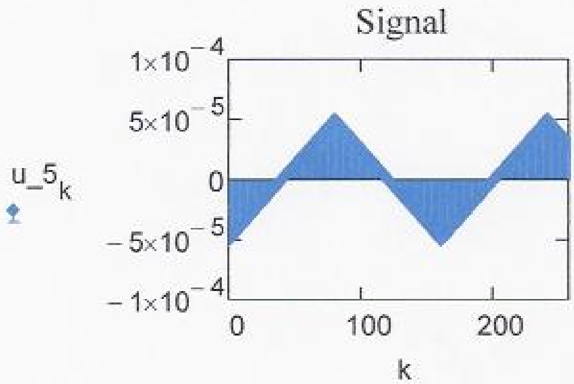


Frequency Responses
dB
0
−20
−40
−60
ωtest
ω0
20log(|A0|)−3
1×10^4
1×10^5
1×10^6
1×10^7
1×10^8
1×10^9
ω

4.3.2.3) Sequence of the Bipolar Triangular wave response:

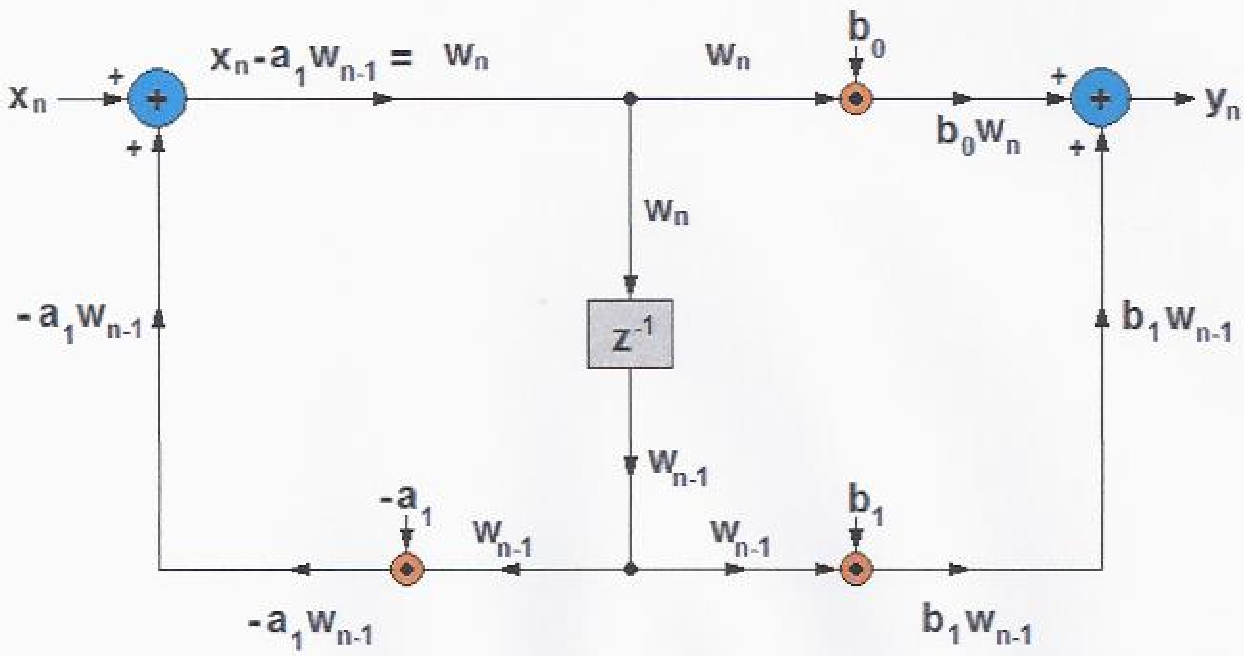
Digital first order High pass filter difference equations:

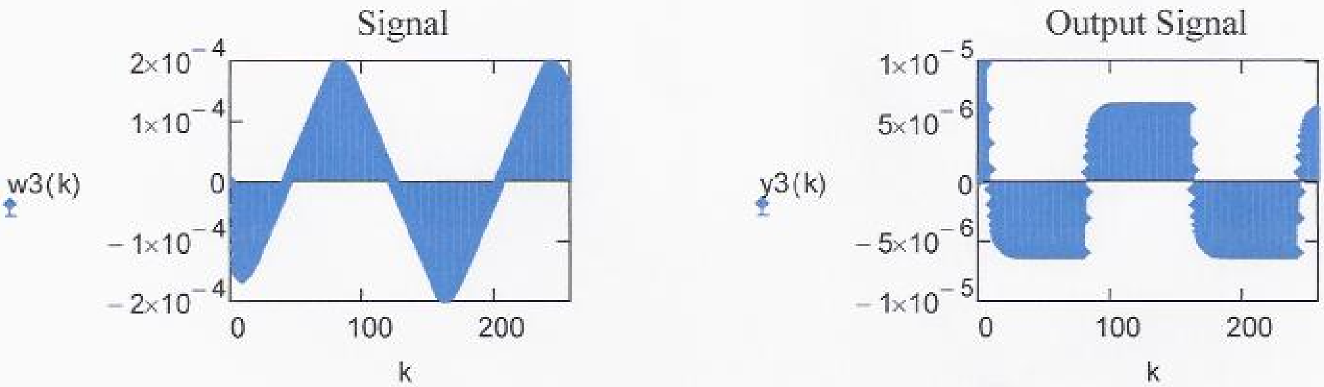
$$x3_i(n) := u\_5_n$$



1) $w3(n) := \begin{cases} x3_i(n) + v0 \cdot w3(n-1) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$

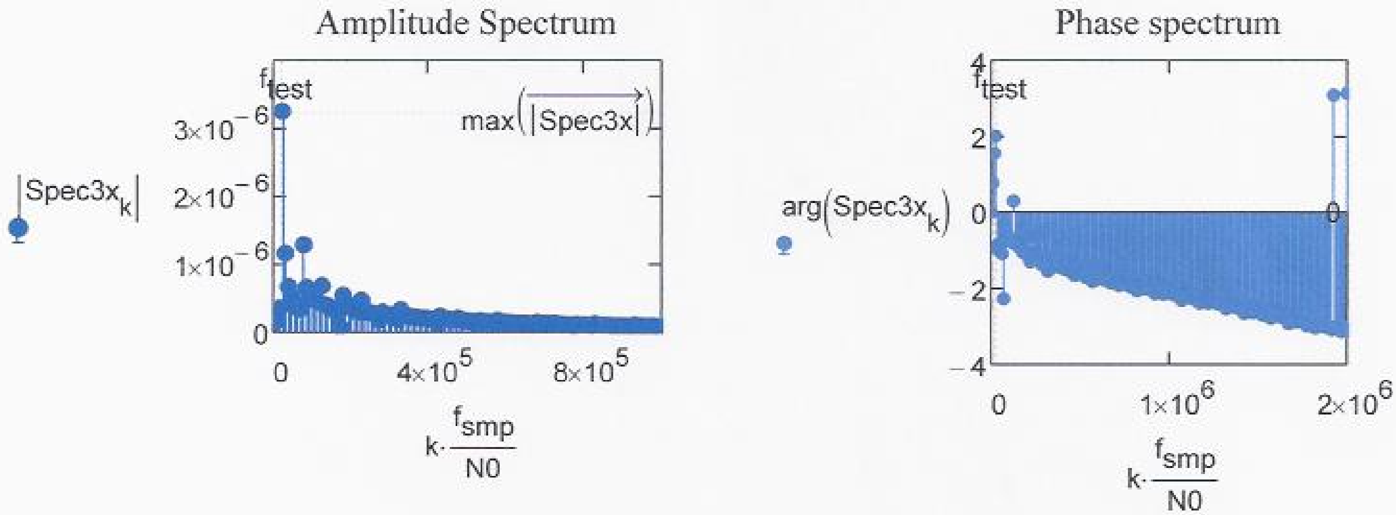
2) $y3(n) := \begin{cases} u0 \cdot (w3(n) - w3(n-1)) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$





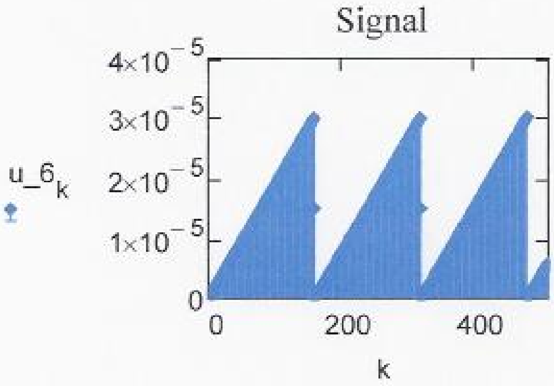
Sampled signal: $v3x_k := y3(k)$

$Spec3x := FFT(v3x)$



4.3.2.4) Sequence of the Sawtooth wave response:

$$u\_6_k := \frac{v1_{sw}(n_k, T_{test}, p)}{volt}$$

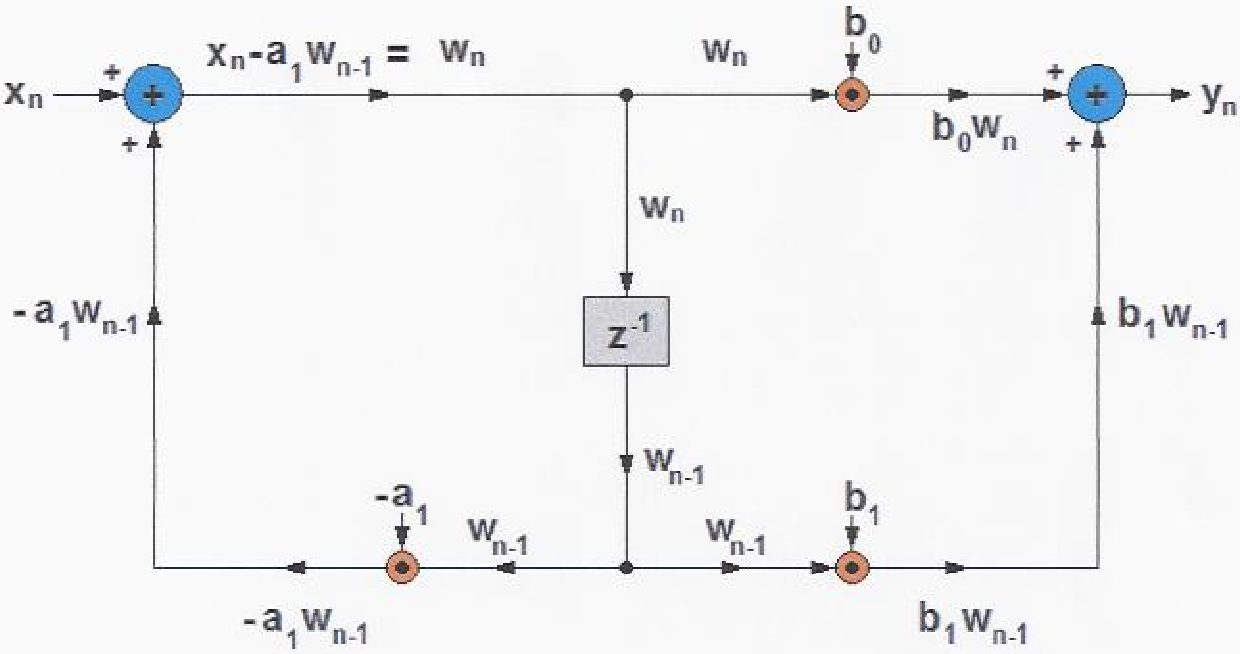


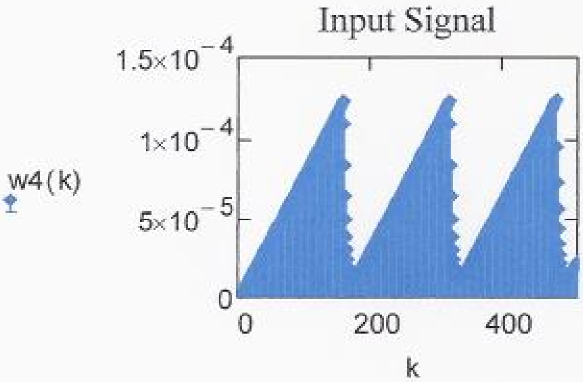
Digital first order High pass filter difference equations:

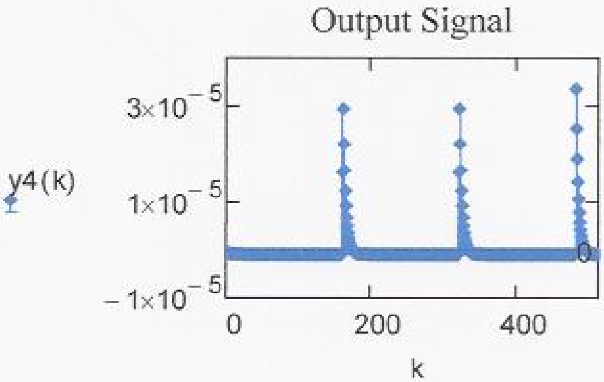
$$x4_i(n) := u\_6_n$$

1) $$w4(n) := \begin{cases} x4_i(n) + v0 \cdot w4(n-1) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

2) $$y4(n) := \begin{cases} u0 \cdot (w4(n) - w4(n-1)) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

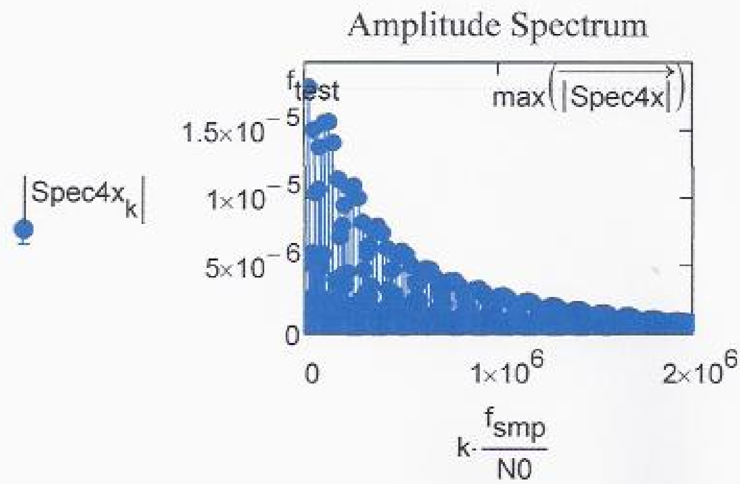


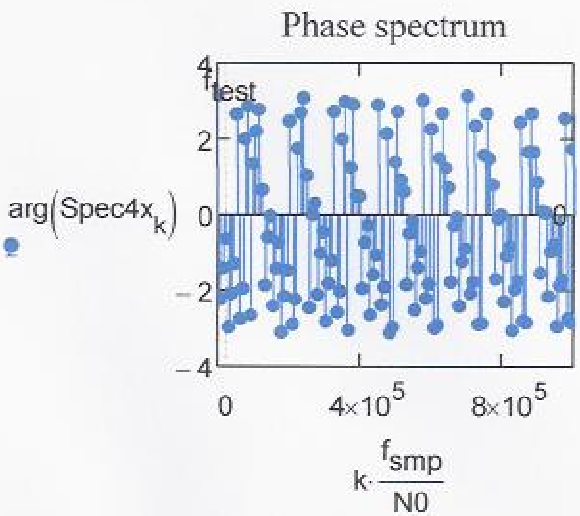




Sampled signal: $v4x_k := y4(k)$

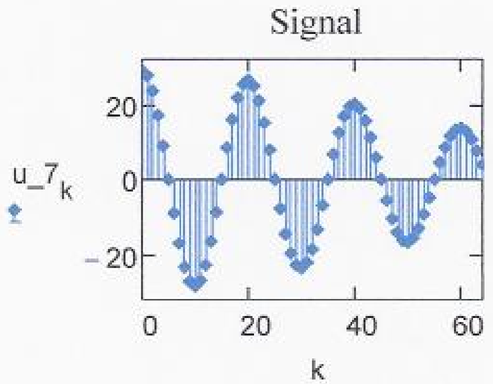
$Spec4x := fft(v4x)$





4.3.2.5) Sequence of the AM Signal response:

$$u\_7_k := \frac{v2_i(n_k, \omega_{test}, \omega_0)}{volt}$$

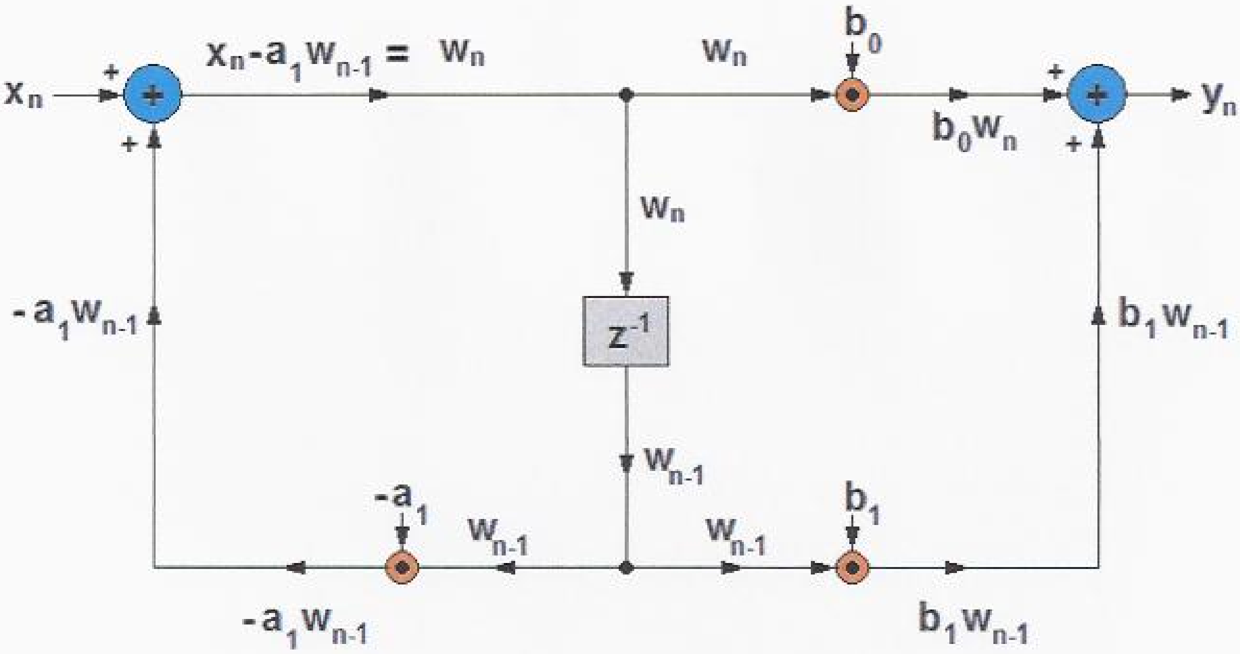


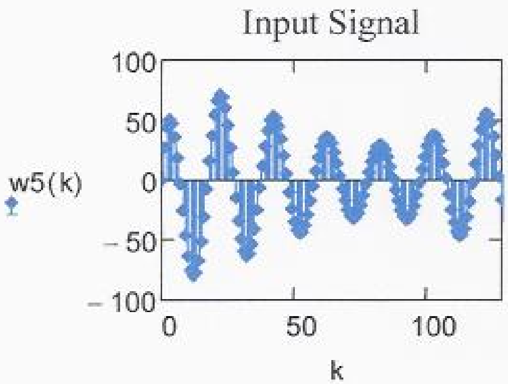
Digital first order High pass filter difference equations:

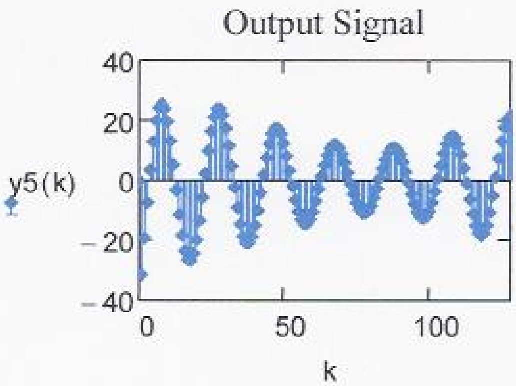
$$x5_i(n) := u\_7_n$$

1) $$w5(n) := \begin{cases} x5_i(n) + v0 \cdot w5(n-1) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

2) $$y5(n) := \begin{cases} u0 \cdot (w5(n) - w5(n-1)) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

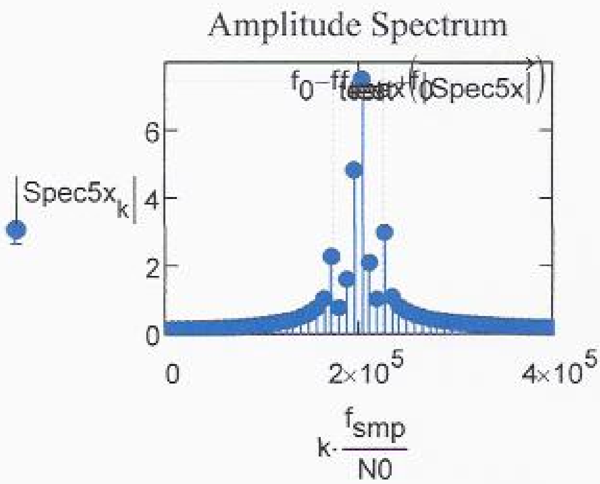


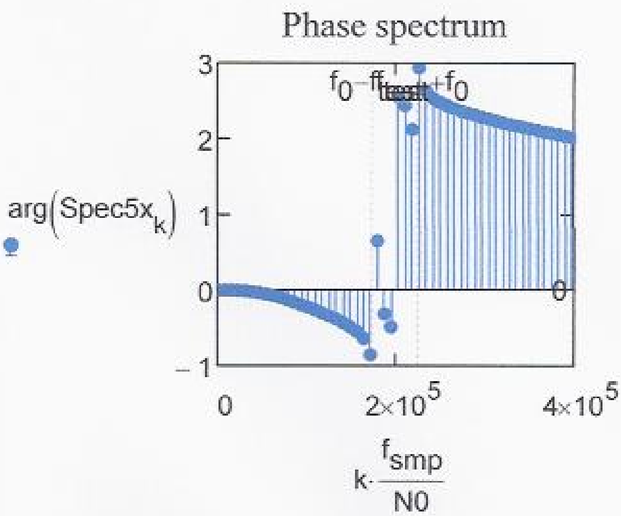




Sampled signal: $v5x_k := y5(k)$

$Spec5x := FFT(v5x)$

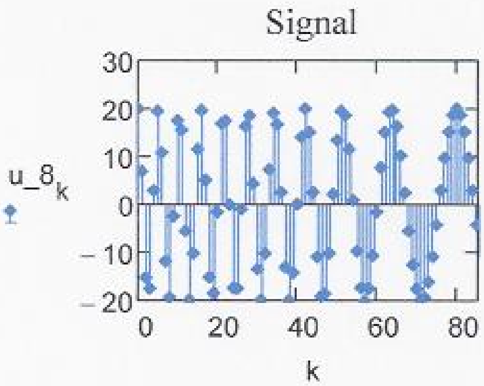




4.3.2.6) Sequence of the Frequency Modulated signal response.

Digital first order High pass filter difference equations:

$$u\_8_k := \frac{v_{fm}(nfm_k, \omega_c, \omega_m, A_{fm}, m_f)}{volt}$$

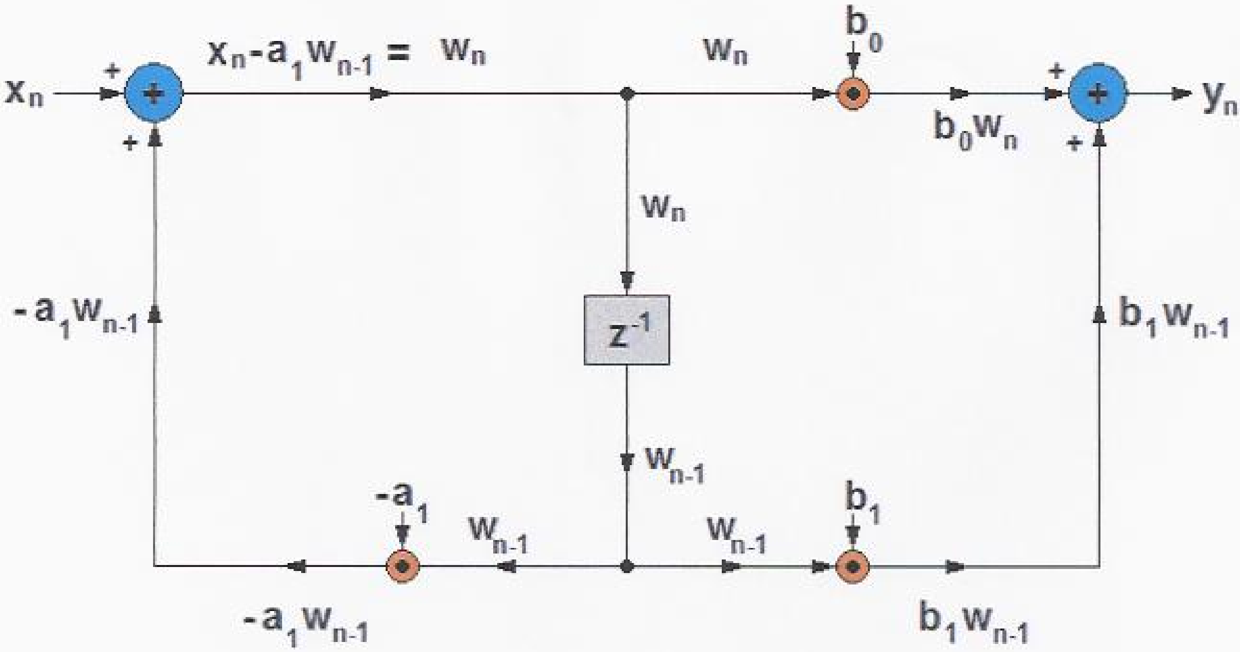


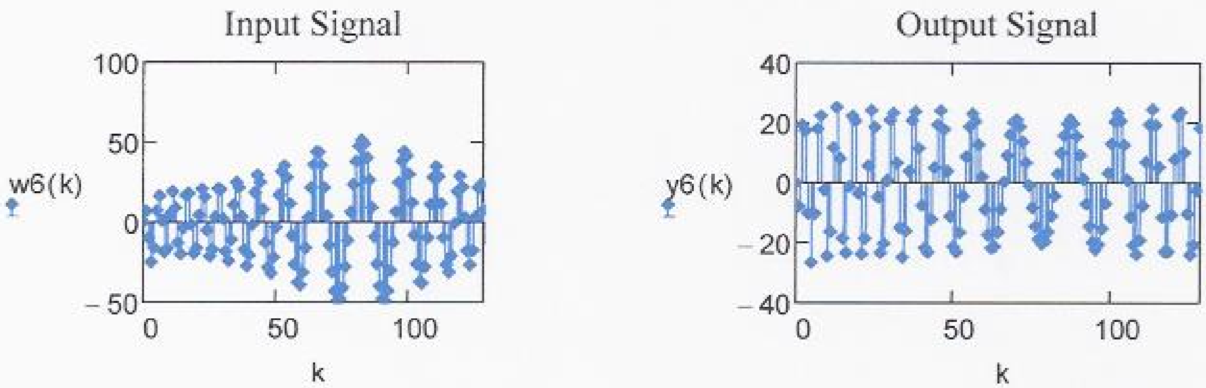
$$x6_i(n) := u\_8_n$$

1) $$w6(n) := \begin{cases} x6_i(n) + v0 \cdot w6(n-1) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

2) $$y6(n) := \begin{cases} u0 \cdot (w6(n) - w6(n-1)) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

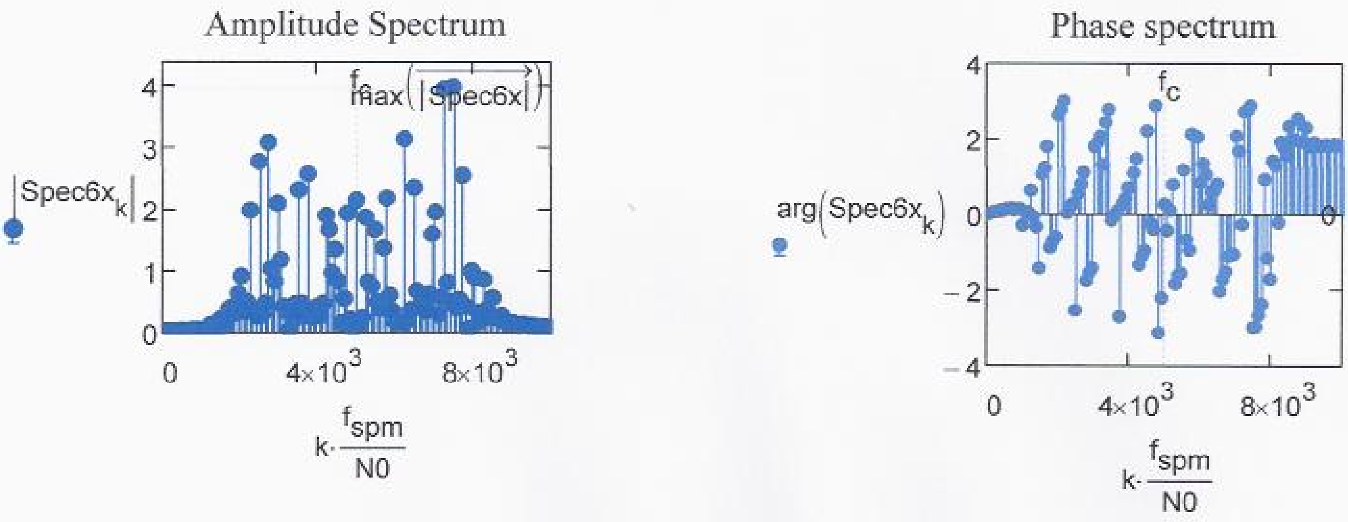
$\omega_c = 0.031 \cdot \frac{Mrads}{sec}$ $\omega_m = 1.571 \times 10^{-3} \cdot \frac{Mrads}{sec}$ $A_0 = -1.5$ $m_f = 11$





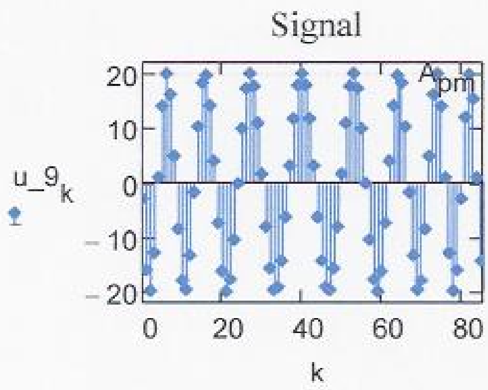
Sampled signal: $v6x_k := y6(k)$

$Spec6x := FFT(v6x)$



4.3.2.7) Sequence of the Phase Modulated signal response.

$$u\_9_k := \frac{v_{pm}(npm_k, \omega_c, \omega_m, A_{pm}, m_p)}{volt}$$



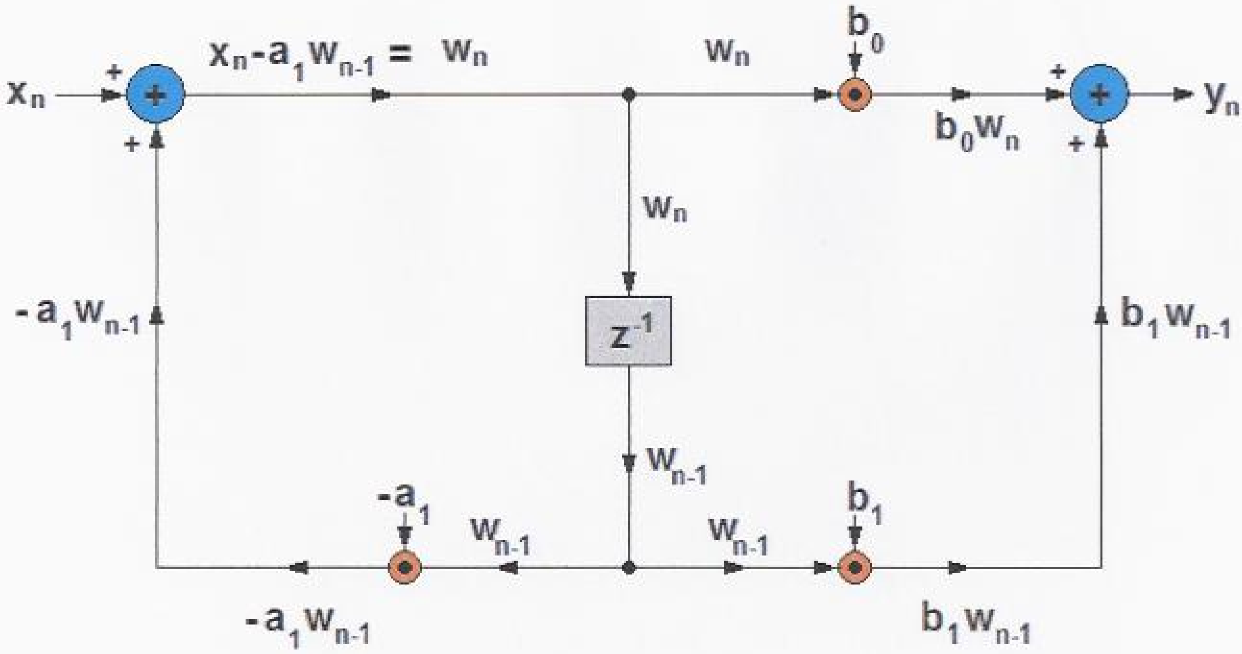
Digital first order High pass filter difference equations:

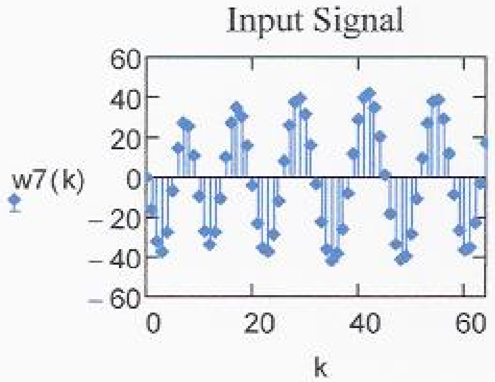
$$x7_i(n) := u\_9_n$$

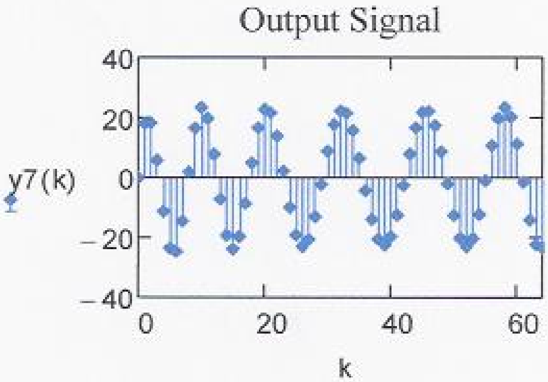
1) $w7(n) := \begin{cases} x7_i(n) + v0 \cdot w7(n-1) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$

2) $y7(n) := \begin{cases} u0 \cdot (w7(n) - w7(n-1)) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$

$\omega_c = 0.031 \cdot \frac{Mrads}{sec}$ $\omega_m = 1.571 \times 10^{-3} \cdot \frac{Mrads}{sec}$ $A_0 = -1.5$ $m_p = 8$

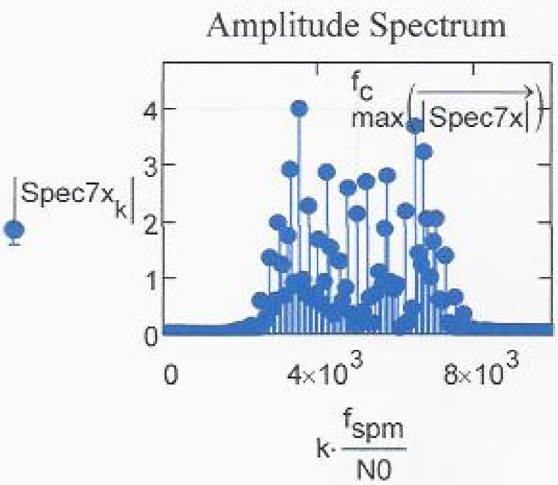


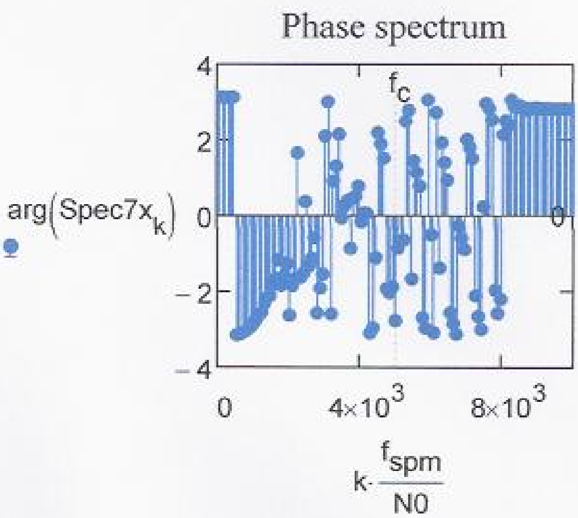




Sampled signal: $v7x_k := y7(k)$

$Spec7x := FFT(v7x)$





4.4
Transfer Function Sequence obtained by an Iterative Algorithm. Convolutional Output

$T_{smp} := T_{smp}$

$\omega_0 := \omega_0 \quad \alpha 0 := \alpha 0 \quad \beta 0 := \beta 0$

$$H_{hp}(z) = u0 \cdot \frac{1 - z^{-1}}{1 - z^{-1} \cdot v0}$$

Numerator degree $N_n := 1$ Denominator degree $M_d := 1$

$N1 := N_n + M_d$ $N0 = 512$ $h1_k := 0$

A generic first order transfer function in the z domain takes this form:

$$H_{hp}(z) = \frac{b_0 + b_1 \cdot z^{-1}}{a_0 + a_1 \cdot z^{-1}}$$

The coefficients of the numerator and denominator can be defined as the elements of two vectors, namely a and b, hence:

Numerator coeff. Denominator coeff.

$n1 := 1 .. N0 - 1$ $b_{n1} := 0.0$ $a_{n1} := 0.0$

$b_0 := u0$ $a_0 := 1$

$b_1 := -u0$ $a_1 := -v0$

and devide the two polinoms by means of the following algorithm:

$N1 = 2 \quad h1_0 := \frac{b_0}{a_0} \qquad h1_{n1} := \frac{1}{a_0} \cdot \left[b_{n1} - \sum_{i=1}^{n1} \left(h1_{n1-i} \cdot a_i \right) \right]$

T. F. Numerator coefficents:

$a^T =$

	0	1	2	3	4	5	6	7
0	1	-0.761	0	0	0	0	0	...

T. F. Denominator coefficents:

$b^T =$

	0	1	2	3	4	5	6	7
0	-1.141	1.141	0	0	0	0	0	...

Sequence of the Impulse response:

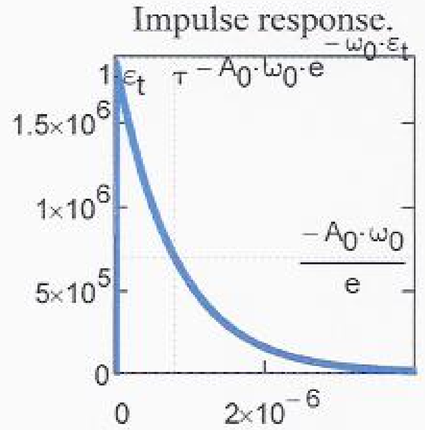
$h1^T =$

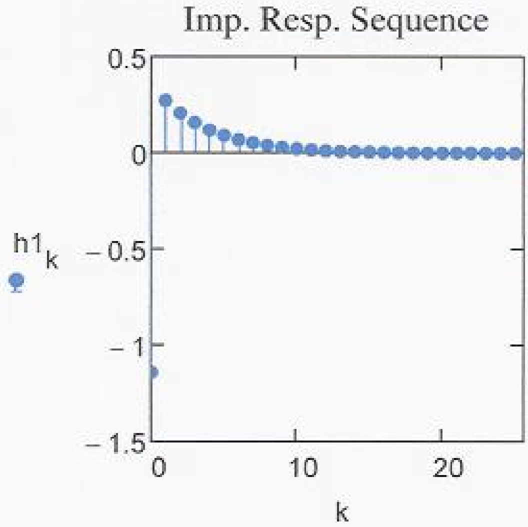
	0	1	2	3	4
0	-1.1414	0.2729	0.2076	0.158	...

Stability (S1<∞): $S1 := \sum_{k=0}^{rows(h1)-1} |h1_k|$ $\qquad S1 = 2.283$

Energy of the sequence h1: $E1 := \sum_{k=0}^{rows(h1)-1} \left(|h1_k|\right)^2$ $\qquad E1 = 1.48$

$t := 0\cdot\tau, \frac{20\cdot\tau}{1000} .. 20\cdot\tau$

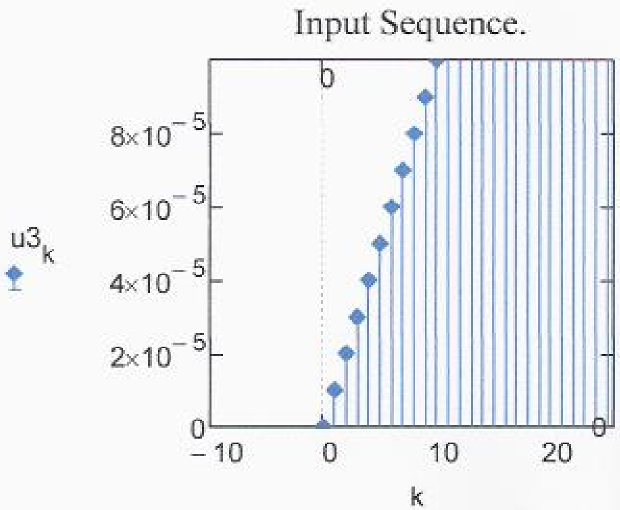


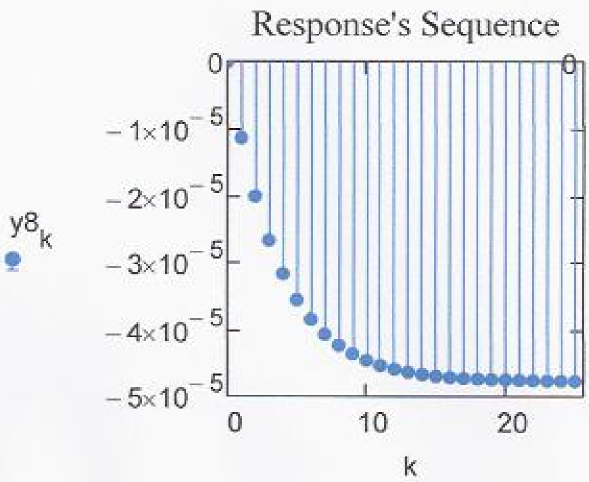


4.4 Transfer Function Sequence obtained by an Iterative Algorithm. Convolutional Output.

4.4.1) Sequence of the voltage ramp response.

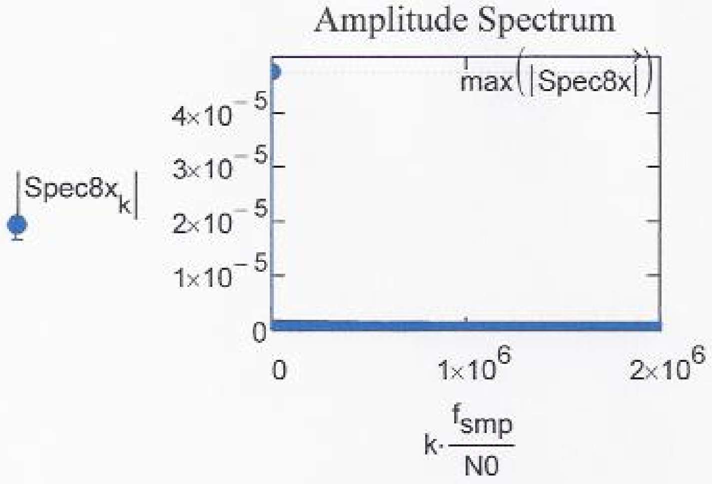
$\nu := n1$ $\qquad y8_\nu := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \ge 0, h1_k \cdot u3_{\nu-k}, 0\right)\right)$

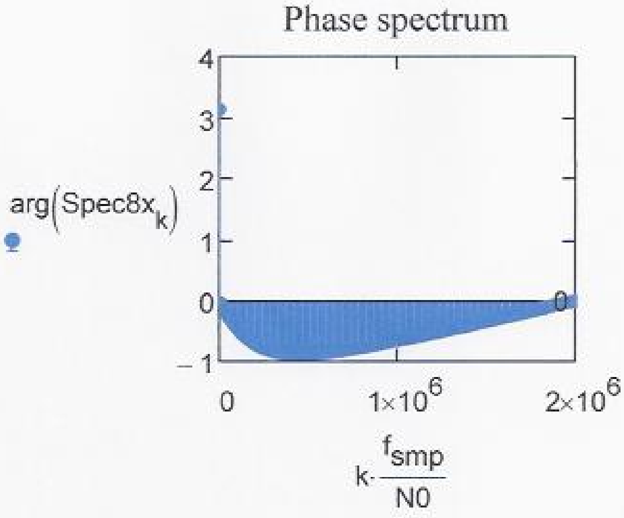




Sampled signal:

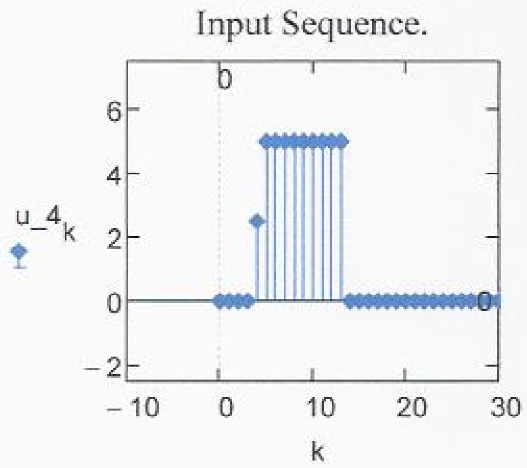
Spec8x := FFT(y8)

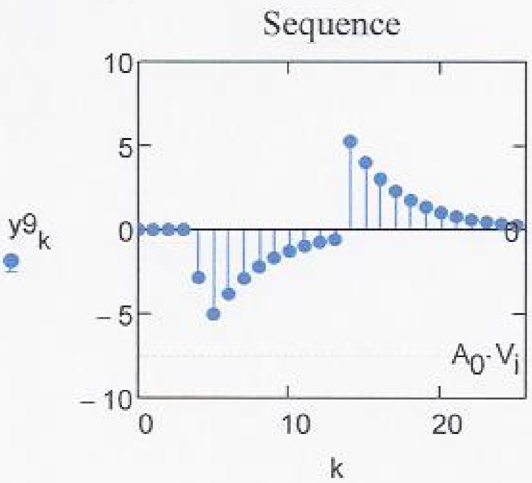




4.4.2) Sequence of the Voltage window response.

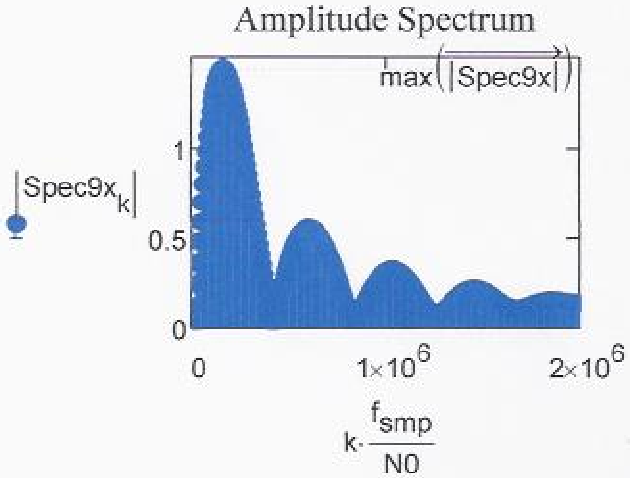
$$y9_\nu := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h1_k \cdot u\_4_{\nu-k}, 0\right)\right)$$

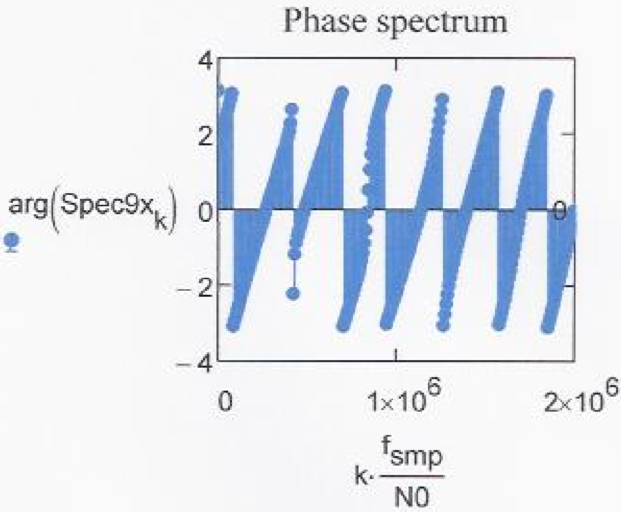




Sampled signal:

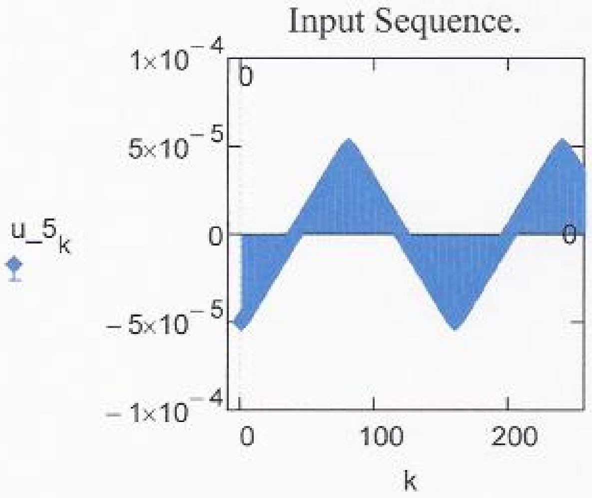
Spec9x := fft(y9)

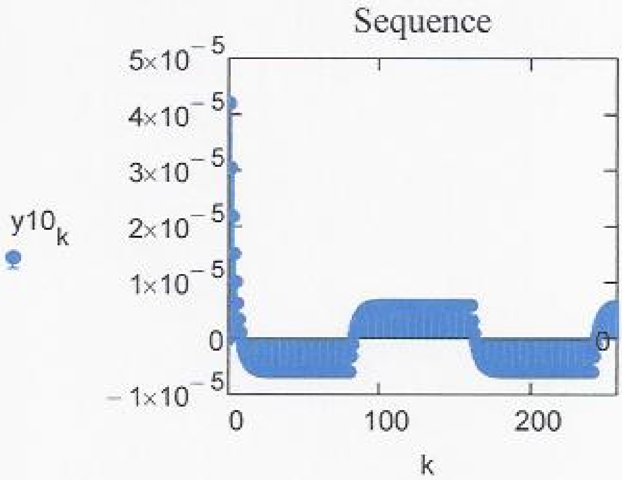




4.4.3) Sequence of the Bipolar Triangular wave response:

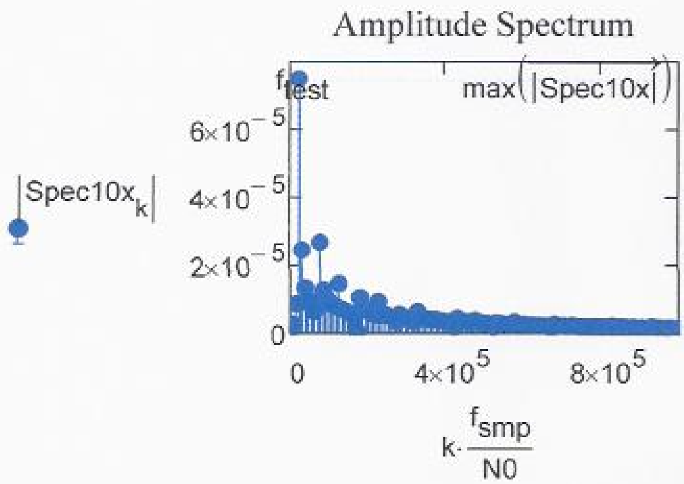
$$y10_{\nu} := \sum_{k=0}^{\nu} \left(\mathrm{if}\left(\nu - k \geq 0, h1_k \cdot u\_5_{\nu-k}, 0 \right) \right)$$

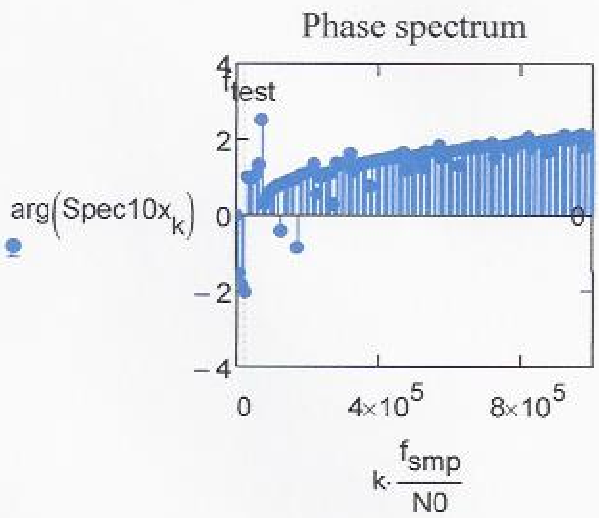




Sampled signal:

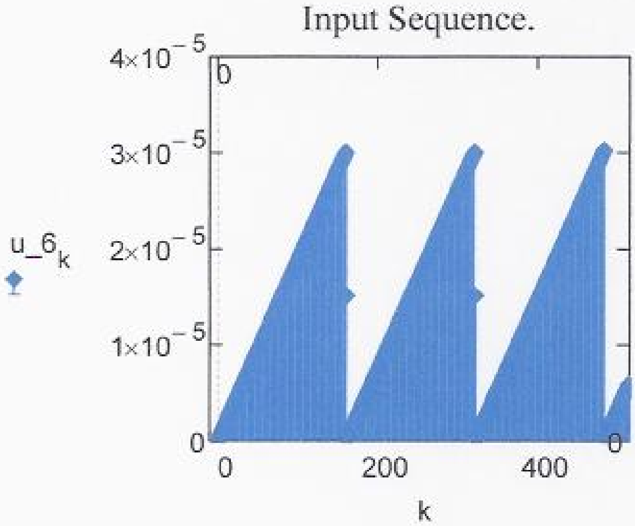
Spec10x := fft(y10)

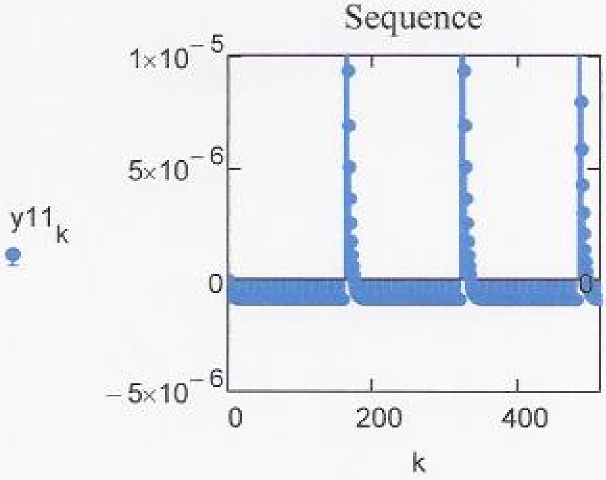




4.4.4) Sequence of the Sawtooth wave response.

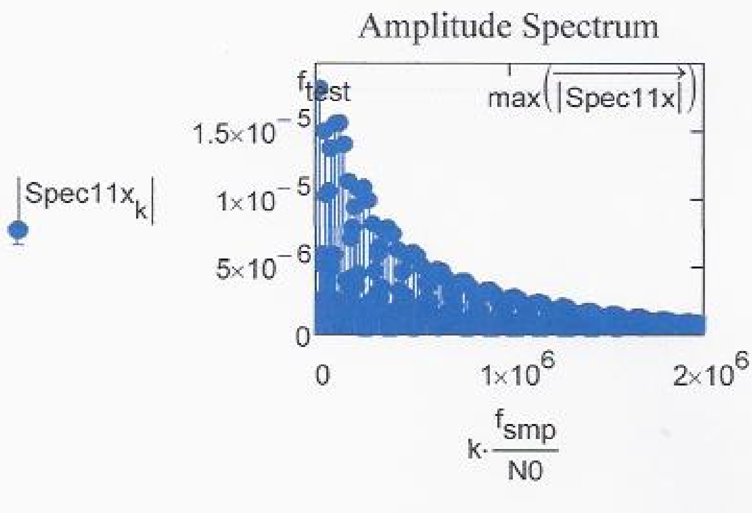
$$y11_{\nu} := \sum_{k=0}^{\nu} \left(\mathrm{if}\left(\nu - k \geq 0, h1_k \cdot u\_6_{\nu-k}, 0\right)\right)$$

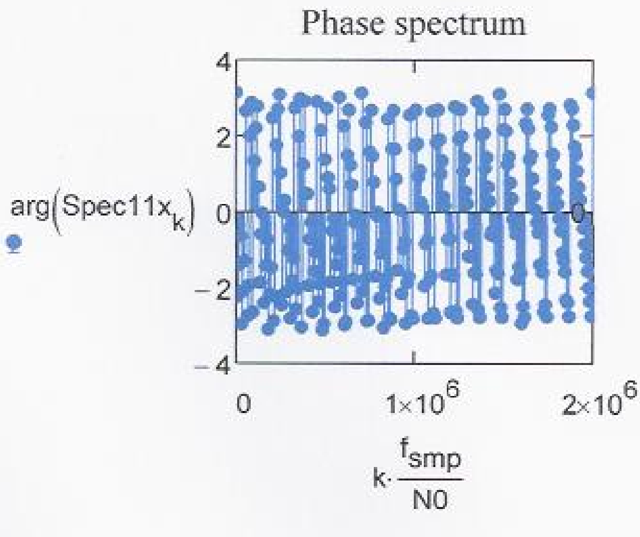




Sampled signal:

$Spec11x := \mathrm{fft}(y11)$

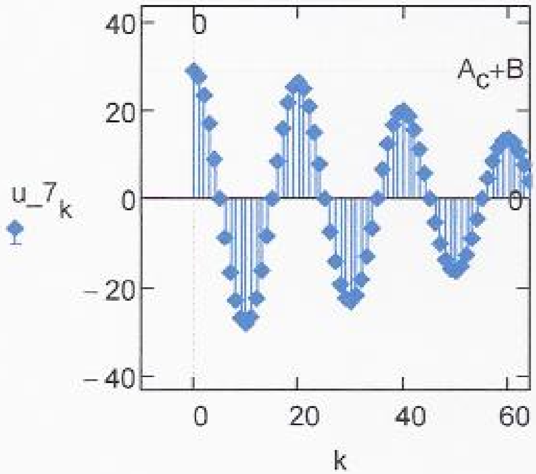




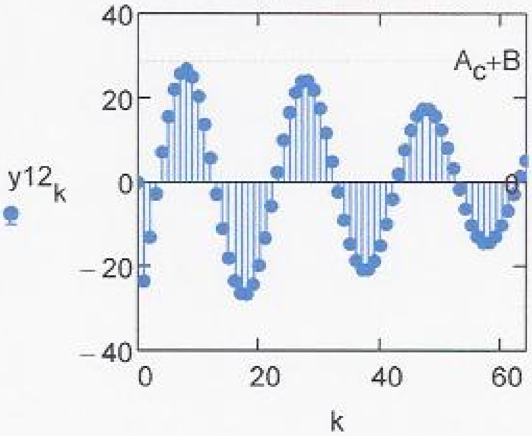
4.4.5) Sequence of the AM Signal response.

$$y12_{\nu} := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \ge 0, h1_k \cdot u\_7_{\nu-k}, 0\right)\right)$$

Input Sequence.



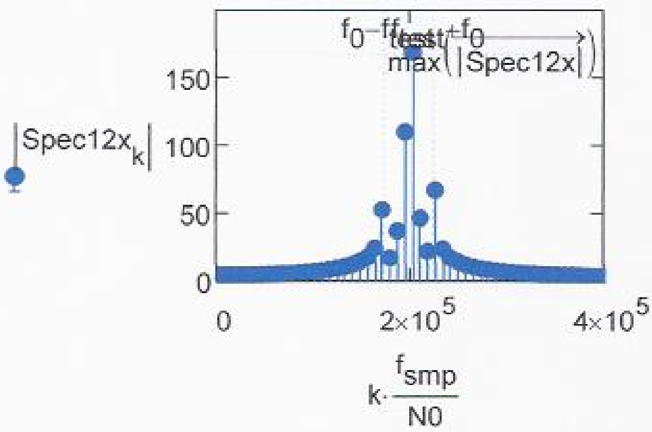
Output Sequence



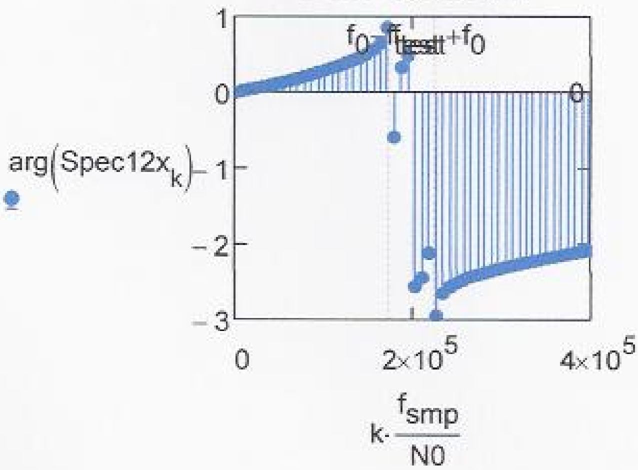
Sampled signal:

Spec12x := fft(y12)

Amplitude Spectrum

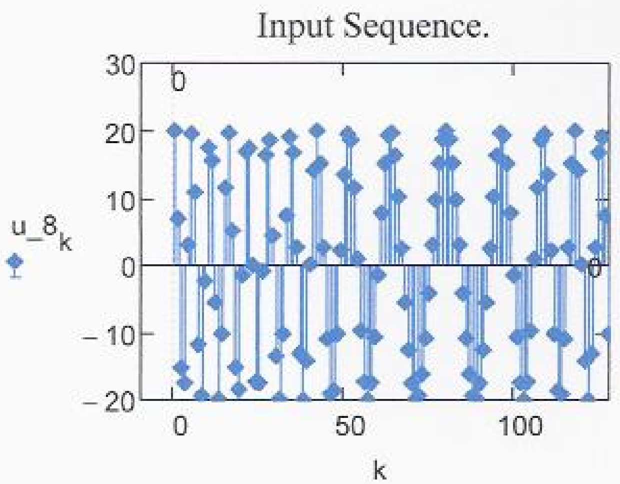


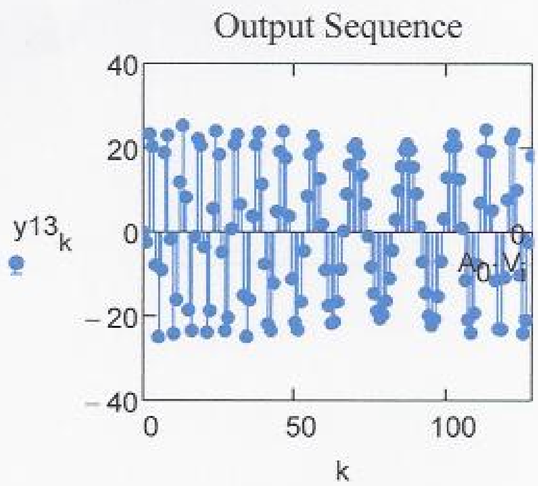
Phase spectrum



4.4.6) Sequence of the Frequency Modulated carrier response.

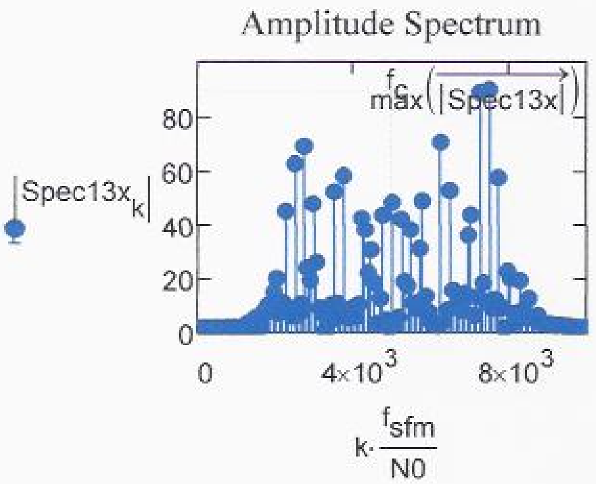
$$y13_{\nu} := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h1_k \cdot u\_8_{\nu-k}, 0\right)\right)$$

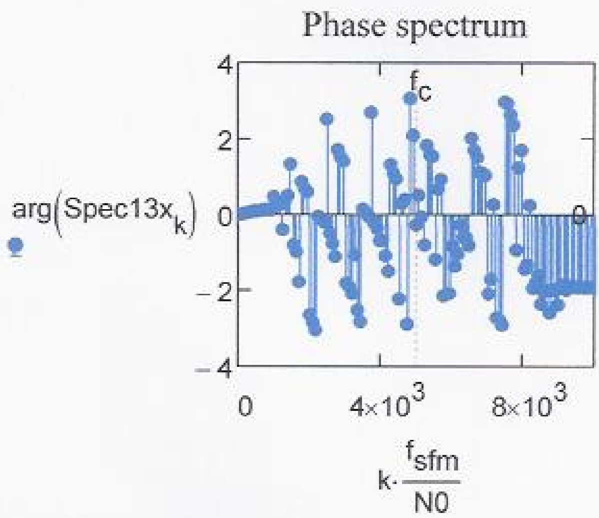




Sampled signal: $m_f = 11$

$Spec13x := \text{fft}(y13)$

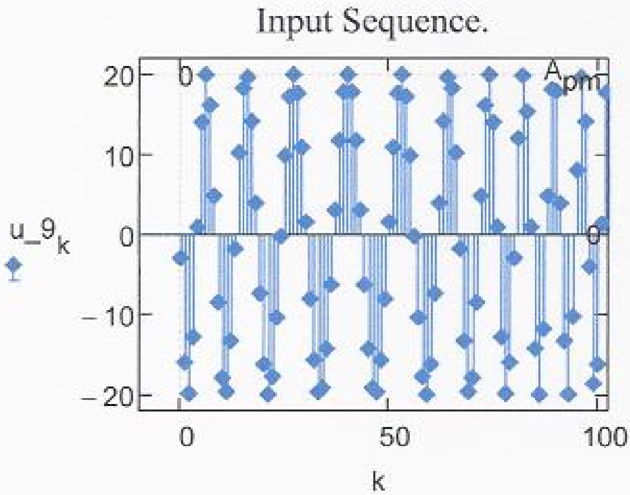


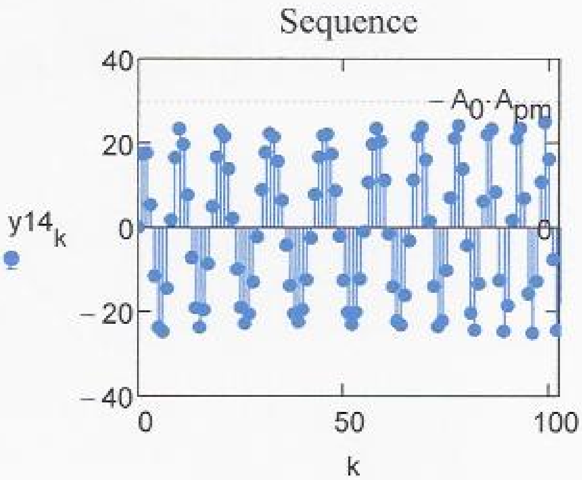


$\omega_c = 0.031 \cdot \frac{Mrads}{sec}$ $\omega_m = 1.571 \times 10^{-3} \cdot \frac{Mrads}{sec}$ $A_0 = -1.5$ $m_f = 11$

4.4.7) Sequence of the Phase Modulated carrier response.

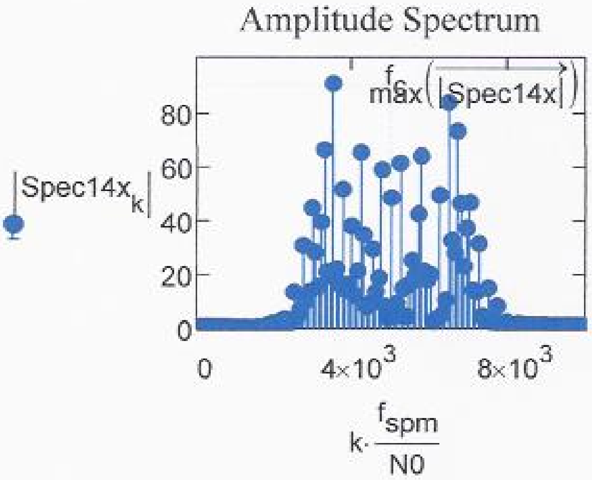
$$y14_\nu := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h1_k \cdot u\_9_{\nu-k}, 0\right)\right)$$

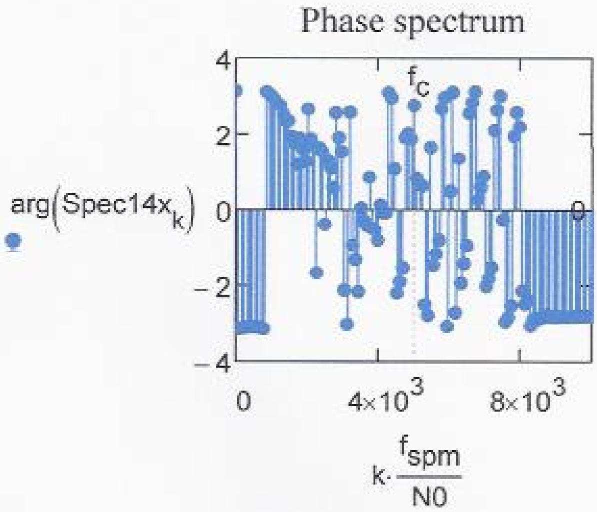




Sampled signal: $m_p = 8$

Spec14x := fft(y14)





$\omega_c = 0.031 \cdot \frac{Mrads}{sec}$ $\omega_m = 1.571 \times 10^{-3} \cdot \frac{Mrads}{sec}$ $A_0 = -1.5$ $m_p = 8$

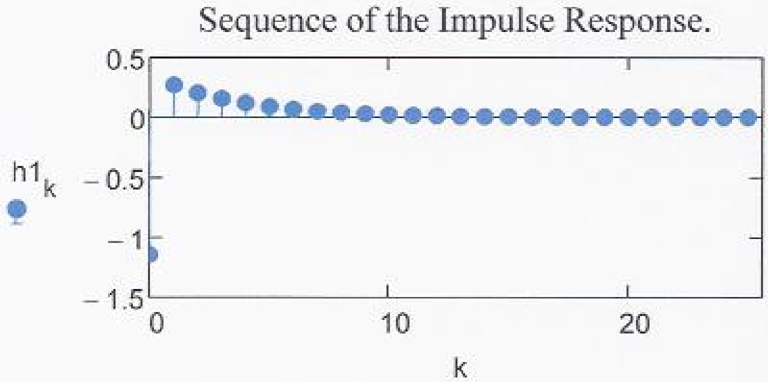
4.5 Search of the output discrete time sequence by a discrete convolution

The sequence corrisponding to the transfer function, can be found using the "**invztrans**" MATHCAD's operator as follows:

$u0 := u0 \qquad v0 := v0 \qquad k := k$

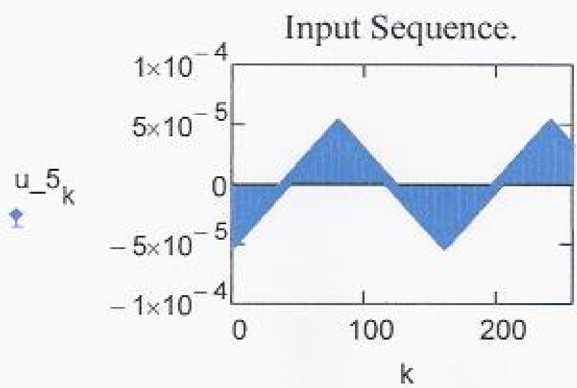
$$h1_k := u0 \cdot \frac{1 - z^{-1}}{1 - z^{-1} \cdot v0} \text{ invztrans}, z, k \rightarrow \frac{u0 \cdot \left(v0^{k+1} + \delta(k,0) - v0^k\right)}{v0}$$

The result is the sequence of the impulse response, here depicted:

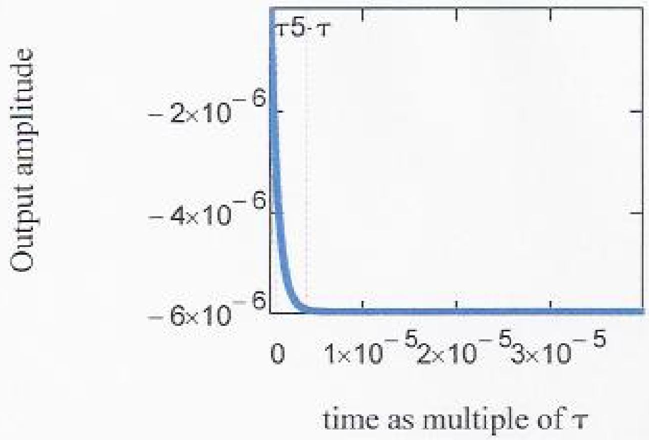


The Output of the Digital System is given by the **discrete convolution** between the discrete time input signal (the discrete time sequence of the *triangular* wave for this example) and the discrete impulse response of the System:

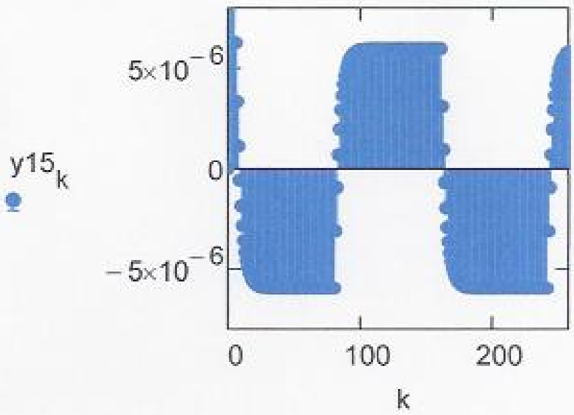
$$y15_{n1} := \sum_{k=0}^{n1} \left(\text{if}\left(n1 - k \geq 0, h1_k \cdot u\_5_{n1-k}, 0\right)\right)$$



Response



Step response.



Knowing the sequences of any input and of the impulse response and the relative Z transforms, the **z-antitransfom of the product** of the two z functions can be determined , it corrisponds to the convolution of the two sequences, as follows:

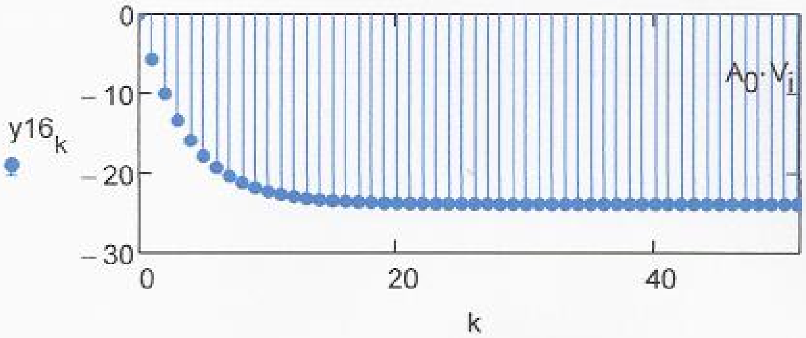
$$X_{hp}(z) := \sum_{n=0}^{N1-1} \left(u1_n \cdot z^{-n}\right) \qquad H_{hps}(z) := \sum_{n=0}^{N0-1} \left(h1_n \cdot z^{-n}\right) \qquad Y_{hp}(z) := H_{hps}(z) \cdot X_{hp}(z)$$

input signal: $V_i := V_i\ ,\ n := n\ ,\qquad V_i \cdot n \text{ ztrans} \rightarrow \dfrac{V_i \cdot z}{(z-1)^2}$

System output corresponding to the z-antitransform of the product:

$$y16_k := u0 \cdot \frac{1-z^{-1}}{1-z^{-1} \cdot v0} \cdot \frac{V_i \cdot z}{(z-1)^2} \left| \begin{array}{l} \text{invztrans}, z, k \\ \text{simplify} \end{array} \right. \rightarrow \frac{V_i \cdot u0 \cdot \left(v0^k - 1\right)}{v0 - 1}$$

Sequence of the ramp response.



sampling frequency: $f_{smp} = 4 \cdot MHz$

sampling period: $T_{smp} = 250 \cdot ns$

sampling angular frequency: $\omega_{smp} = 25.133 \cdot \dfrac{Mrads}{sec}$

$T_{test} = 40\cdot\mu s$ $\quad t := T_{test}\cdot 0, T_{test}\cdot 0 + \frac{T_{test}}{100} .. 10\cdot T_{test}$ $\quad n_k := \frac{k}{f_{smp}}$

Example: Sinusoidal input:

Z transform of the input signal: $\omega := \omega_{test}$ $\quad \omega = 0.157\cdot\frac{Mrads}{sec}$ $\quad \Delta T = 39.063\cdot ns$

$\Delta T := \Delta T \quad \omega := \omega \quad V_i = 5V$

$n := n \quad V_i := V_i \quad V_i\cdot\sin(\omega\cdot n\cdot\Delta T)$ ztrans $\rightarrow \frac{V_i\cdot z\cdot\sin(\omega\cdot\Delta T)}{z^2 - 2\cdot\cos(\omega\cdot\Delta T)\cdot z + 1}$

We place: $K2 := \sin(\Delta T\cdot\omega)$ $\quad \cos(\Delta T\cdot\omega) = \sqrt{1-K2^2}$ $\quad \sqrt{1-K2^2} = 1$

$K2 := K2$

$$poles1 := 1 - 2\cdot\sqrt{1-K2^2}\cdot z^{-1} + z^{-2} \text{ solve}, z \rightarrow \begin{pmatrix} \sqrt{1-K2^2} + K2\cdot i \\ \sqrt{1-K2^2} - K2\cdot i \end{pmatrix}$$

$p1_0 := poles1_0 \quad p1_0 = 1 + 6.136i \times 10^{-3} \quad p1_1 := poles1_1 \quad p1_1 = 1 - 6.136i \times 10^{-3}$

$$\frac{V_i\cdot z\cdot K2}{z^2 - 2\cdot\sqrt{1-K2^2}\cdot z + 1} = \frac{K2\cdot V_i\cdot z}{(p1_0 - z)\cdot(p1_0 - \overline{p1_0})} - \frac{K2\cdot V_i\cdot z}{(p1_0 - \overline{p1_0})\cdot(z - \overline{p1_0})}$$

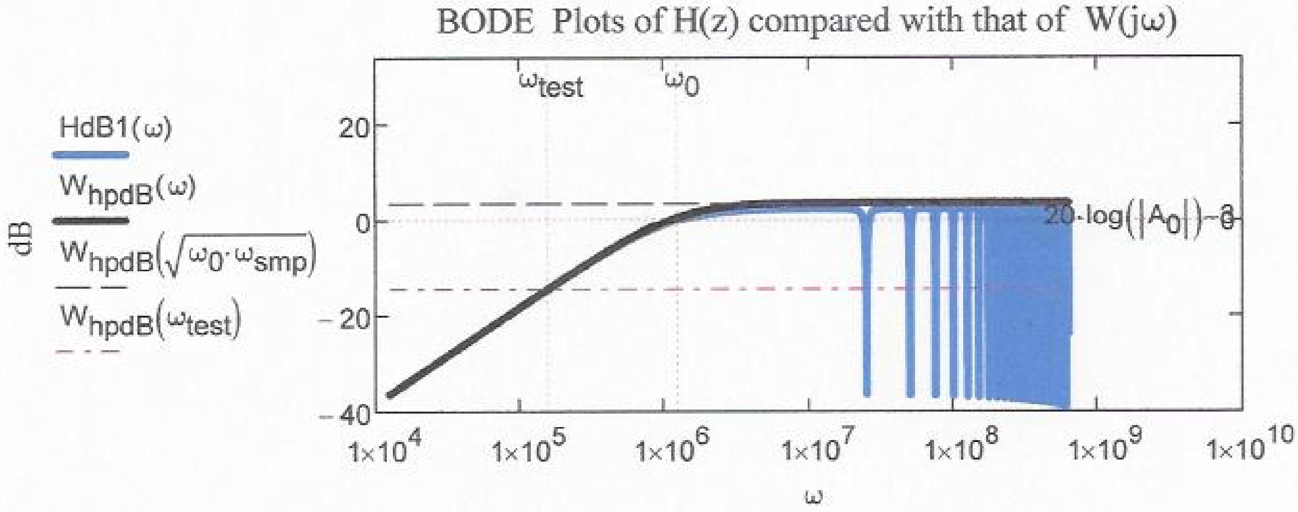
Computing the corresponding sequence the result returned for the symbolic operation is too large to be displayed, but can be used for other calculations if assigned to a function.

$$y17_k := u0\cdot\frac{1 - z^{-1}}{1 - z^{-1}\cdot v0}\cdot\left[\frac{K2\cdot V_i\cdot z}{(p1_0 - z)\cdot(p1_0 - \overline{p1_0})} \cdots + \frac{-K2\cdot V_i\cdot z}{(p1_0 - \overline{p1_0})\cdot(z - \overline{p1_0})}\right] \begin{vmatrix} \text{invztrans}, z, \text{using}, n = k \\ \text{simplify} \end{vmatrix} \rightarrow$$

$$y17_k := K2\cdot V_i\cdot u0\cdot\left[\frac{\left[(p1_0)^k - p1_0\cdot(p1_0)^k + v0\cdot v0^k - v0^k\right]}{(\overline{p1_0} - p1_0)\cdot(v0 - p1_0)} \cdots + (-1)\cdot\frac{\left[\overline{p1_0}\cdot(\overline{p1_0})^k - (\overline{p1_0})^k - v0\cdot v0^k + v0^k\right]}{(v0 - \overline{p1_0})\cdot(\overline{p1_0} - p1_0)}\right]$$

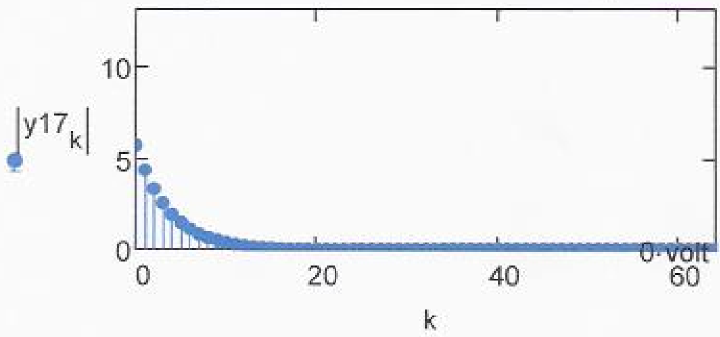
$A_0 = -1.5 \qquad 20\cdot\log\left(\left|W_{hp}\left(j\cdot\sqrt{\omega_0\cdot\omega_{smp}}\right)\right|\right) = 3.31$

$$\omega := \frac{\omega_0}{U}, \frac{\omega_0}{U} + \frac{\omega_{smp}\cdot U - \frac{\omega_0}{U}}{4\cdot U^2} .. \frac{U}{4}\cdot\omega_{smp}$$



ω

Sequence of the sinusoidal response.

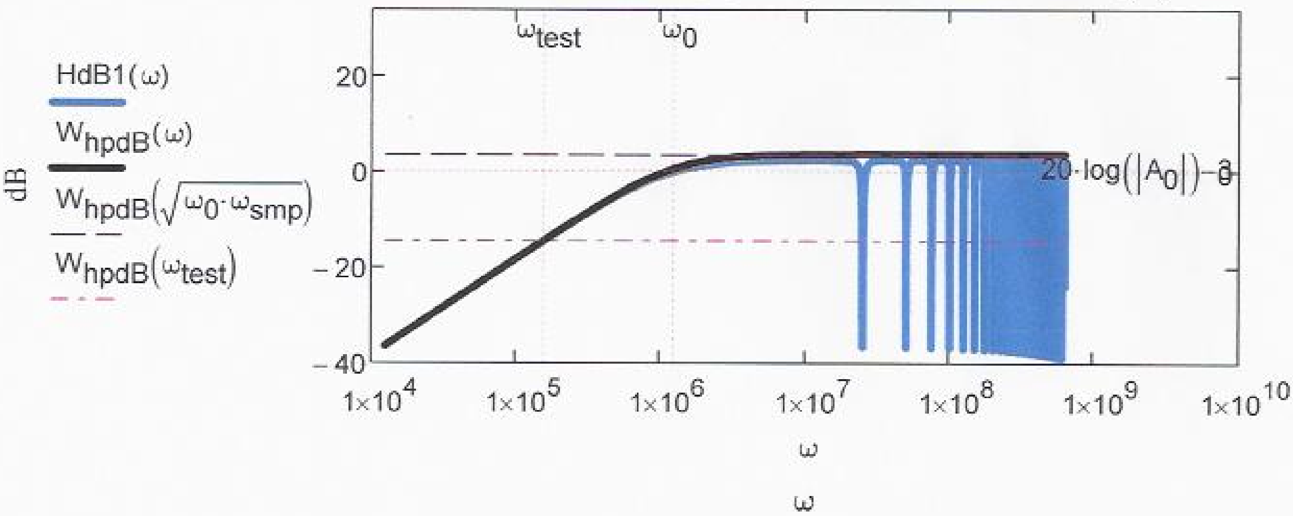


$$1.1 \cdot \left| \frac{K2 \cdot V_i \cdot u0}{\left(\overline{p1_0} - p1_0\right) \cdot \left(v0 - p1_0\right)} \right| = 13.127\,V$$

$A_0 = -1.5$ $\qquad 20 \cdot \log\left(\left|W_{hp}\left(j \cdot \sqrt{\omega_0 \cdot \omega_{smp}}\right)\right|\right) = 3.31$

$$\omega := \frac{\omega_0}{U}, \frac{\omega_0}{U} + \frac{\omega_{smp} \cdot U - \frac{\omega_0}{U}}{4 \cdot U^2} .. \frac{U}{4} \cdot \omega_{smp}$$

BODE Plots of H(z) compared with that of W(jω)



4.6
The bilinear transformation

4.6.1) Z-transfer function of the I° Order High Pass Digital Filter.

$$s = \frac{2}{T_{smp}} \cdot \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right),$$

the amplitude response of the analog function is preserved.

$$A_0 := A_0 \qquad \omega_0 := \omega_0 \qquad \omega_{smp} := \omega_{smp}$$

$$\omega_{smp} = \frac{2 \cdot \pi}{T_{smp}}$$

$$\frac{2}{T_{smp}} = \frac{\omega_{smp}}{\pi}$$

$$H11_t(z) := \frac{A_0 \cdot s}{s + \omega_0} \left| \begin{array}{l} \text{substitute}, s = \frac{\omega_{smp}}{\pi} \cdot \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) \\ \text{collect}, z \\ \text{collect}, A_0 \cdot \omega_{smp} \end{array} \right. \rightarrow$$

$$\frac{1 - z^{-1}}{\omega_{smp} + \pi \cdot \omega_0 - z^{-1} \cdot (\omega_{smp} - \pi \cdot \omega_0)} \cdot (A_0 \cdot \omega_{smp}) = \frac{1 - z^{-1}}{1 - z^{-1} \cdot \left(\frac{\omega_{smp} - \pi \cdot \omega_0}{\omega_{smp} + \pi \cdot \omega_0} \right)} \cdot \left(\frac{A_0 \cdot \omega_{smp}}{\omega_{smp} + \pi \cdot \omega_0} \right)$$

$$H11(z) = \frac{1 - z^{-1}}{1 - \left(\frac{\omega_{smp} - \pi \cdot \omega_0}{\omega_{smp} + \pi \cdot \omega_0} \right) \cdot z^{-1}} \cdot \left[\frac{A_0 \cdot \omega_{smp}}{(\omega_{smp} + \pi \cdot \omega_0)} \right]$$

The following new parameters are necessary for the design of the digital filter:

$$\delta 0 := \frac{\omega_{smp} - \pi \cdot \omega_0}{\omega_{smp} + \pi \cdot \omega_0}, \qquad \chi 0 := \frac{A_0 \cdot \omega_{smp}}{(\omega_{smp} + \pi \cdot \omega_0)},$$

$$\omega_0 = 1.257 \times 10^3 \cdot \frac{\text{krads}}{\text{s}}, \qquad \delta 0 = 0.728489504, \qquad \chi 0 = -1.2963671278$$

the new t. f. is:

$$H11(z) := \chi 0 \cdot \frac{1 - z^{-1}}{1 - \delta 0 \cdot z^{-1}}$$

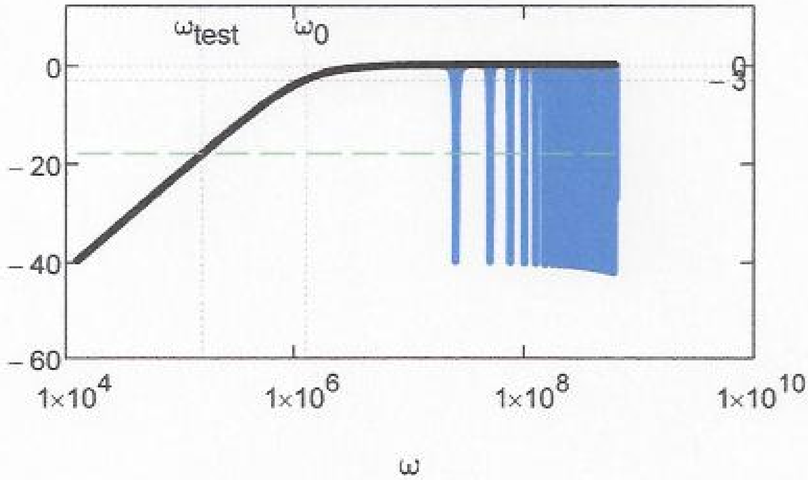
$$\delta 0 := \delta 0 \qquad \chi 0 := \chi 0$$

Z T. Initial value theorem: $\lim_{z \to \infty} \left(\chi 0 \cdot \frac{1 - z^{-1}}{1 - \delta 0 \cdot z^{-1}} \right) \to \chi 0$ $\qquad \chi 0 = -1.296$

Z T. Final value theorem: $\lim_{z \to 0} \left(\chi 0 \cdot \frac{1 - z^{-1}}{1 - \delta 0 \cdot z^{-1}} \right) \to \begin{cases} \frac{\chi 0}{\delta 0} & \text{if } \delta 0 \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$

$$H11NdB(\omega) := 20 \cdot \log\left(\frac{\left| H11\left(e^{j \cdot \omega \cdot T_{smp}}\right) \right|}{|A_0|} \right) \qquad WhpNdB(\omega) := 20 \cdot \log\left(\frac{\left| W_{hp}(j \cdot \omega) \right|}{|A_0|} \right)$$

Normalized Magnitude of H11(z) and W(jω)



4.6.2) Difference equations (High Pass filter(I°order)). Canonical form

$$H11(z) = \chi 0 \cdot \frac{1 - z^{-1}}{1 - \delta 0 \cdot z^{-1}} = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$\frac{Y(z)}{W(z)} = \chi 0 \cdot \left(1 - z^{-1}\right)$$

$$Y(z) = \chi 0 \cdot W(z) - \chi 0 \cdot z^{-1} \cdot W(z)$$

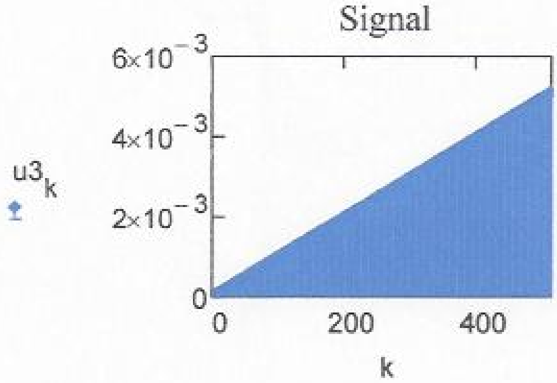
$$y(\nu) = \chi 0 \cdot (w(\nu) - w(\nu - 1))$$

$$\frac{W(z)}{X(z)} = \frac{1}{\left(1 - \delta 0 \cdot z^{-1}\right)}$$

$$X(z) = \left(1 - \delta 0 \cdot z^{-1}\right) \cdot W(z) = W(z) - \delta 0 \cdot z^{-1} \cdot W(z)$$

$$w(n) = x(n) + \delta 0 \cdot w(n - 1)$$

4.6.2.1) Sequence of the voltage ramp response.

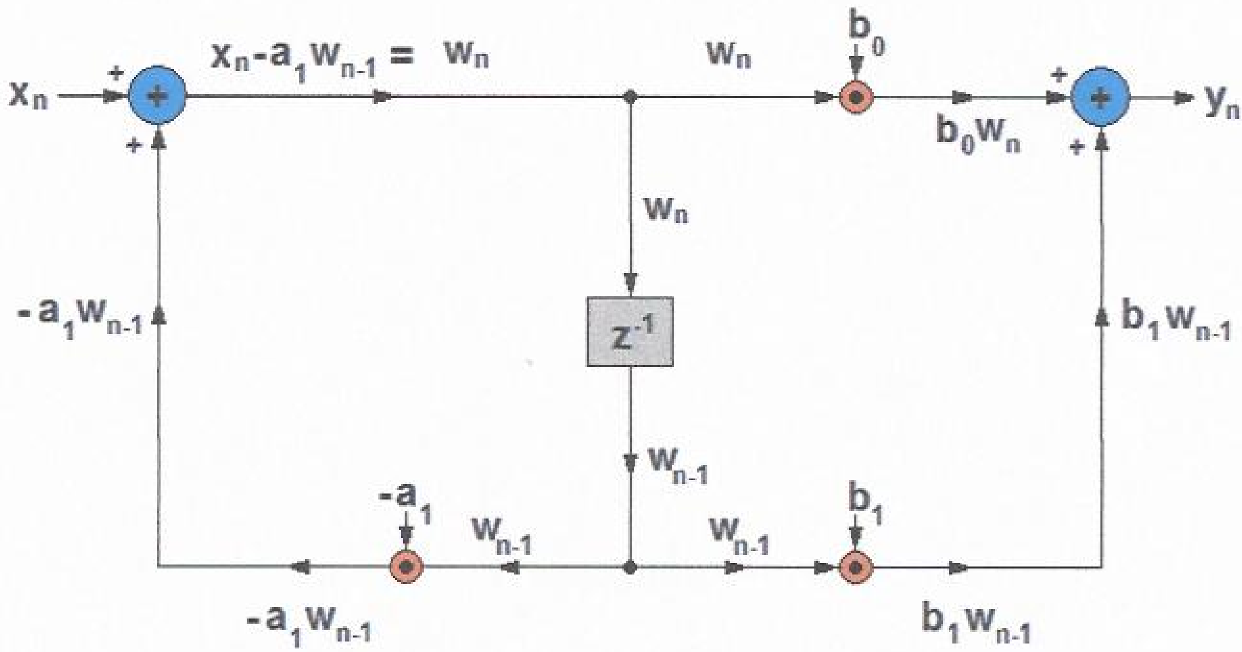


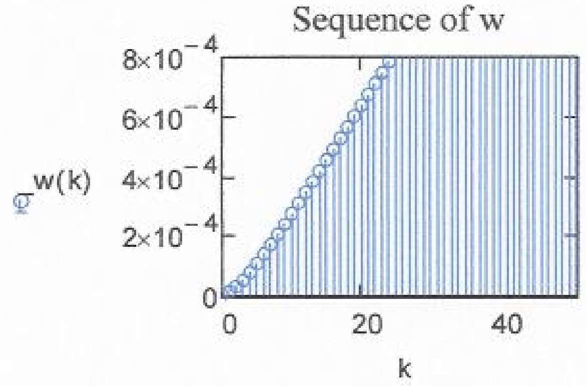
The corresponding set of difference equations:

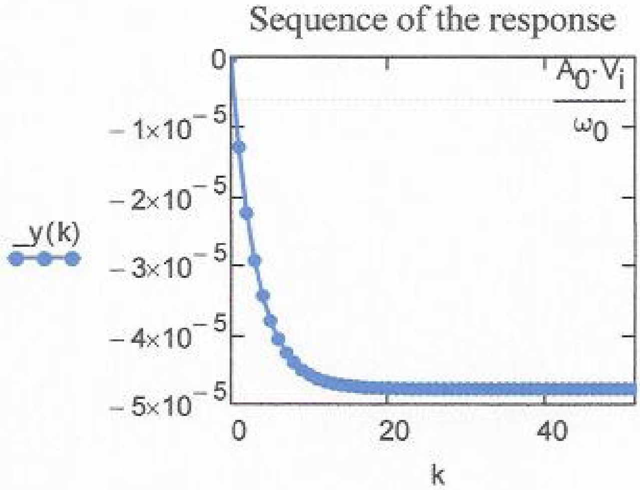
$$v_{ni}(\nu) := \frac{u3_\nu}{volt}$$

1) $$\_w(\nu) := \begin{cases} v_{ni}(\nu) + \delta 0 \cdot \_w(\nu - 1) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$$

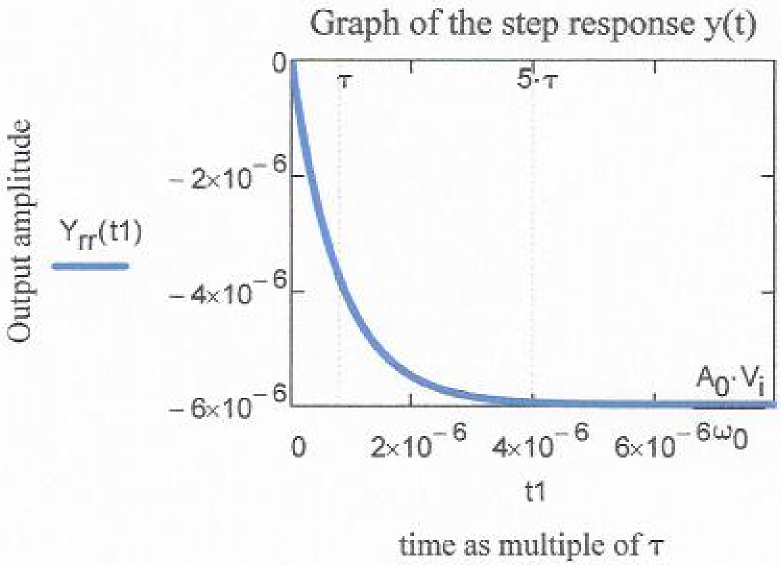
2) $$\_y(\nu) := \begin{cases} [\chi 0 \cdot (\_w(\nu) - \_w(\nu - 1))] & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$$







$$t1 := 0\cdot\tau, 0\cdot\tau + \frac{10\cdot\tau}{1000} .. 10\cdot\tau$$



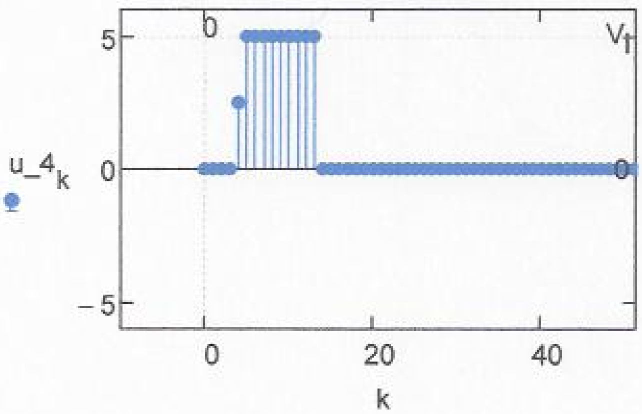
4.6.2.2) Sequence of the Voltage window response.

$T_{test} = 4 \times 10^4 \cdot ns$ $T_{smp} = 250 \cdot ns$ $\tau = 0.796 \cdot \mu s$

Chosen test signal period, $T_{test} = 4 \times 10^4 \cdot ns$ $\frac{1}{T_{test}} = 0.025 \cdot MHz$

Sort pulse sequence of amplitude V_i:

Pulses Sequence.

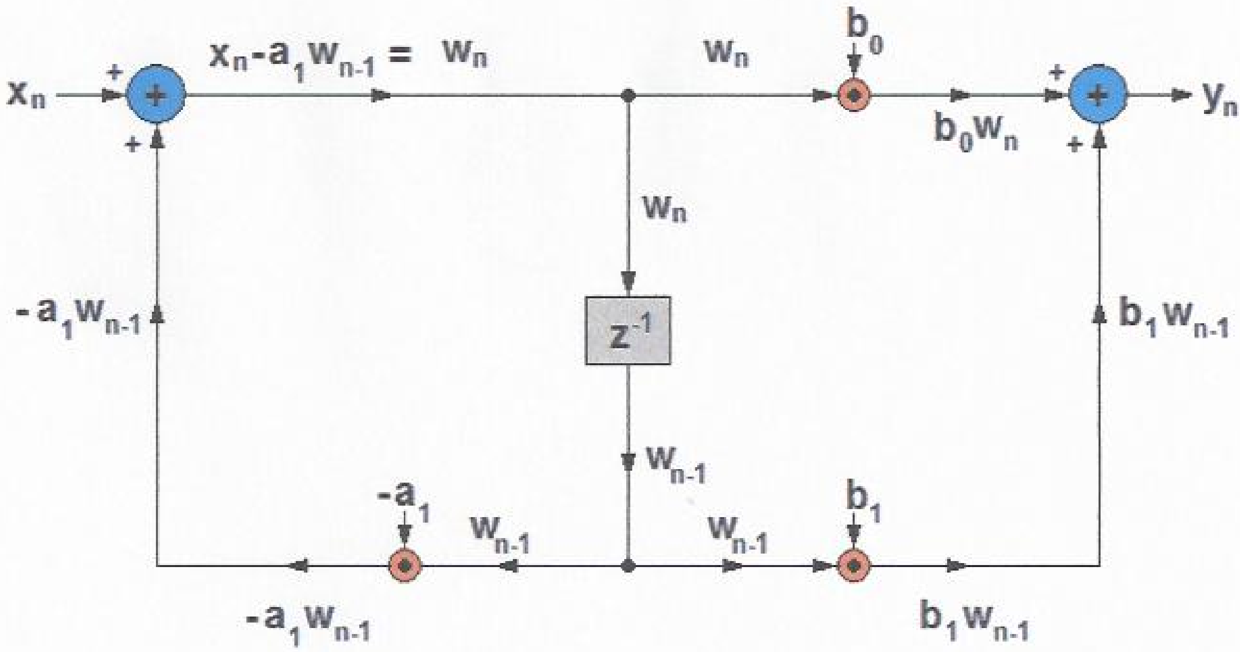


Digital first order High pass filter difference equations:

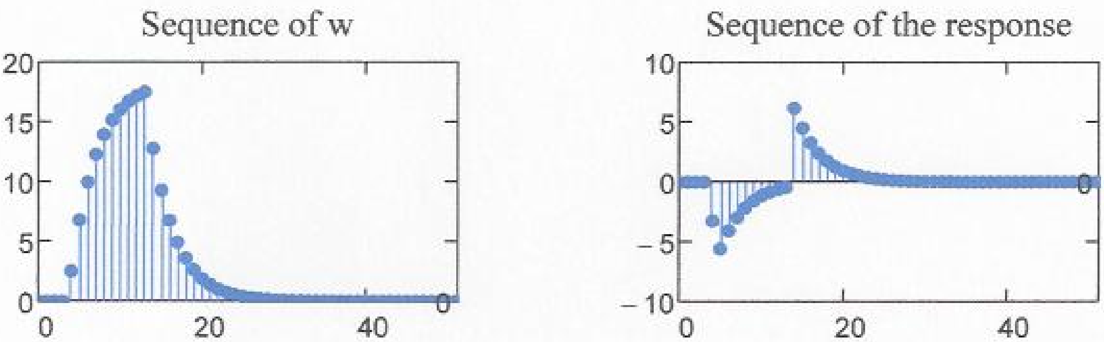
adimensional input signal: $v_{i18}(\nu) := \frac{u\_4_{\nu}}{volt}$

1) $w18(\nu) := \begin{cases} v_{i18}(\nu) + \delta0 \cdot w18(\nu - 1) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$

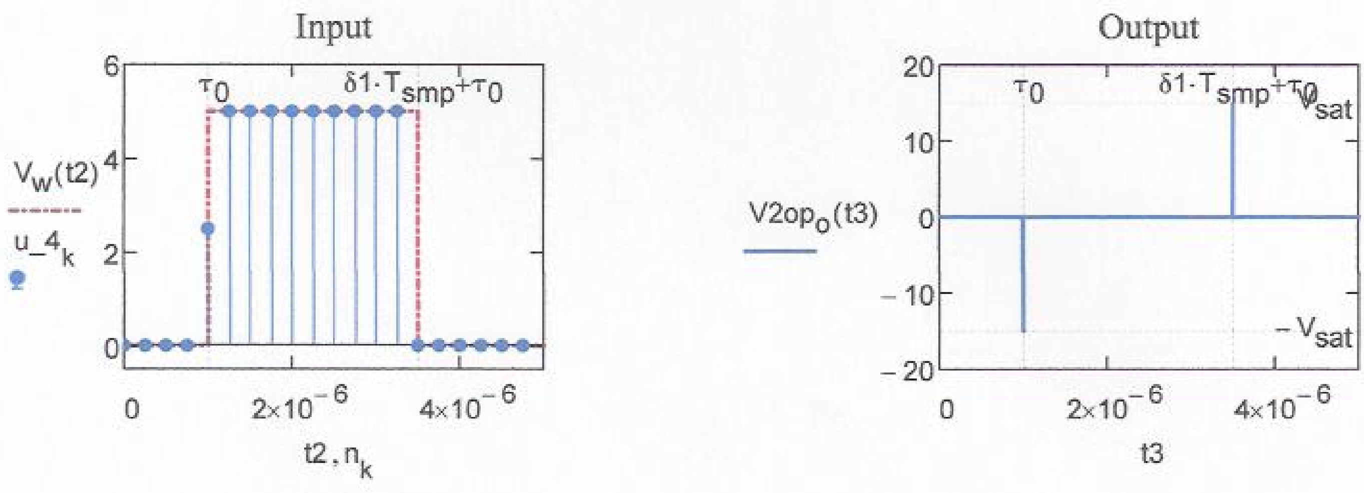
2) $y18(\nu) := \begin{cases} \chi0 \cdot (w18(\nu) - w18(\nu - 1)) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$



N0 = 512

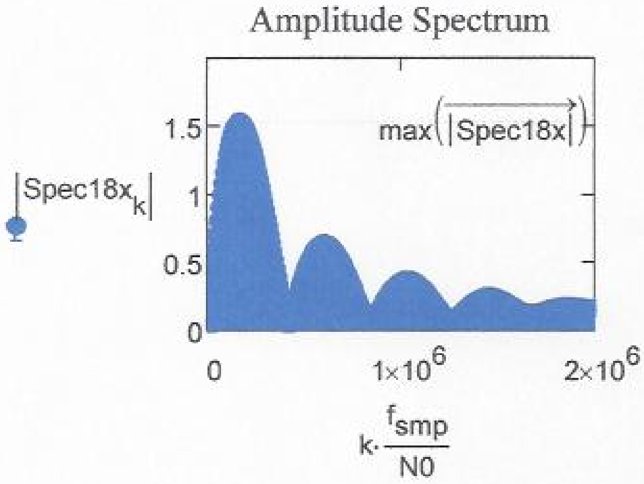


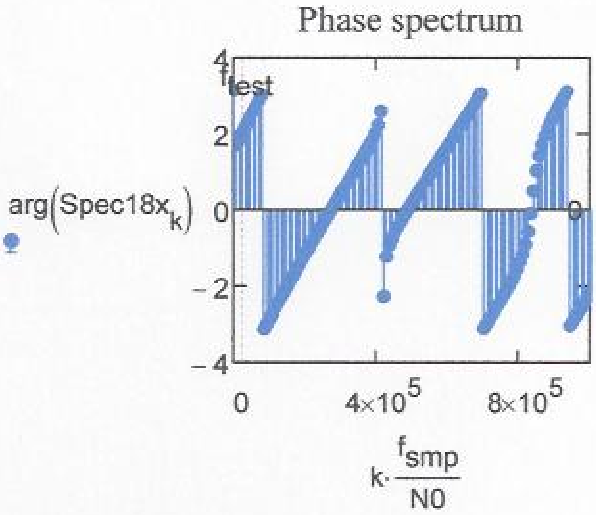
$$t2 := 0, \frac{\tau_0 + 2\cdot(\delta1\cdot T_{smp})}{20000} .. \tau_0 + 2\cdot(\delta1\cdot T_{smp})$$



Sampled signal: $v18x_k := y18(k)$

$Spec18x := fft(v18x)$





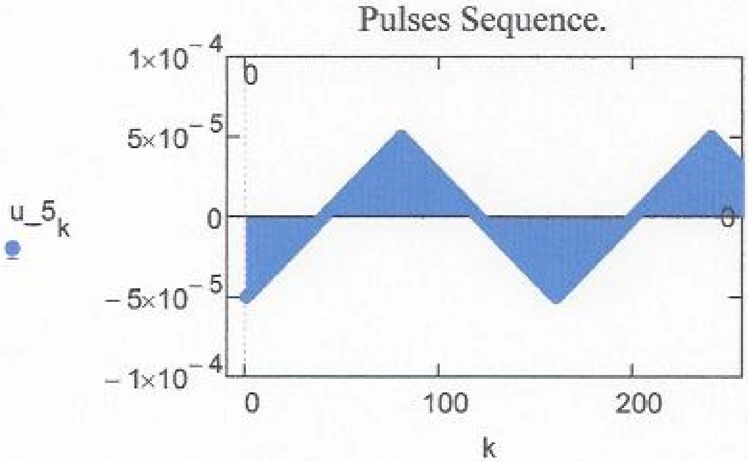
$$\frac{\max(|Spec2|)}{\max(|Spec18x|)} = \blacksquare$$

4.6.2.3) Sequence of the Bipolar Triangular wave response:

$T_{test} = 4 \times 10^4 \cdot ns$ $\quad T_{smp} = 250 \cdot ns$ $\quad \tau = 0.796 \cdot \mu s$

Chosen test signal period, $T_{test} = 4 \times 10^4 \cdot ns$ $\quad \frac{1}{T_{test}} = 0.025 \cdot MHz$

Triangular wave sequence:



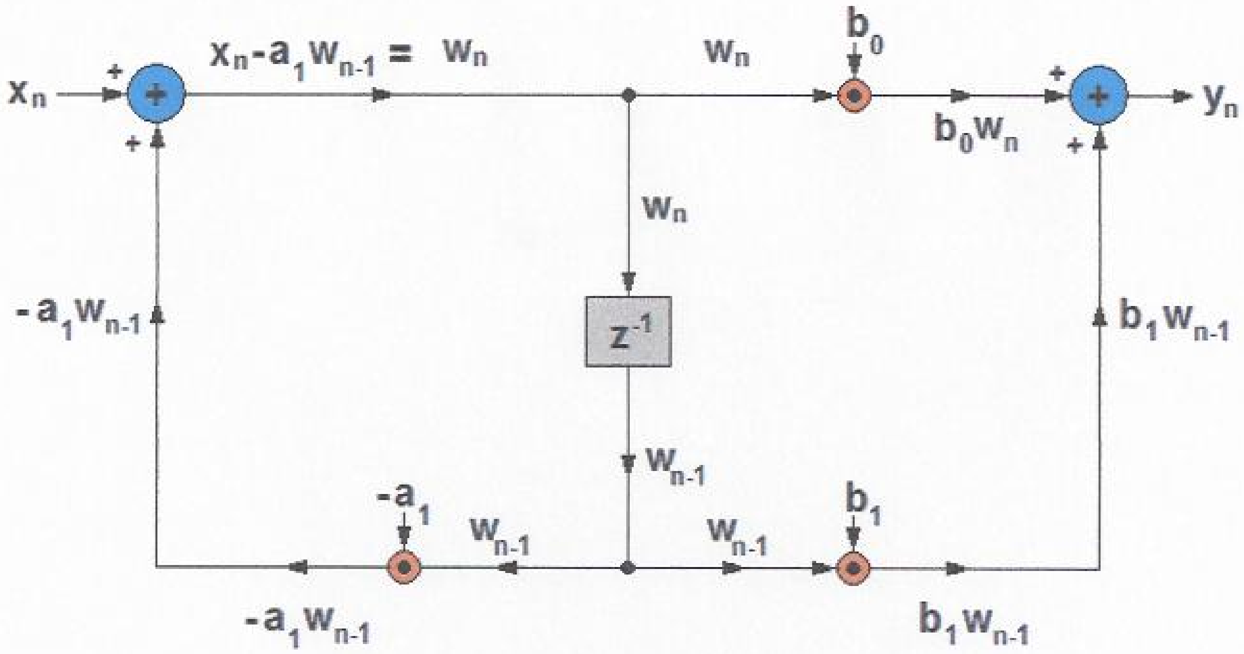
Digital first order High pass filter recurrence relations:

adimensional input signal: $v19_i(\nu) := u\_5_\nu$

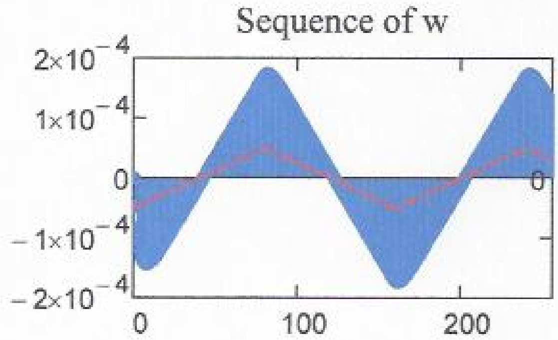
$$\frac{f_{smp}}{f_{test}} = 160$$

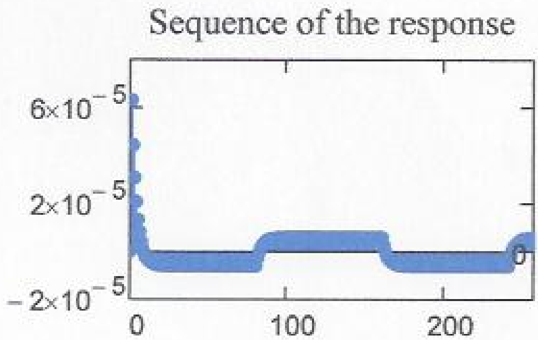
1) $w19(\nu) := \begin{cases} v19_i(\nu) + \delta 0 \cdot w19(\nu - 1) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$

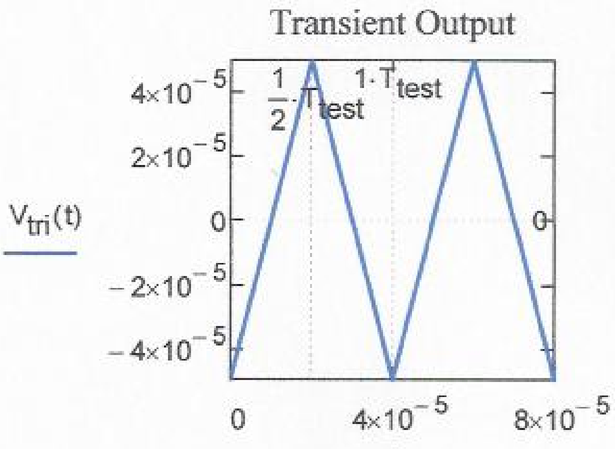
2) $y19(\nu) := \begin{cases} \chi 0 \cdot (w19(\nu) - w19(\nu - 1)) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$



N0 = 512

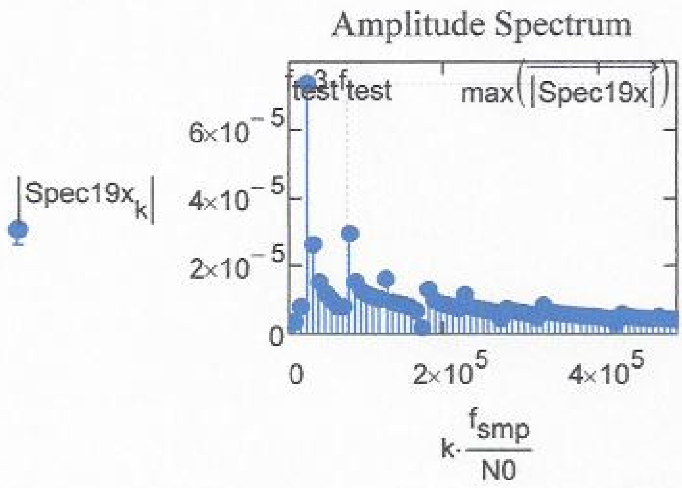


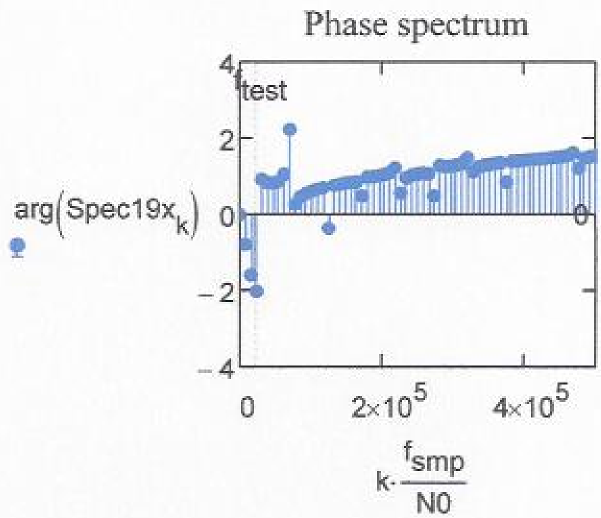




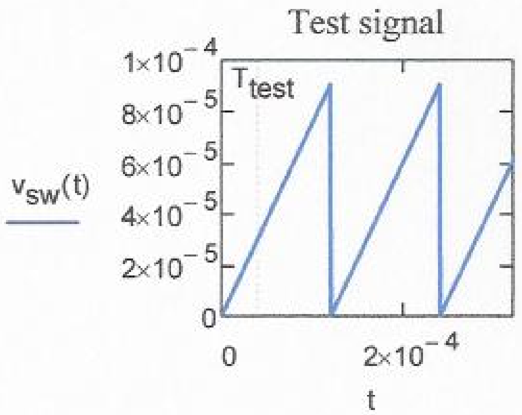
Sampled signal: $v19x_k := y19(k)$

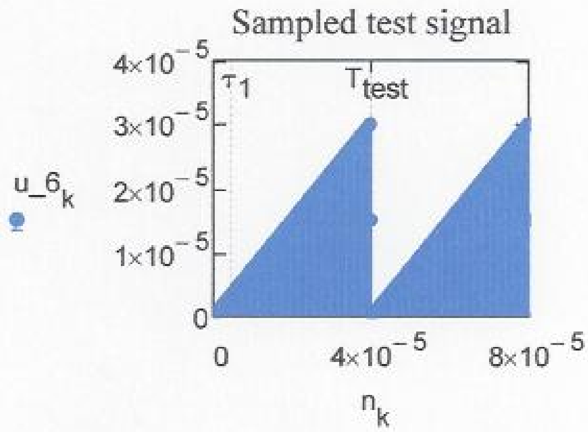
$Spec19x := fft(v19x)$


Amplitude Spectrum
|Spec19x_k|
6×10^-5
4×10^-5
2×10^-5
0
max(|Spec19x|)
f_test
3·f_test
0
2×10^5
4×10^5
k·f_smp/N0


Phase spectrum
arg(Spec19x_k)
4
2
0
−2
−4
f_test
0
2×10^5
4×10^5
k·f_smp/N0

4.6.2.4) Sequence of the Sawtooth wave response.





Step sequence of amplitude V_i:

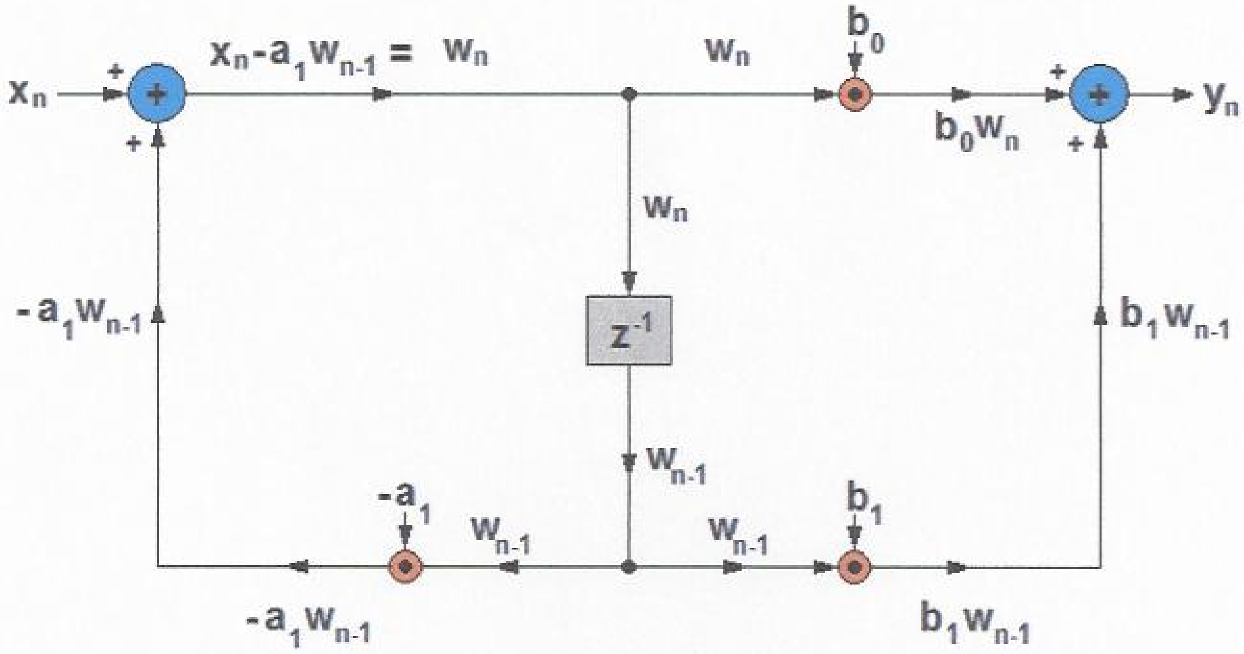
Digital first order High pass filter recurrence relations:

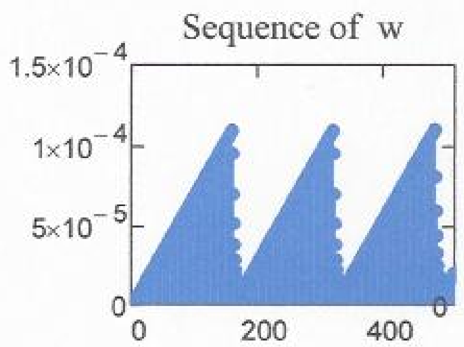
adimensional input signal: $v20_i(\nu) := u\_6_\nu$

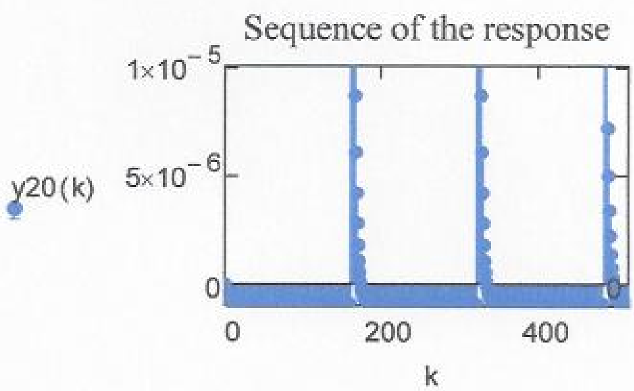
$$\frac{f_{smp}}{f_{test}} = 160$$

1) $w20(\nu) := \begin{cases} v20_i(\nu) + \delta0 \cdot w20(\nu - 1) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$

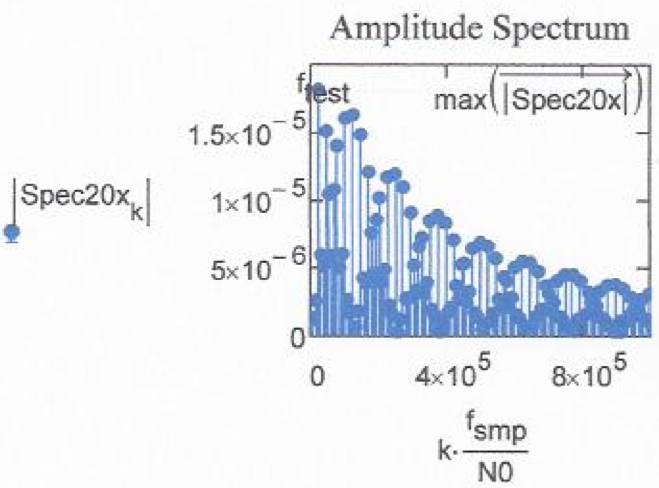
2) $y20(\nu) := \begin{cases} \chi0 \cdot (w20(\nu) - w20(\nu - 1)) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$

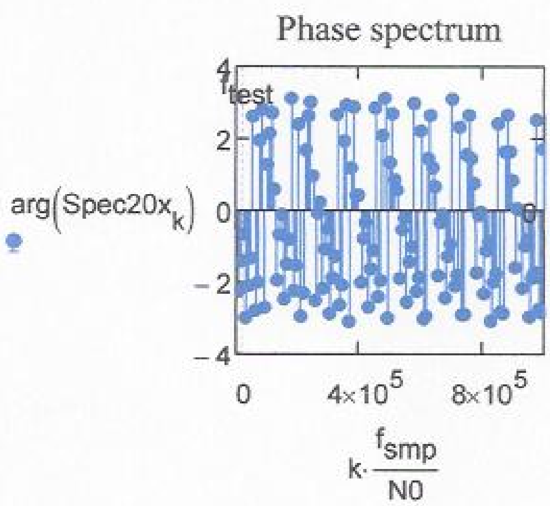






Sampled signal: $v20x_k := y20(k)$ Spec20x := fft(v20x)

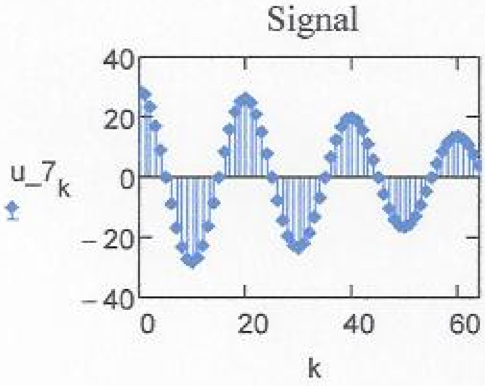




$$\frac{\max(|\blacksquare|)}{\max\left(\left|\overrightarrow{Spec20x}\right|\right)} = \blacksquare \qquad \max\left(\left|\overrightarrow{Spec20x}\right|\right) = 18.191\cdot\mu s$$

4.6.2.5) Sequence of the AM Signal response.

AM Signal sequence of amplitude V_i:



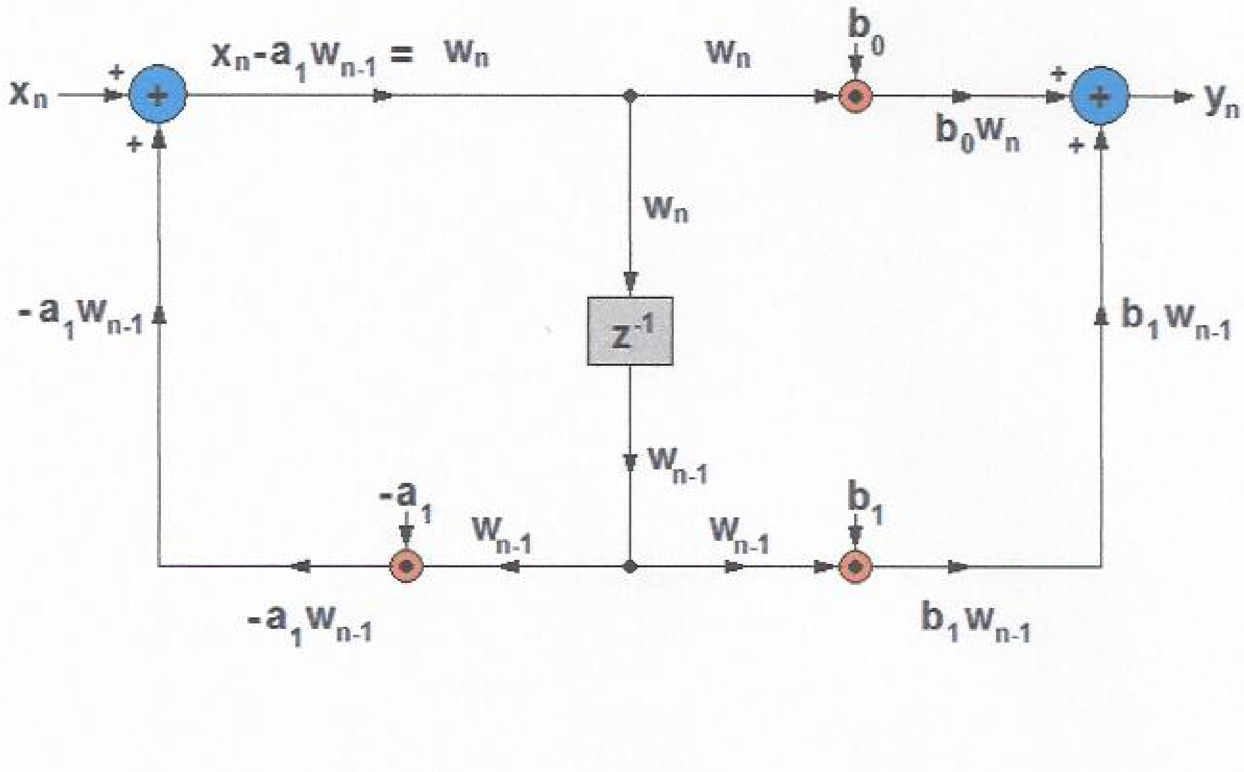
Digital first order High pass filter recurrence relations:

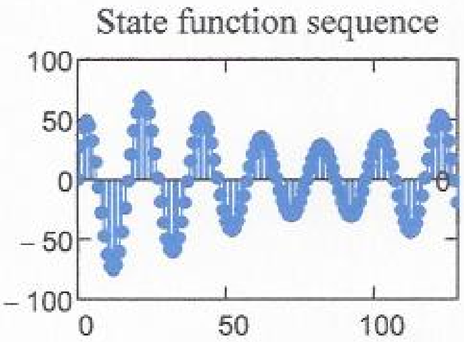
adimensional input signal: $v21_i(\nu) := u\_7_\nu$

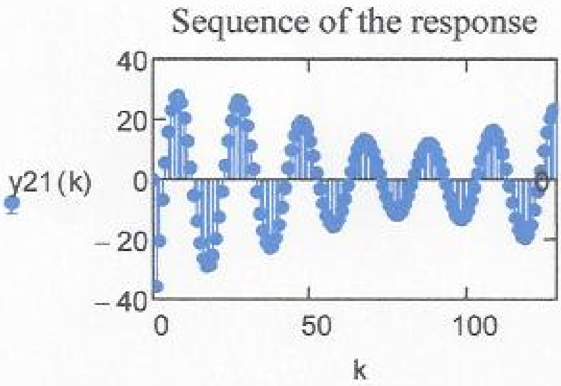
$$\frac{f_{smp}}{f_c} = 800$$

$$1)\quad w21(\nu) := \begin{cases} v21_i(\nu) + \delta 0 \cdot w21(\nu - 1) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$2)\quad y21(\nu) := \begin{cases} \chi 0 \cdot (w21(\nu) - w21(\nu - 1)) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$$

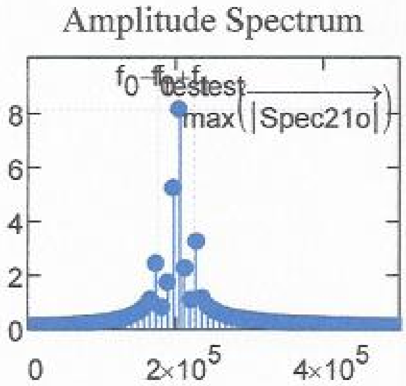


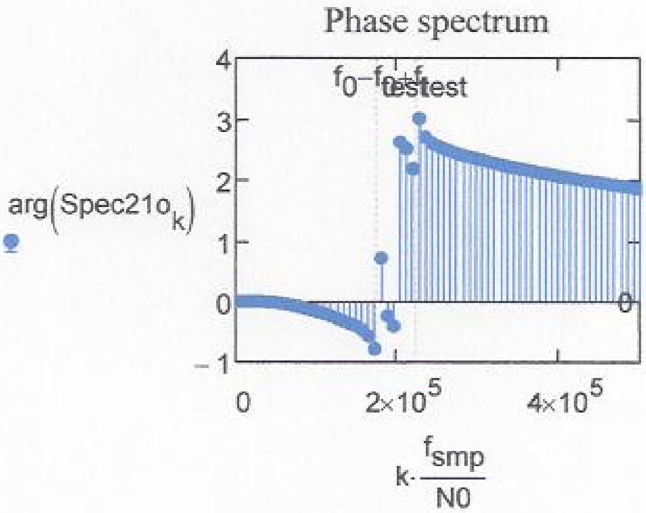




Sampled signal: $v21x_k := y21(k)$

$Spec21o := FFT(v21x)$

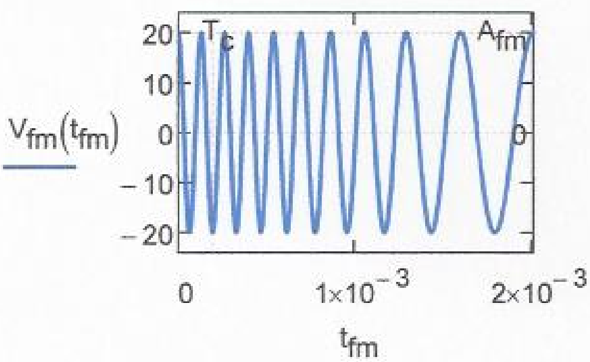




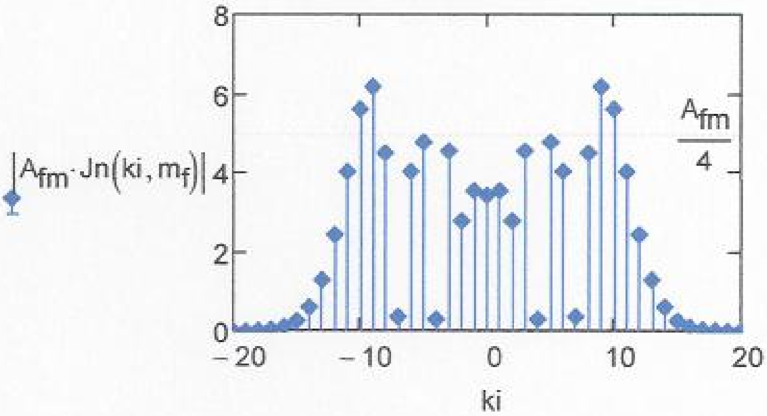
$$\frac{\max\left(\overrightarrow{|Spec5|}\right)}{\max\left(\overrightarrow{|Spec21o|}\right)} = 2.741 \times 10^{-5} V$$

4.6.2.6) Sequence of the Frequency Modulated carrier response. $m_f = 11$

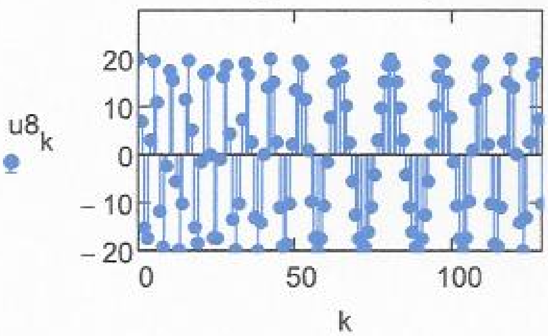
FM signal



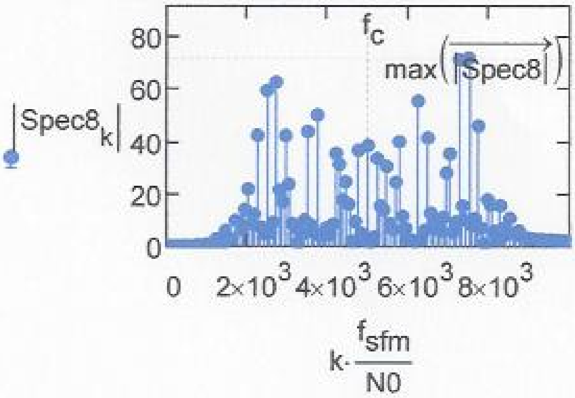
FM Spectrum (sinus. test signal)



Sampled FM signal



FM Test Signal spectrum



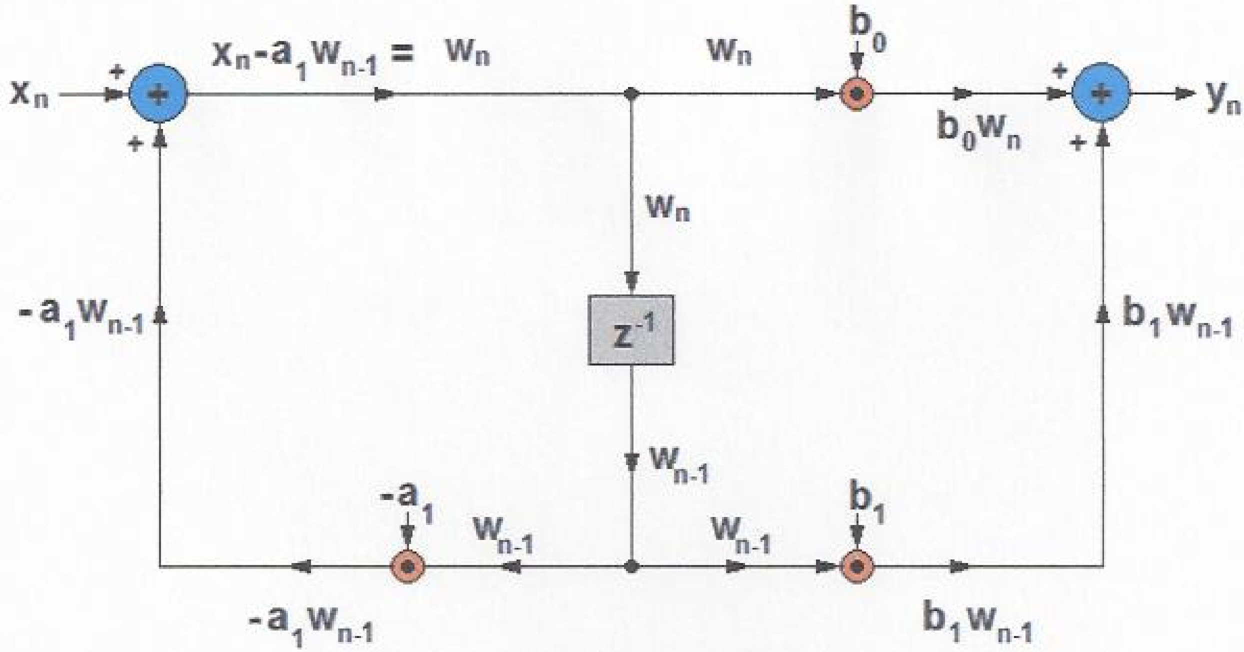
$\max\left(\overrightarrow{|Spec8|}\right) = 71.472$ $\qquad \frac{f_{sfm}}{f_c} = 8$ $\qquad u8_{12} = -5.556$

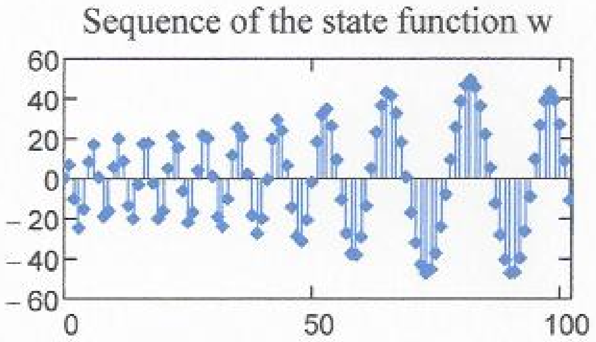
Digital first order High pass filter difference relations:

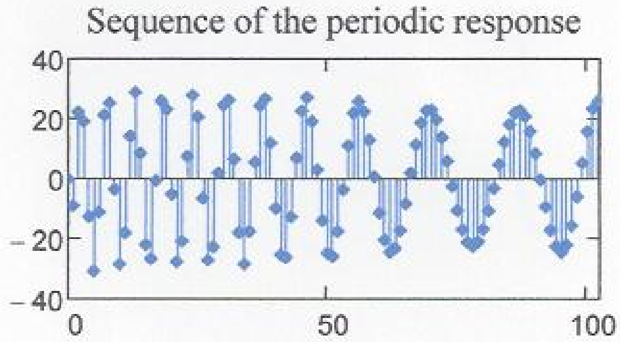
adimensional input signal: $v22_i(\nu) := u\_8_\nu$

$$1)\quad w22(\nu) := \begin{cases} v22_i(\nu) + \delta 0 \cdot w22(\nu - 1) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$2)\quad y22(\nu) := \begin{cases} \chi 0 \cdot (w22(\nu) - w22(\nu - 1)) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$$



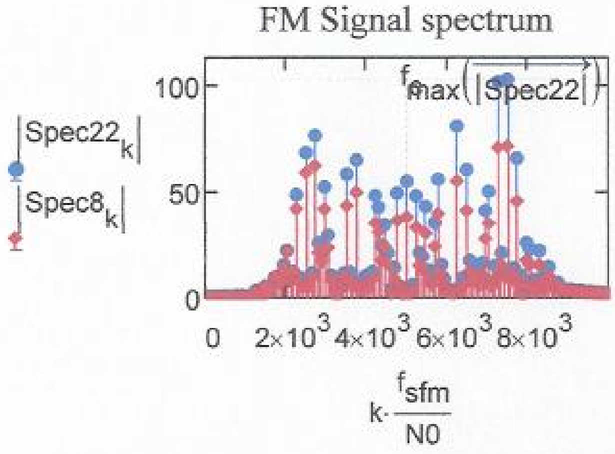


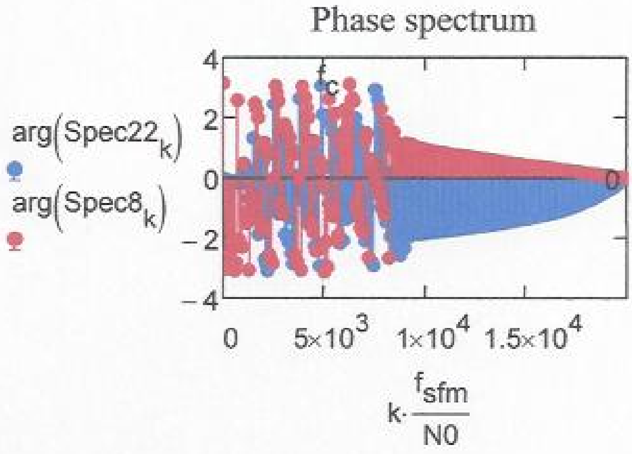


$$f_c = 5 \times 10^{-3} \cdot \text{MHz} \qquad \frac{f_{smp}}{f_c} = 800$$

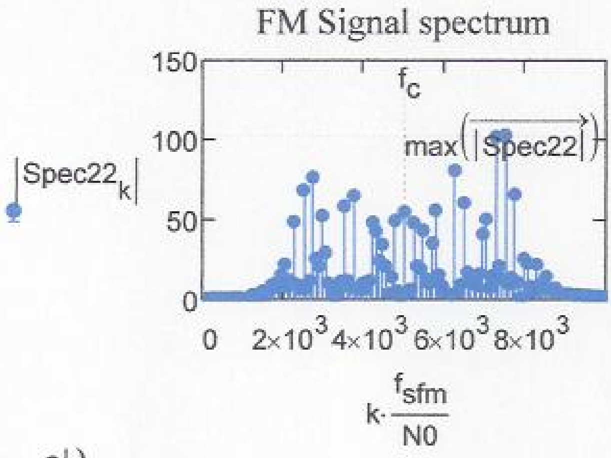
$$U22_k := y22(k)$$

$$\text{Spec22} := \text{fft}(U22) \quad m_f = 11 \qquad \omega_m = 1.571 \times 10^{-3} \cdot \frac{\text{Mrads}}{\text{sec}}$$





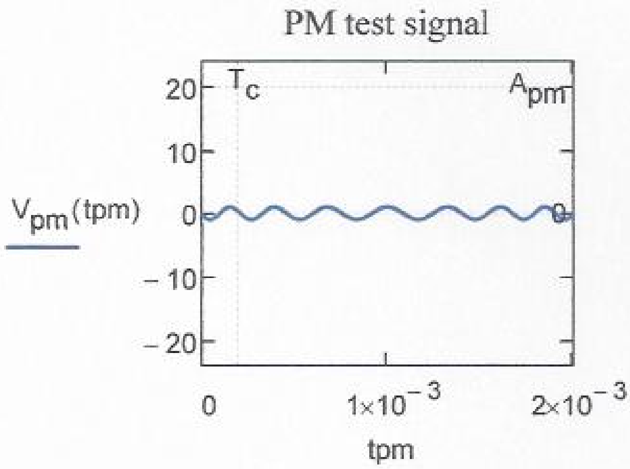
$$\max\left(\overrightarrow{|Spec8|}\right) = 71.472 \qquad \max\left(\overrightarrow{|Spec22|}\right) = 102.899$$

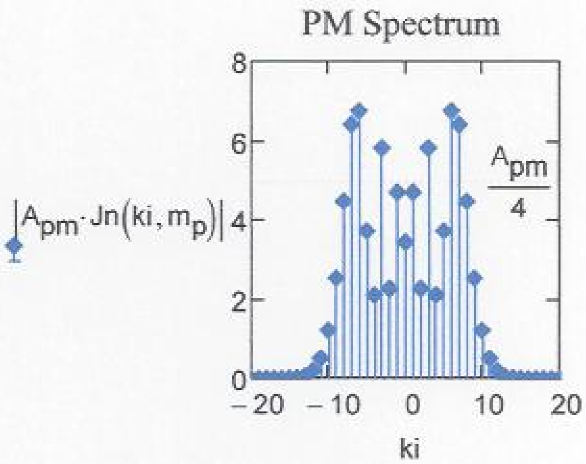


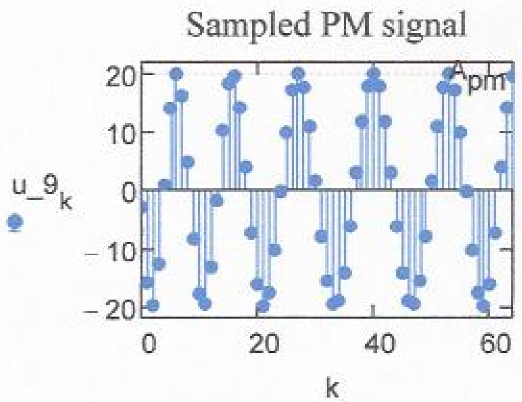
$$\frac{\max(|Spec8|)}{|\max(Spec22)|} = 2.45 \qquad \max(|Spec22|) = 308.154$$

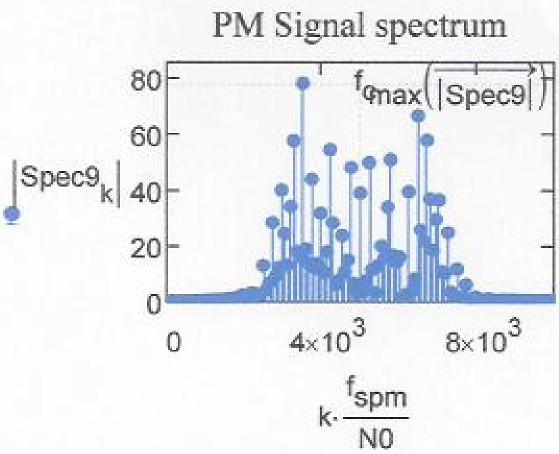
4.6.2.7) Sequence of the Phase Modulated carrier response.

$m_p = 8$





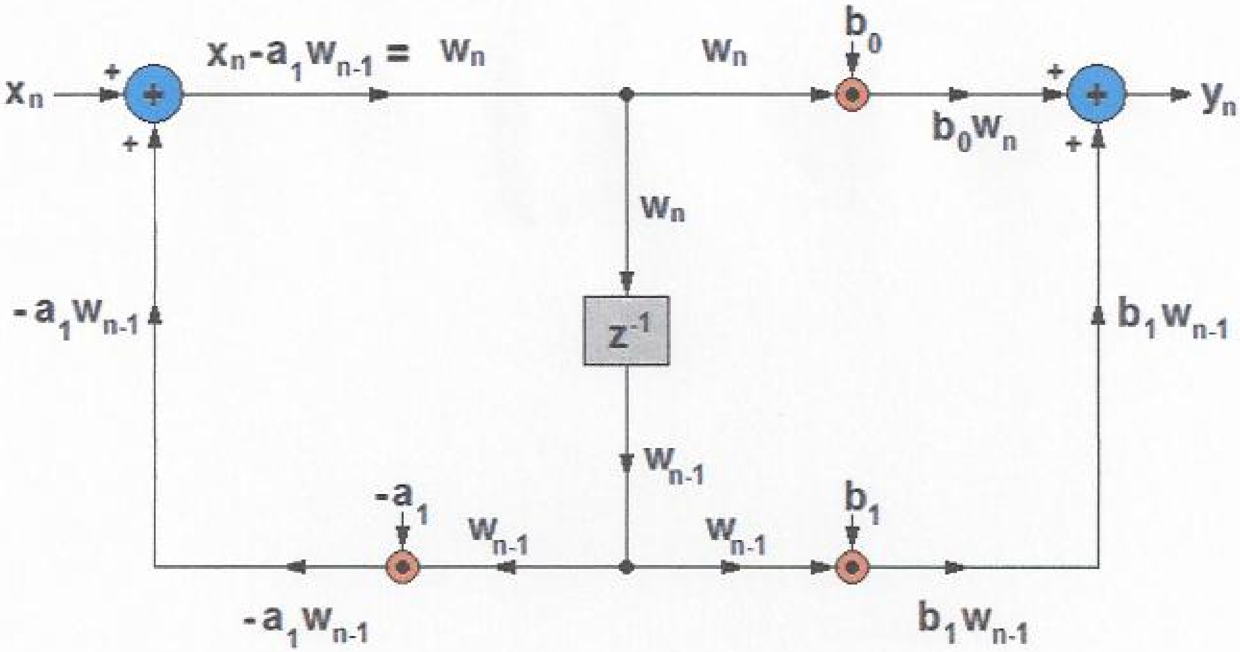


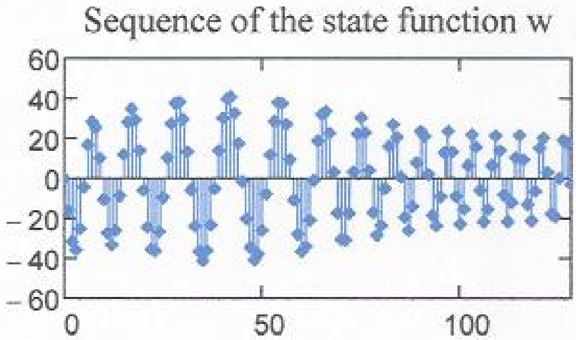


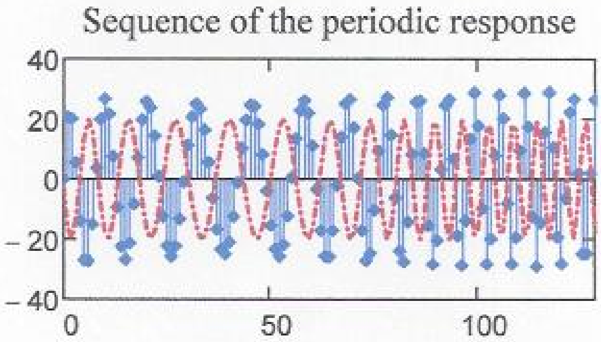
adimensional input signal: $v23_i(\nu) := u\_9_\nu$

1) $w23(\nu) := \begin{cases} v23_i(\nu) + \delta 0 \cdot w23(\nu - 1) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$ $\delta 0 = 0.728489504$

2) $y23(\nu) := \begin{cases} \chi 0 \cdot (w23(\nu) - w23(\nu - 1)) & \text{if } \nu > 0 \\ 0 & \text{otherwise} \end{cases}$ $\chi 0 = -1.296367$

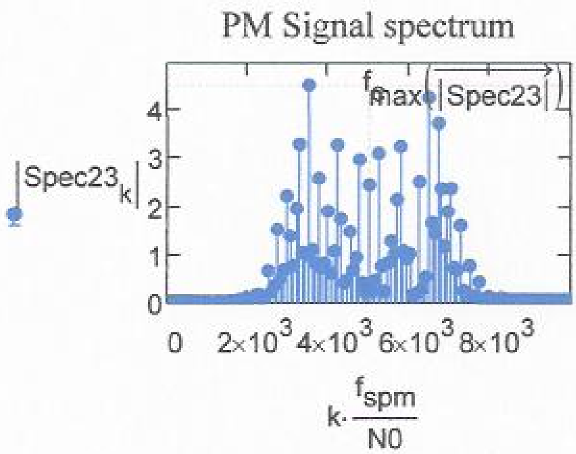


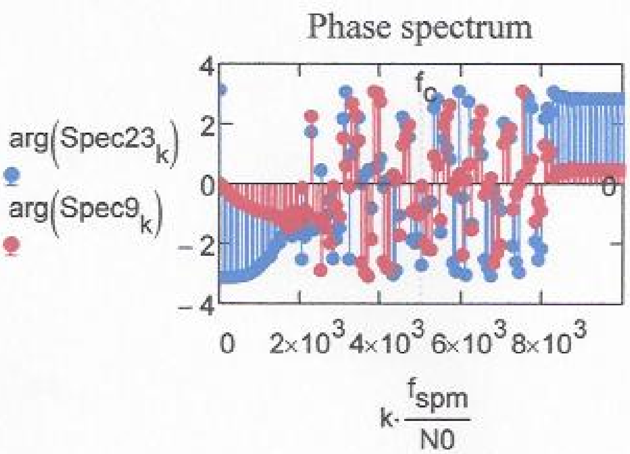




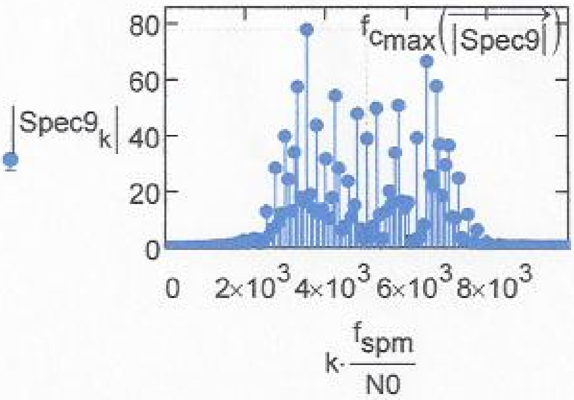
$$U23_k := y23(k)$$

$Spec23 := FFT(U23)$ $\qquad m_p = 8$ $\qquad \omega_m = 1.571 \times 10^{-3} \cdot \frac{Mrads}{sec}$





PM Signal spectrum



$$\frac{max\left(\overrightarrow{|Spec9|}\right)}{max\left(\overrightarrow{|Spec23|}\right)} = 17.32 \qquad max\left(\overrightarrow{|Spec23|}\right) = 4.481$$

4.7
Iterative algorithm (considering the bilinear transformation)

The sequence corresponding to the following t. f. :

$m_p = 8$ $$H11(z) = \chi0 \cdot \frac{1 - z^{-1}}{1 - \delta0 \cdot z^{-1}} \qquad z_0 := -1 \qquad p_0 := \delta0 \qquad p_0 = 0.728$$

is realized using an iterative method.

Numerator degree $Nu_n := 1$ Denominator degree $Md_d := 1$

$N2 := Nu_n + Md_d$ $N0 = 512$ $h11_k := 0$

$N2 = 2$ $$H11(z) = \frac{b_0 + b_1 \cdot z^{-1}}{a_0 + a_1 \cdot z^{-1}}$$

we can define the coefficients of the numerator and denominator as elements of two vectors, namely a and b:

Numerator coeff.	Denominator coeff.
$b_k := 0.0$	$a_k := 0.0$
$b_0 := \chi0$	$a_0 := 1$
$b_1 := -\chi0$	$a_1 := -\delta0$

and devide the two polinoms by means of the following algorithm:

$$h11_0 := \frac{b_0}{a_0} \qquad h11_\nu := \frac{1}{a_0} \cdot \left[b_\nu - \sum_{i=1}^{\nu} \left(h11_{\nu-i} \cdot a_i \right) \right]$$

T. F. Numerator coefficents:

$a^T =$

	0	1	2	3	4	5	6	7
0	1	-0.728	0	0	0	0	0	...

T. F. Denominator coefficents:

$b^T =$

	0	1	2	3	4	5	6	7
0	-1.296	1.296	0	0	0	0	0	...

Sequence Impulse Response:

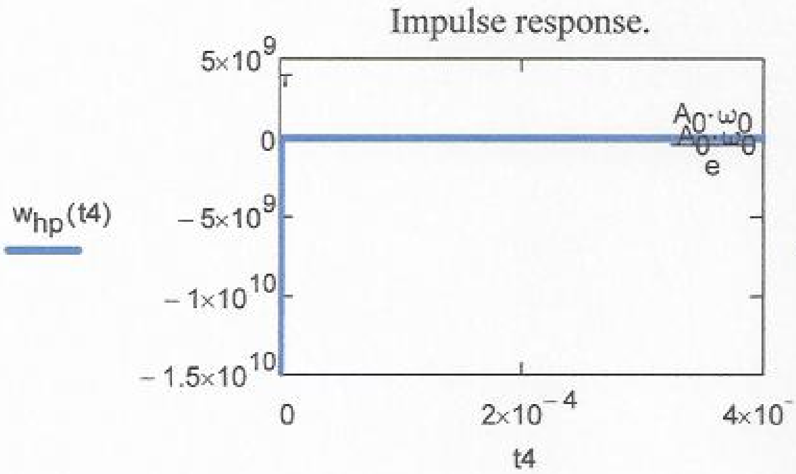
$h11^T =$

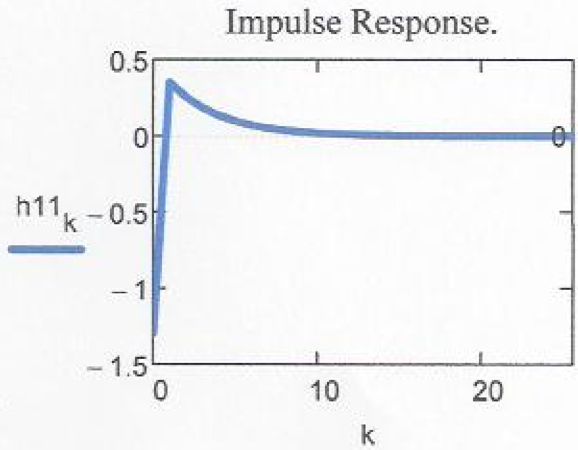
	0	1	2	3	4
0	-1.2964	0.352	0.2564	0.1868	...

Stability (S1<∞): $S2 := \sum_{k=0}^{rows(h11)-1} |h11_k| \qquad S2 = 2.593$

Energy of the sequence h11: $E11 := \sum_{k=0}^{rows(h11)-1} (|h11_k|)^2 \quad E11 = 1.945$

$\tau = 0.796 \cdot \mu s \qquad 100 \cdot T_{test} = 4 \times 10^3 \cdot \mu s \qquad t4 := 0 \cdot T_{test}, \frac{T_{test}}{100} .. 1000 \cdot T_{test} \qquad T_{test} = 40 \cdot \mu s$



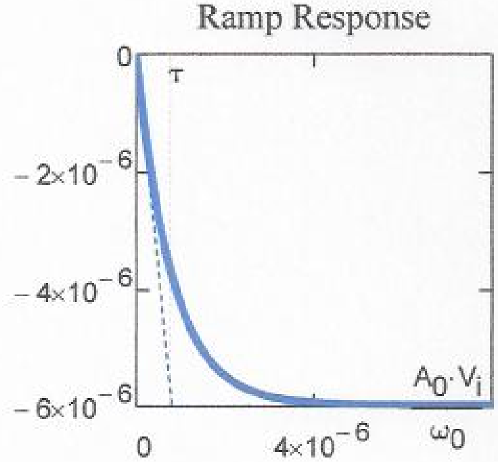


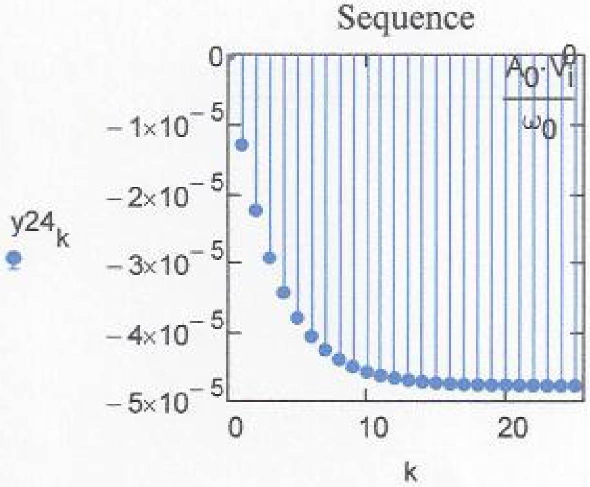
$f_c = 5 \times 10^{-3} \cdot MHz \qquad \frac{f_{smp}}{f_c} = 800$

4.7.1) Sequence of the voltage ramp response.

$t := 0\cdot\tau, \frac{20\cdot\tau}{1000} .. 20\cdot\tau$

$$y24_{\nu} := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \ge 0, h11_k \cdot u3_{\nu-k}, 0\right)\right)$$

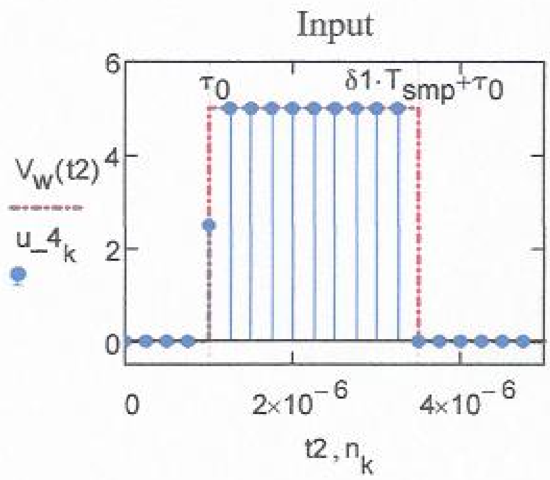


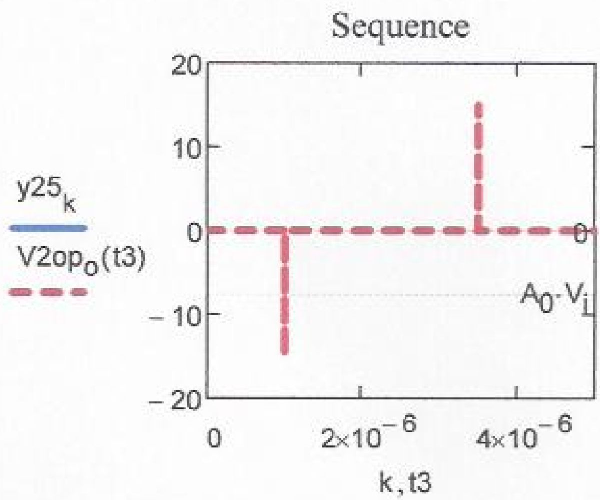


4.7 Iterative algorithm (considering the bilinear transformation)

4.7.2) Sequence of the Voltage window response.

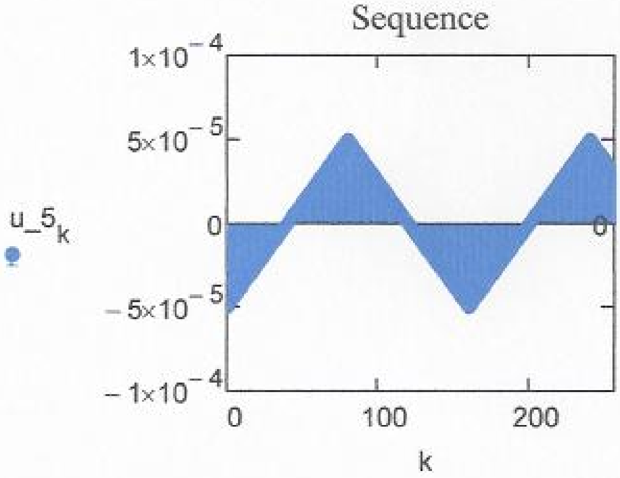
$$y25_{\nu} := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \ge 0, h11_k \cdot u\_4_{\nu-k}, 0\right)\right)$$

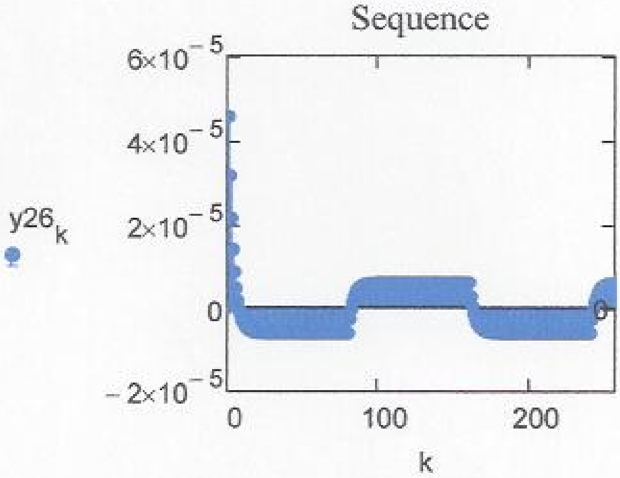




4.7.3) Sequence of the triangular wave response:

$$y26_{\nu} := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h11_k \cdot u\_5_{\nu-k}, 0\right)\right)$$

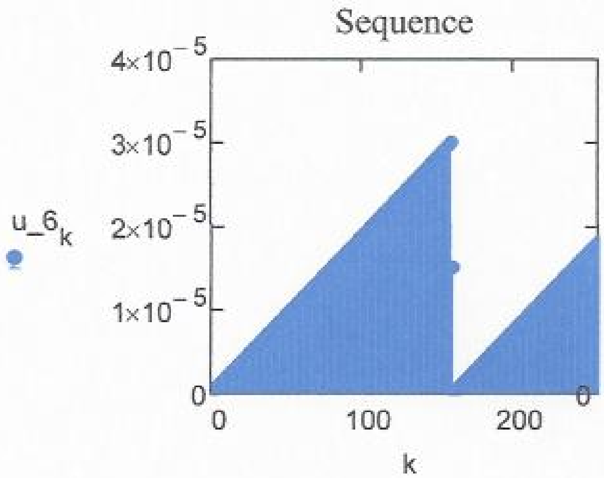


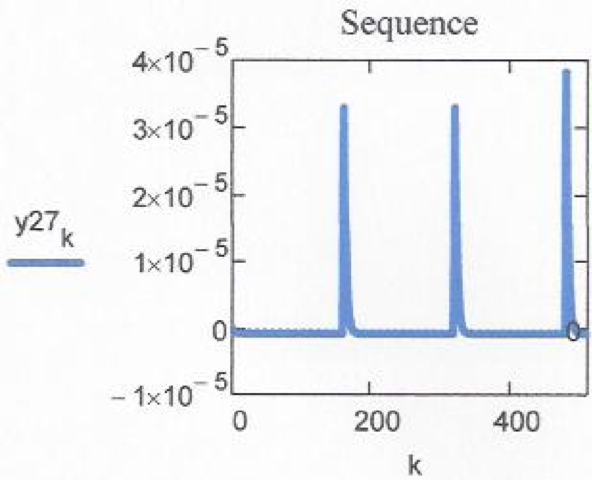


4.7 Iterative algorithm (considering the bilinear transformation)

4.7.4) Sequence of the Sawtooth wave response.

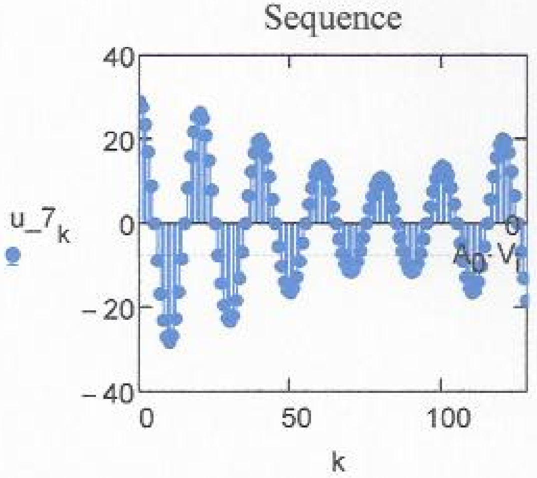
$$y27_{\nu} := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h11_k \cdot u\_6_{\nu-k}, 0\right)\right)$$

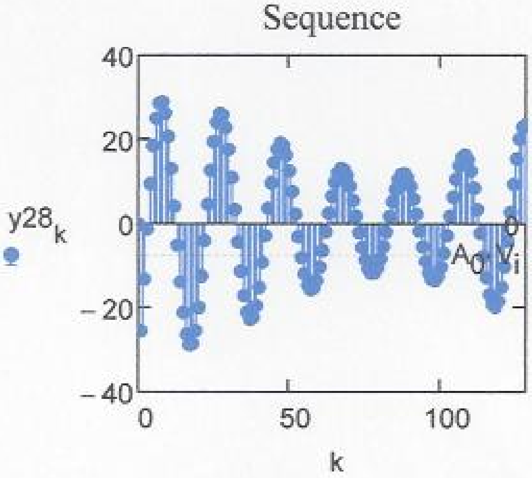




4.7.5) Sequence of the AM Signal response.

$$y28_{\nu} := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h11_k \cdot u\_7_{\nu-k}, 0\right)\right)$$

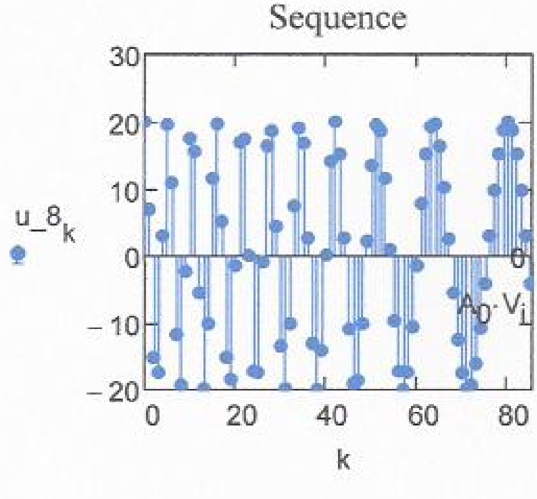


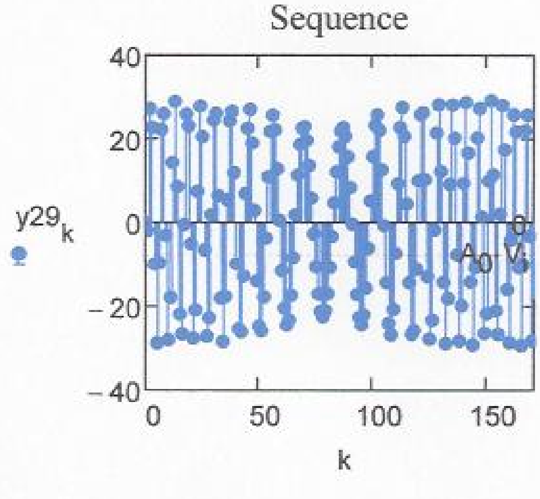


4.7 Iterative algorithm (considering the bilinear transformation)

4.7.6) Sequence of the Frequency Modulated carrier response.

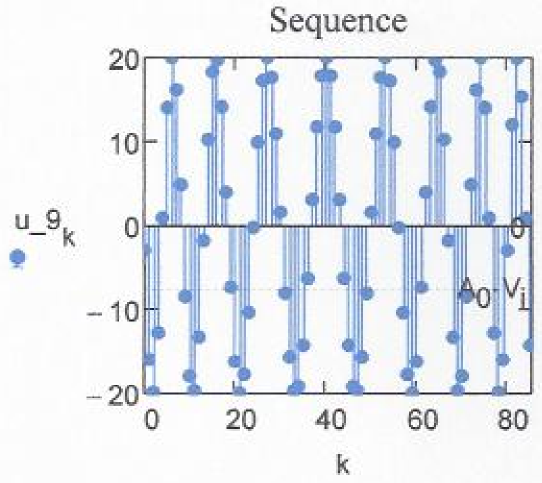
$$y29_{\nu} := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h11_k \cdot u\_8_{\nu-k}, 0\right)\right)$$

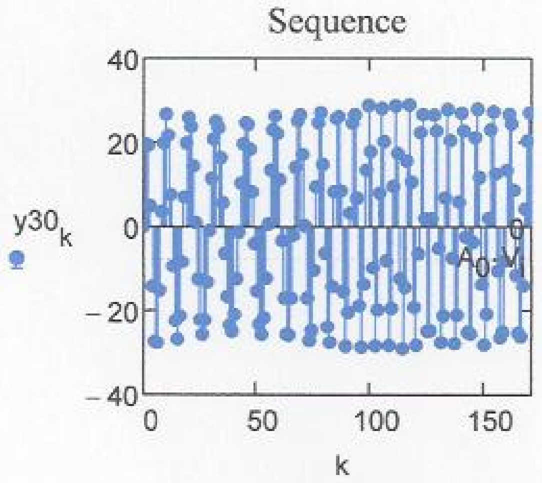




4.7.7) Sequence of the Phase Modulated carrier response.

$$y30_{\nu} := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \ge 0, h11_k \cdot u\_9_{\nu-k}, 0\right)\right)$$





4.8
Analytical search of the output sequence by means of the residues method (considering the bilinear transformation)

Poles and zeroes of $H11(z) = \chi 0 \cdot \dfrac{1 - z^{-1}}{1 - \delta 0 \cdot z^{-1}}$:

$\delta 0 := \delta 0 \quad \chi 0 := \chi 0$

$$v := \text{numer}(H11(z)) \text{ coeffs}, z \rightarrow \begin{pmatrix} 0 \\ A_0 \cdot \omega_{smp}^{2} + \pi \cdot A_0 \cdot \omega_0 \cdot \omega_{smp} \\ -A_0 \cdot \omega_{smp}^{2} - \pi \cdot A_0 \cdot \omega_0 \cdot \omega_{smp} \end{pmatrix}$$

zeros := polyroots(v) $\qquad$ $\text{zeros} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

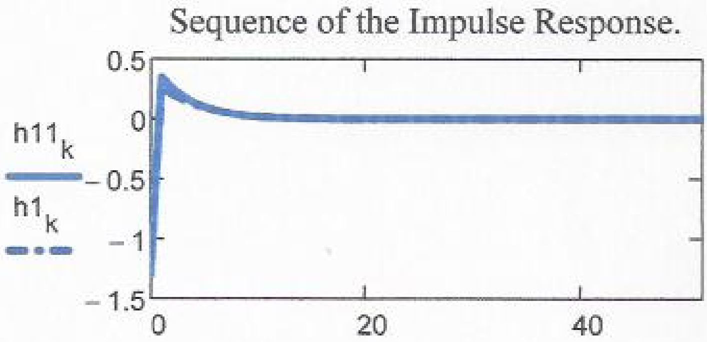
$$v := \text{denom}(H11(z)) \text{ coeffs}, z \rightarrow \begin{pmatrix} 0 \\ \omega_{smp}^{2} - \pi^{2} \cdot \omega_0^{2} \\ -\pi^{2} \cdot \omega_0^{2} - 2 \cdot \pi \cdot \omega_0 \cdot \omega_{smp} - \omega_{smp}^{2} \end{pmatrix}$$

poles := polyroots(v) $\qquad$ $\text{poles} = \begin{pmatrix} 0 \\ 0.728 \end{pmatrix}$

$\delta 0 = 0.728$, $\qquad \chi 0 = -1.296$

The calculation gives:

$h11_k :=$ ▮



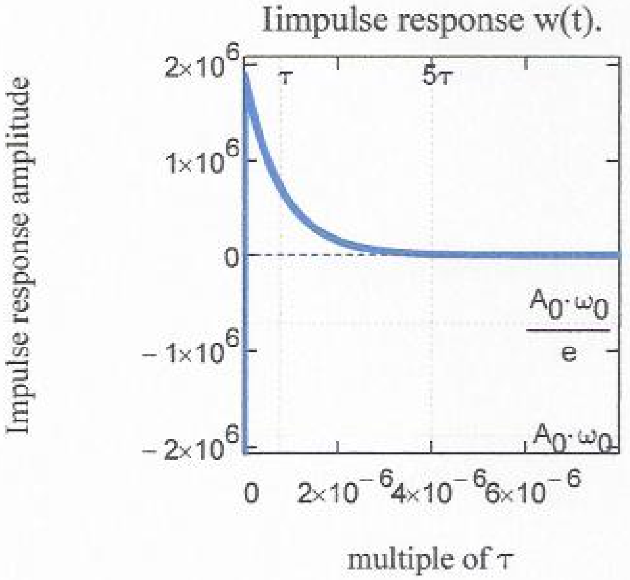
k

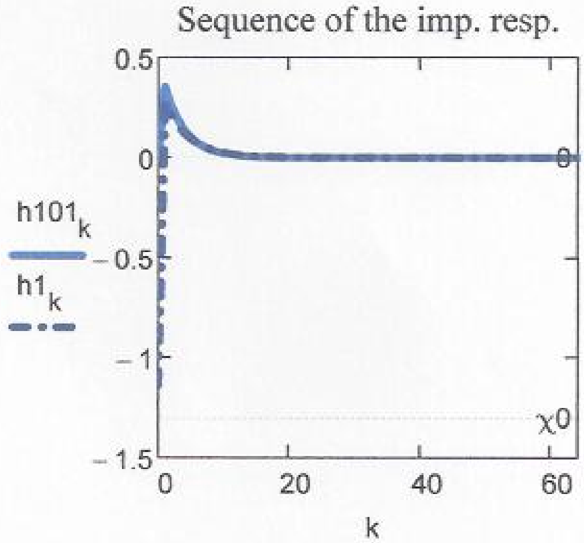
1.6 iii) By using the "invztrans" operator:

$$\chi 0 := \chi 0 \qquad \delta 0 := \delta 0 \qquad \nu := \nu$$

$$h101_{\nu} := \chi 0 \cdot \frac{1 - z^{-1}}{1 - \delta 0 \cdot z^{-1}} \text{ invztrans}, z, \nu \rightarrow \frac{\chi 0 \cdot \left(\delta 0^{\nu+1} + \delta(\nu, 0) - \delta 0^{\nu}\right)}{\delta 0}$$

$$t4 := 0 \cdot \tau, \frac{\tau}{1000} .. 50 \cdot \tau$$





Stability (S11<∞): $S11 := \sum_{j=0}^{rows(h101)-1} \left|h101_j\right| \quad S11 = 1.296$

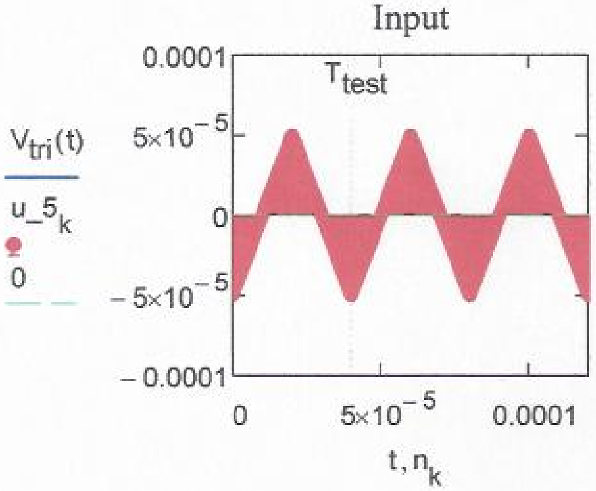
Energy of the sequence h11: $E101 := \sum_{j=0}^{rows(h101)-1} \left(\left|h101_j\right|\right)^2 \qquad E101 = 0.264$

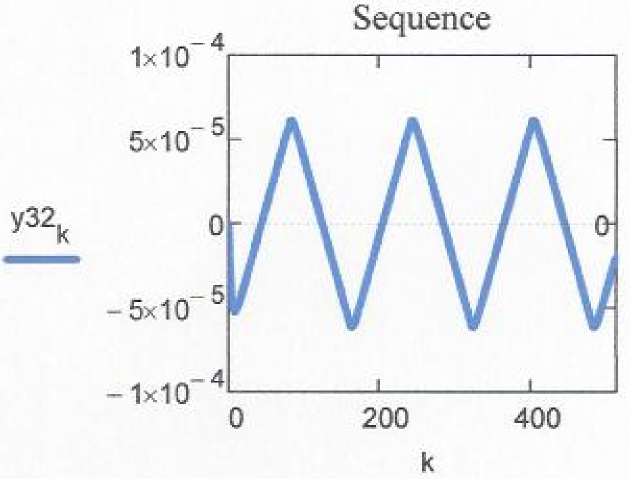
The Output of the Digital System is given by the discrete convolution between the input signal (in this case the sequence of a step function) and the impulse response of the System:

$$y31_{\nu} := \sum_{j=0}^{\nu} \left(\text{if}\left(\nu - j \geq 0, h1_j \cdot u50_{\nu-j}, 0\right)\right)$$

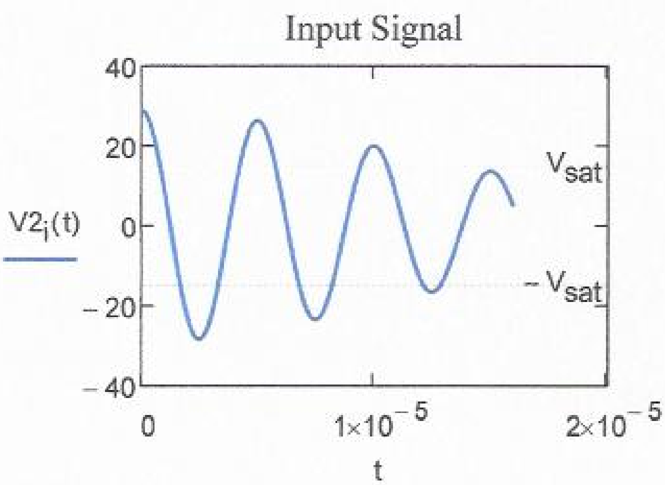
$$y32_{\nu} := \sum_{k=0}^{\nu} \left(\text{if}\left(\nu - k \geq 0, h101_k \cdot u\_5_{\nu-k}, 0\right)\right)$$

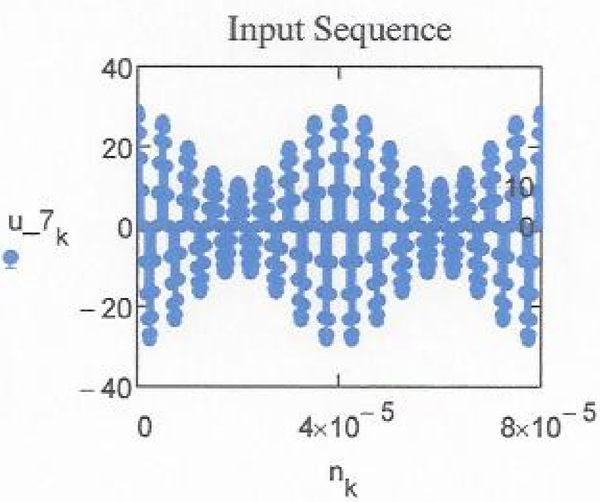
Example 1) Triangular wave



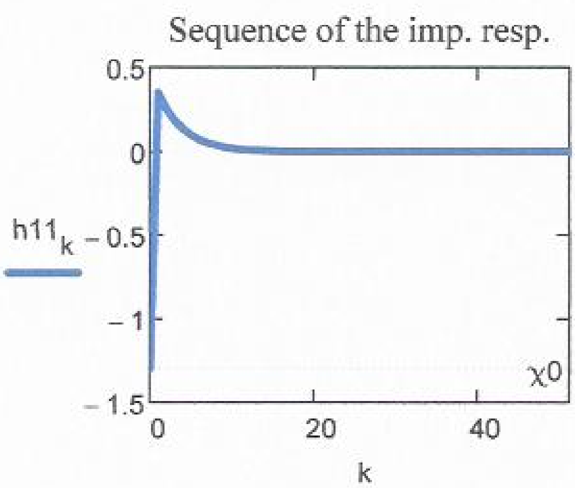


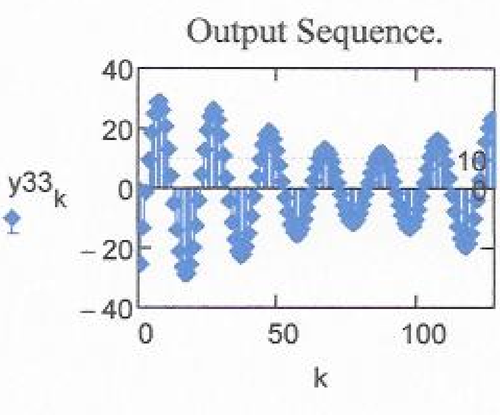
Example 2) AM Signal input:





$$y33_{\nu} := \sum_{k=0}^{\nu} \left(\mathrm{if}\left(\nu - k \geq 0, h11_k \cdot u\_7_{\nu-k}, 0\right)\right)$$



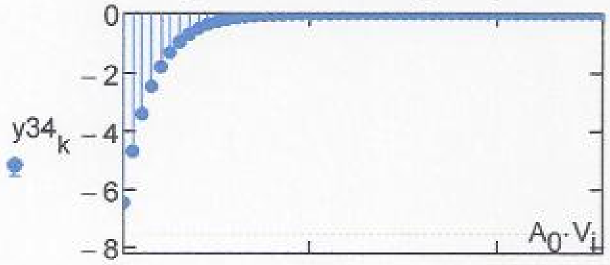


Knowing the sequences of any input and of the impulse response and the relative Z transforms, we can determine the inverse of the product of the two Z transformed , corrisponding to the convolution of the two sequences, as follows:

$X4(z) := \sum_{n=0}^{N1-1} \left(u1_n \cdot z^{-n}\right)$ $H4(z) := \sum_{n=0}^{N-1} \left(h11_n \cdot z^{-n}\right)$ $Y4(z) := H4(z) \cdot X4(z)$

$V_i := V_i$ $V_i \text{ ztrans } \rightarrow \frac{V_i \cdot z}{z-1}$ $\delta0 = 0.728489504$

$$y34_k := \chi0 \cdot \frac{1 - z^{-1}}{1 - \delta0 \cdot z^{-1}} \cdot \frac{V_i \cdot z}{z-1} \left| \begin{array}{l} \text{invztrans}, z, k \\ \text{simplify} \end{array} \right. \rightarrow V_i \cdot \chi0 \cdot \delta0^k$$



$n := n$ $V_i = 5V$ $\Delta T = 39.063 \cdot ns$

Example 3) $\omega_2 := \omega_{test}$. System Input: $x2_k := \frac{V_i}{volt} \cdot \sin\left(k \cdot \omega_2 \cdot \Delta T\right)$

$V_i := V_i$ $\Delta T := \Delta T$ $\omega_2 := \omega_2$

$K3 := \sin\left(\Delta T \cdot \omega_2\right)$ $\sqrt{1 - K3^2} = 1$ $K3 = 6.136 \times 10^{-3}$ $\omega_{0dB} = 1.124 \cdot \frac{Mrads}{sec}$

$$X_{sin}(z) := \frac{V_i \cdot z \cdot K3}{z^2 - 2 \cdot \sqrt{1 - K3^2} \cdot z + 1}$$ $K3 := K3$

$$poles1 := z^2 - 2 \cdot \sqrt{1 - K3^2} \cdot z + 1 \text{ solve}, z \rightarrow \begin{pmatrix} \sqrt{1 - K3^2} + K3 \cdot i \\ \sqrt{1 - K3^2} - K3 \cdot i \end{pmatrix} \qquad poles1 = \begin{pmatrix} 1 + 6.136i \times 10^{-3} \\ 1 - 6.136i \times 10^{-3} \end{pmatrix}$$

$$\frac{V_i \cdot z \cdot K3}{z^2 - 2 \cdot \sqrt{1 - K3^2} \cdot z + 1} = \frac{V_i \cdot z \cdot K3}{\left(z - poles1_0\right) \cdot \left(z - poles1_1\right)}$$

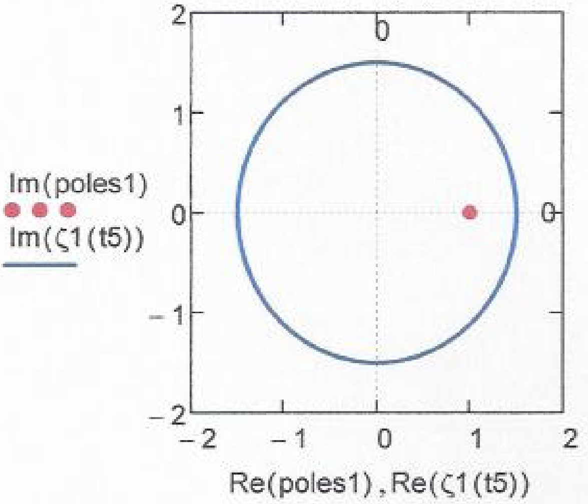
$r := 1.5$ $p2_0 := poles1_0$ $p2_0 = 1 + 6.136i \times 10^{-3}$

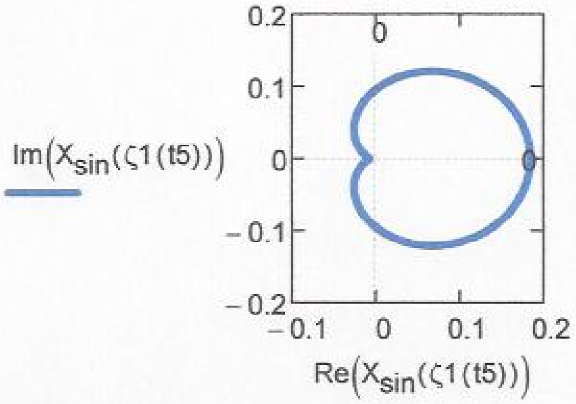
$p2_1 := poles1_1$ $p2_1 = 1 - 6.136i \times 10^{-3}$

$\xi(t5) := r \cdot \cos(t5)$

$\psi(t5) := r \cdot \sin(t5)$

$\zeta1(t5) := \xi(t5) + j \cdot \psi(t5)$



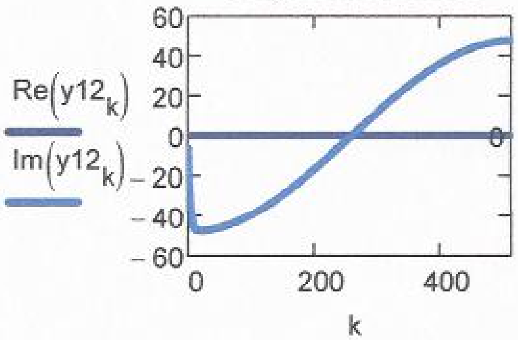


The sequence of the result returned for the symbolic operation is too large to be displayed.
It require some seconds.

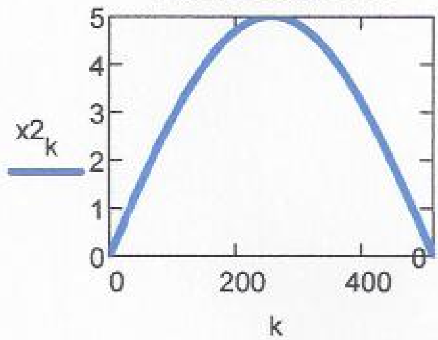
$$y12_k := \chi 0 \cdot \frac{1 - z^{-1}}{1 - \delta 0 \cdot z^{-1}} \cdot \left[\frac{K2 \cdot V_i \cdot z}{\left(p2_0 - z\right)\cdot\left(p2_0 - \overline{p2_0}\right)} \cdots + (-1)\cdot\frac{K2 \cdot V_i \cdot z}{\left(p2_0 - \overline{p2_0}\right)\cdot\left(z - \overline{p2_0}\right)} \right] \begin{array}{|l} \text{invztrans}, z, \text{using}, n = k \\ \text{simplify} \end{array} \rightarrow$$

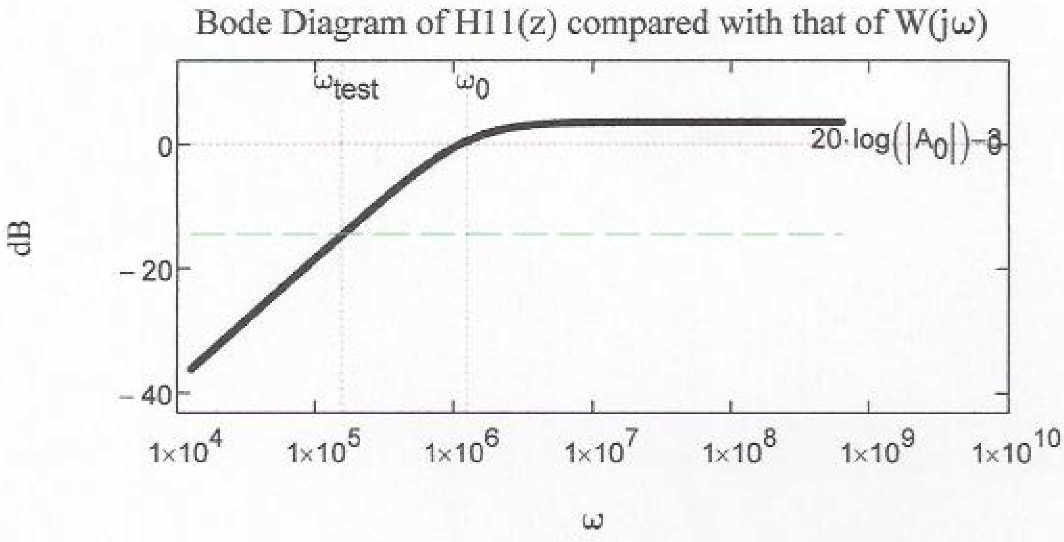
$$y12_k := \frac{K2 \cdot V_i \cdot \chi 0 \cdot \left[\begin{array}{l} \delta 0 \cdot \left(\overline{p2_0}\right)^k - {p2_0}^k \cdot \overline{p2_0} - p2_0 \cdot \left(\overline{p2_0}\right)^k + \delta 0^k \cdot \overline{p2_0} + p2_0 \cdot \delta 0^{k+1} \cdots \\ + {p2_0}^{k+1} \cdot \delta 0 - 2 \cdot \delta 0^{k+1} - 2 \cdot \delta 0^{k+2} - p2_0 \cdot \left(\overline{p2_0}\right)^{k+1} - {p2_0}^{k+1} \cdot \overline{p2_0} \cdots \\ + \delta 0 \cdot \left(\overline{p2_0}\right)^{k+1} + \delta 0^{k+1} \cdot \overline{p2_0} + p2_0 \cdot \delta 0^k + {p2_0}^k \cdot \delta 0 \end{array} \right]}{\left(p2_0 - \delta 0\right)\cdot\left(p2_0 - \overline{p2_0}\right)\cdot\left(\delta 0 - \overline{p2_0}\right)\cdot \text{volt}}$$

Output Sequence.



Input Sequence.





$20\cdot\log\left(\left|H11\left(e^{j\cdot\omega_{test}\cdot T_{smp}}\right)\right|\right) = -14.606\cdot dB$ $\qquad$ $20\cdot\log\left(\left|W_{hp}\left(j\cdot\omega_{test}\right)\right|\right) = -14.607\cdot dB$