Section 4 Network Analysis Using an Admittance Matrix

The following document uses an admittance matrix to solve for voltages in a network and plots the voltage gain and phase shift (Bode plots) for a range of frequencies. A demonstration of a transfer function calculation appears at the end. You provide:

- the network configuration
- impedances for all elements in the network
- the frequency range for the Bode plots

For more information on transfer functions, see **Section 5: Feedback Circuits and Stability Criteria**.

References

William H. Hayt, Jr., and Jack E. Kemmerly, *Engineering Circuit Analysis*, McGraw-Hill Book Company (New York, 1978), and *HP-41C Circuit Analysis Pack*, Hewlett-Packard, 1984. See also Stephen Senturia and Bruce Wedlock, *Electronic Circuits and Applications*, John Wiley & Sons, Inc. (New York, 1975).See also Joseph A. Edminister and Mahmood Nahvi, *Schaum's Outline of Theory and Problems of Electric Circuits, 3e*, chapter 4, McGraw-Hill (New York, 1997).

Background

The voltages (or currents) in a linear RLC network can be determined between each node by applying Kirchoff's voltage and current laws to the various impedances. The simultaneous system of equations developed in this way can be expressed as a single matrix equation, and solved in a straightforward manner using matrix arithemetic. The matrix of interest is the admittance matrix, whose elements are the admittances associated with each pair of nodes.

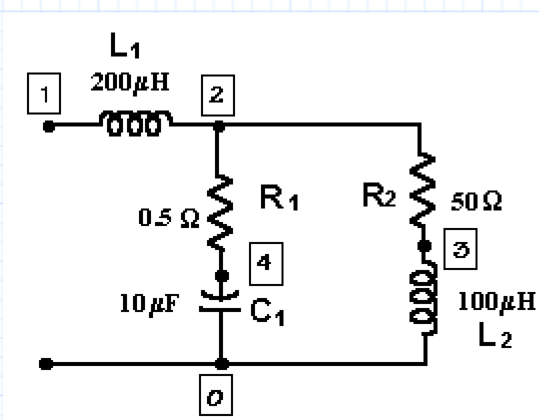


Fig. 4.1 A sample network. This example is a four node circuit, requiring a 4-by-4 admittance matrix.

Admittances for the various elements are simply the inverses of their impedances. The expressions for inductors and capacitors should be expressed in terms of the operating frequency for the circuit. Using the admittance matrix, it is possible to find the resonant frequency for the network, and the general frequency response. Admittance formulae for capacitors and inductors are shown below:

$$Y_C(\omega) = 1\mathrm{j} \cdot \omega \cdot C \qquad\qquad Y_L(\omega) = \frac{1}{1\mathrm{j} \cdot \omega \cdot L}$$

Circuit Diagrams in Mathcad

The circuit diagram in this document was implemented using a copy/paste operation. Bitmap graphics can be copied and pasted into Mathcad, as well as other types of image file formats. Another option to insert graphics into a Mathcad document is to use the *Image* button located on the *Math* tab of the ribbon. The *Image* button will let you browse for your picture, and once you have selected the desired picture file, will import the picture into the Mathcad document. Note that the Image button imports the picture one time.

Mathcad Implementation

This document uses Mathcad's matrix operators to build the admittance matrix for a simple linear network. It solves for the node voltages corresponding to a current source of a given frequency, and generates Bode plots for a range of frequencies.

Creating the Admittance Matrix

To enter information about a circuit, edit the definition of the matrix **A(w)** so that the entry in row **m** and column **n** gives the complex admittance between nodes **m** and **n**. To reduce the amount of typing, you only need to enter admittances in one direction (i.e. from node 1 to 2 and NOT 2 to 1). Mathcad will add the additional numbers later. Using this technique, only the upper triangular part of the matrix will be filled.

$$A(\omega) := \begin{bmatrix} 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & \frac{1}{1\mathrm{j}\cdot\omega\cdot 100\ \mu H} & 1\mathrm{j}\cdot\omega\cdot 10\ \mu F \\ 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & \frac{1}{1\mathrm{j}\cdot\omega\cdot 200\ \mu H} & 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} \\ 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & \frac{1}{50\ \Omega} & \frac{1}{.5\ \Omega} \\ 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} \\ 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} & 0\,\frac{1}{\Omega} \end{bmatrix}$$

Note: The network in this example has four nodes in addition to node zero, resulting in 5 rows and columns for **A(w)**. If your network has a different number of nodes, adjust the matrix size by clicking on it and using the ***Matrices/Tables*** **tab** to add or remove rows and columns.

The following commands create the additional matrix entries in the lower triangular portion of the matrix, and then build the admittance matrix by adding conductive legs of the network together.

$$AS(\omega) := A(\omega) + A(\omega)^{T}$$

$$i := 0 .. \text{rows}(A(1\ Hz)) - 2$$

$$k := 0 .. \text{rows}(A(1\ Hz)) - 2$$

$$F(\omega, i, k) := \textbf{if}\left(i \neq k, (-AS(\omega))_{i+1,k+1}, \sum\left(AS(\omega)^{\langle i+1\rangle}\right)\right)$$

The admittance matrix is:

$$Y(\omega) := \begin{bmatrix} F(\omega,0,0) & F(\omega,0,1) & F(\omega,0,2) & F(\omega,0,3) \\ F(\omega,1,0) & F(\omega,1,1) & F(\omega,1,2) & F(\omega,1,3) \\ F(\omega,2,0) & F(\omega,2,1) & F(\omega,2,2) & F(\omega,2,3) \\ F(\omega,3,0) & F(\omega,3,1) & F(\omega,3,2) & F(\omega,3,3) \end{bmatrix}$$

For any of these matrices, the first index, 0, corresponds to node 1, the second index, 1, corresponds to node 2, and so on. If you wish to start your origins at 1,1, see **Tutorial Task 4-2: Defining Matrices**. The Tutorial tasks can be found here: **Mathcad Ribbon: *Getting Started* tab: *Tutorial* button**

If your network has a different number of nodes than this example, you will need to redefine the final admittance matrix **Y(w)** so that it has as many rows and columns as there are nodes in the network. To do this, position your cursor inside the matrix on a row or column to use as a point of reference, then use the ***Matrices/Tables* tab** to add or remove rows and columns. Insert as many additional rows and columns as you need; fill the new matrix elements with the expression **C(w, i, k)**, with appropriate **i**'s and **k**'s, as in the 4 node example above.

Solving for Node Voltages

Use the inverse of the admittance matrix **Y(w)** to find a node voltage vector **V(w)**. First, invert the admittance matrix and multiply it by a node current vector. This example places a current source at node 1:

$$V(\omega) := Y(\omega)^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

V(w) can be used to generate a Bode plot for any pair of nodes (the whole network can be represented by the voltage gain across nodes 0 and 1). First, choose a frequency range by defining the minimum frequency **fmin** and maximum frequency **fmax**. For convenience in plotting, use powers of 10 for both frequencies. This example plots the gain and phase shift between nodes 1 and 4.

Graphing Parameters

Choose a **frequency range** (for convenience in plotting, use powers of 10):

Minimum frequency: $f_{min} := 100\ \mathbf{Hz}$

Maximum frequency: $f_{max} := 0.5\ \mathbf{MHz}$

$n := 0..50$

$$r := \log\left(\frac{f_{max}}{\mathbf{Hz}}\right) - \log\left(\frac{f_{min}}{\mathbf{Hz}}\right)$$

$$f_n := f_{min} \cdot 10^{.02 \cdot r \cdot n}$$

The voltage gain can be defined as follows for any two nodes (see subscripts).

$$G_n := \frac{V\left(2 \cdot \pi \cdot f_n\right)_3}{V\left(2 \cdot \pi \cdot f_n\right)_0}$$

The first plot shows the voltage gain in dB, and the second shows the phase shift for the same pair of nodes over the same frequency range.

$$dB(z) := 20 \cdot \log(|z|)$$

$$\phi(z) := \frac{180}{\pi} \cdot \arg(z) - 360 \cdot (\mathbf{if}(\arg(z) \geq 0, 1, 0))$$

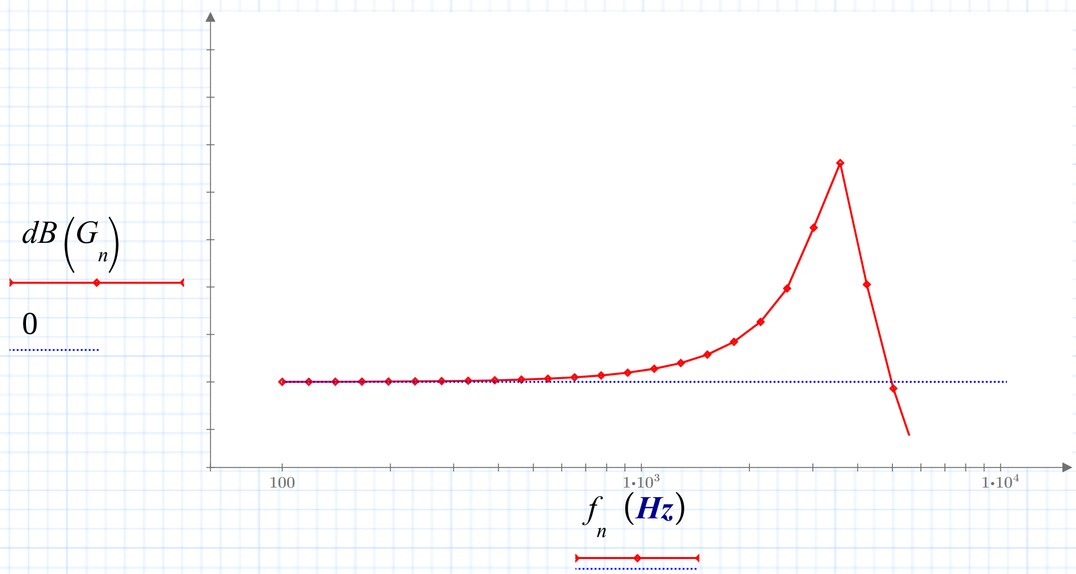


Fig. 4.2 Plot of gain magnitude in dB vs. frequency

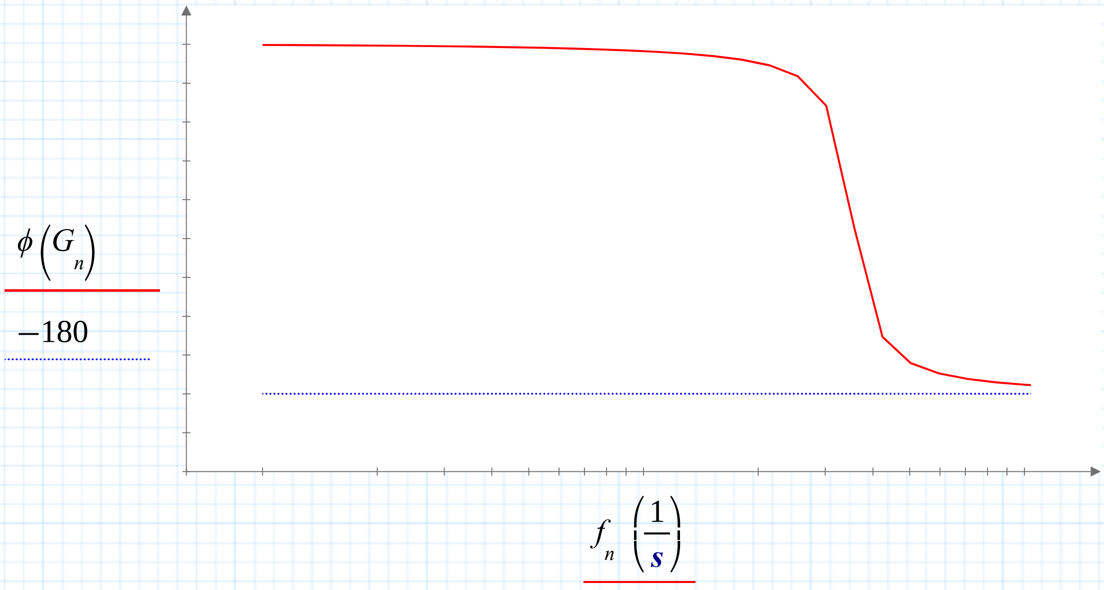


Fig. 4.3 Relative voltage phase shift

In this plot, the tick mark values for the horizontal and vertical scales are not displayed. To activate the display of the tick mark values, first put your cursor on the input data to the plot for the scale you wish to work with, then use the ***Plots* tab: *Axes* button**. You can activate and deactivate the tick marks and tick mark values for each axis separately.

Frequency Characteristics

Find the height and location of the resonance peak:

$$I_n := n \qquad h(x, y, M) := \textbf{if}(x = M, y, 0)$$

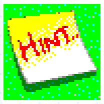
$$M := \max\left(\overrightarrow{dB(G)}\right) \qquad f_p := f_{\max\left(\overrightarrow{h(dB(G), I, M)}\right)}$$

The peak magnitude in dB is

$$M = 13.84$$

The peak is at a frequency of

$$f_p = \left(3.577 \cdot 10^3\right)\ \boldsymbol{Hz}$$



It may be useful to define admittances in terms of general variables **C, R** or **L** if they appear repeatedly in your network. You can also set the frequency constant, and cast the equations in terms of the impedances. For example,

$$C := 50\ \mu F \qquad R := 100\ \Omega \qquad L := 100\ \mu H$$

$$Y_C(\omega) := 1j \cdot \omega \cdot C \qquad Y_L(\omega) := \frac{1}{1j \cdot \omega \cdot L}$$

You can then proceed as usual.

$$A(\omega) := \begin{bmatrix} 0\ \frac{1}{\Omega} & \frac{1}{R} & Y_L(\omega) & \frac{Y_L(\omega)}{R \cdot \Omega^{-1}} \\ 0\ \frac{1}{\Omega} & 0\ \frac{1}{\Omega} & Y_C(\omega) & 0\ \frac{1}{\Omega} \\ 0\ \frac{1}{\Omega} & 0\ \frac{1}{\Omega} & 0\ \frac{1}{\Omega} & 2 \cdot Y_C(\omega) \\ 0\ \frac{1}{\Omega} & 0\ \frac{1}{\Omega} & 0\ \frac{1}{\Omega} & 0\ \frac{1}{\Omega} \end{bmatrix}$$