13.2 Analog Elliptic Filter Design

This document carries out design of an elliptic IIR lowpass analog filter. You define the following parameters:

- **fp**, the passband edge
- **fs**, the stopband edge frequency
- δ, the maximum ripple

Mathcad then calculates the filter order and finds the zeros, poles, and coefficients for the filter transfer function.

Background

Elliptic filters, so called because they employ elliptic functions to generate the filter function, are useful because they have equiripple characteristics in the pass and stopband regions. This ensures that the lowest order filter can be used to meet design constraints.

For information on how to transform a continuous-time IIR filter to a discrete-time IIR filter, see **Section 13.1: Analog/Digital Lowpass Butterworth Filter**.

For information on the lowest order polynomial (as opposed to elliptic) filters, see **Section 14: Chebyshev Polynomials**.

Mathcad Implementation

This document implements an analog elliptic lowpass filter design. The definitions below of elliptic functions are used throughout the document to calculate filter characteristics.

Elliptic Function Definitions

$$\mathbf{TOL} \equiv 10^{-5}$$

$$U(\phi, k) \equiv 1i \cdot \int_0^{\mathrm{Im}(\phi)} \frac{1}{\sqrt{1 - k^2 \cdot \sin(1j \cdot y)^2}} \, dy + \int_0^{\mathrm{Re}(\phi)} \frac{1}{\sqrt{1 - k^2 \cdot \sin(1j \cdot \mathrm{Im}(\phi) + y)^2}} \, dy$$

$$\phi \equiv 1i \qquad sn(u, k) \equiv \sin(\mathbf{root}(U(\phi, k) - u, \phi))$$

$$L(k) \equiv \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \cdot \sin(y)^2}} \, dy$$

$$V(k) \equiv \frac{2}{1 + \sqrt{1 - k^2}} \cdot L\left(\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right)\right)$$

$$K(k) \equiv \mathbf{if}(k < .9999, L(k), V(k))$$

Design Specifications

First, the passband edge is normalized to 1 and the maximum passband response is set at one. Set the stopband edge and the maximum ripple in the passband and stopband. The filter order will be calculated from these constraints.

Pass band edge: $\omega p_p := 1$

Stop band edge: $\omega s_s := 1.2$

Pass band ripple: $\delta_1 := .05$

Stop band ripple: $\delta_2 := .05$

Calculation of Filter Order

Definitions of moduli:

$$\varepsilon := \sqrt{\frac{2 \cdot \delta_1 - \delta_1^{\,2}}{1 - 2 \cdot \delta_1 + \delta_1^{\,2}}}$$

$$k_1 := \frac{\varepsilon}{\sqrt{\frac{1}{\delta_2^{\,2}} - 1}} \qquad k'_1 := \sqrt{1 - k_1^{\,2}}$$

$$k := \frac{\omega p_p}{\omega s_s} \qquad k' := \sqrt{1 - k^2}$$

For correct calculations, check to see that **k** and **k'** are not too close to 1. For accuracy, they should be less than 1 - 10-9.

$$check := k < 1 - 10^{-9} = 1 \qquad check_1 := k'_1 < 1 - 10^{-9} = 1$$

$$N := \text{ceil}\left(\frac{K(k) \cdot K\langle k'_1 \rangle}{K(k') \cdot K\langle k_1 \rangle}\right)$$

The minimum filter order is:

$$N = 5$$

Finding the Zeros and Poles of the Transfer Function

Zeros and poles of the transfer function are calculated using the elliptic functions. If the filter order is odd, there will be one real pole of order 1, in addition to several complex conjugate poles.

$$KK := K(k) = 2.0673$$

$$odd := \text{mod}(N, 2) \qquad I := \textbf{if}(odd, 2, 1)$$

$$m := 0 \,..\, N - I$$

$$c_m := I + 2 \cdot \text{floor}\left(\frac{m}{2}\right)$$

$$sr_m := \frac{\langle -1 \rangle^m \cdot 1j}{k \cdot sn\left(c_m \cdot \frac{KK}{N}, k\right)}$$

$l := 0..N-1 \qquad conj(z,l) := \textbf{if}\left(\text{mod}(l,2) = 0, z, \overline{z}\right)$

$$d_l := 1 - \text{mod}(N,2) + 2 \cdot \text{floor}\left(\frac{l + \text{mod}(N,2)}{2}\right)$$

$$v := \frac{KK}{N \cdot K(k_1)} \cdot U\left(\text{atan}\left(\frac{1}{\varepsilon}\right), k'_1\right)$$

$$sp_l := conj\left(1j \cdot sn\left(\frac{d_l \cdot KK}{N} + 1j \cdot v, k\right), l\right)$$

Poles:

$$sp_l = \begin{bmatrix} -0.5162 + 3.6468i \cdot 10^{-14} \\ -0.2768 - 0.783i \\ -0.2768 + 0.783i \\ -0.0594 - 1.0143i \\ -0.0594 + 1.0143i \end{bmatrix}$$

Zeros:

$$sr_m = \begin{bmatrix} 1.0941 \cdot 10^{-9} + 1.7229i \\ -1.0941 \cdot 10^{-9} - 1.7229i \\ -1.658 \cdot 10^{-12} + 1.2333i \\ 1.658 \cdot 10^{-12} - 1.2333i \end{bmatrix}$$

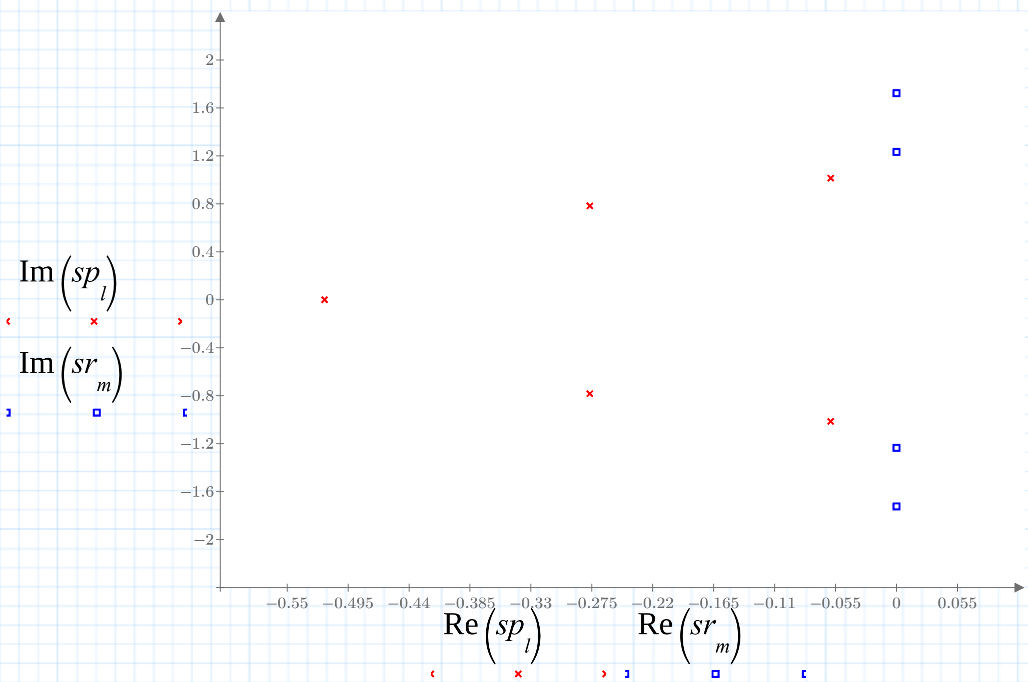

Fig. 13.2.1 Poles and zeros in the complex s-plane

Use the poles and zeros of the system to construct the transfer function. It will be written as a factored expression of each pole and zero in the form **(s - p1)(s - p2)(...)**, etc..

$$A := \text{if}\left(\text{mod}(N,2) = 0, 1-\delta_l, 1\right) \cdot \frac{\prod_l sp_l}{\prod_m sr_m}$$

The transfer function, **H**, normalized so that max **H(jω)** = 1:

$$H(s) := A \cdot \frac{\prod_m \left(s - sr_m\right)}{\prod_l \left(s - sp_l\right)}$$

Define dB magnitude:

$$dbm(\omega) := 20 \cdot \log\left(\left|H(1j \cdot \omega)\right|\right)$$

Frequency range to plot:

$$\omega := 0, .01 .. 3$$

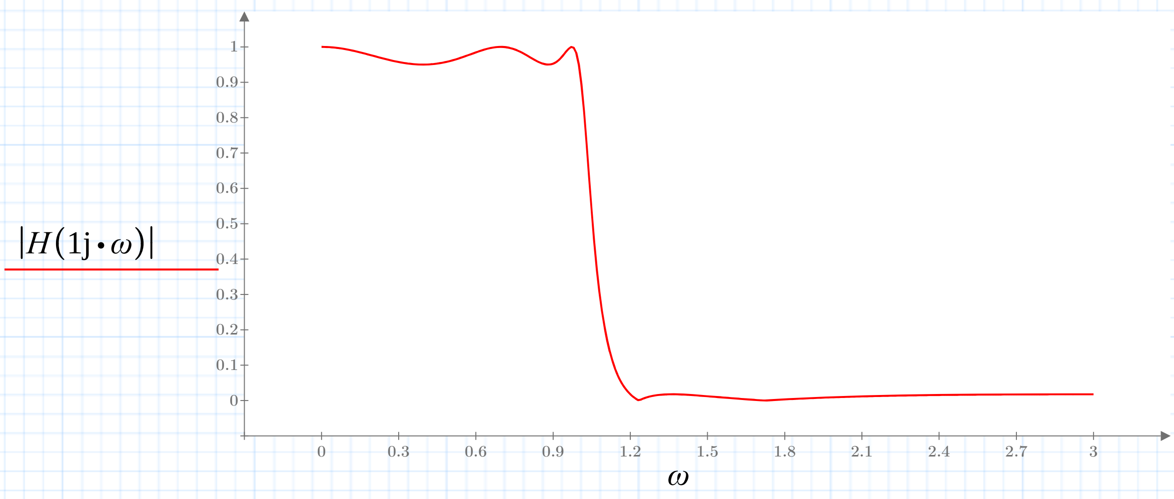

Fig. 13.2.2 Magnitude frequency response of filter

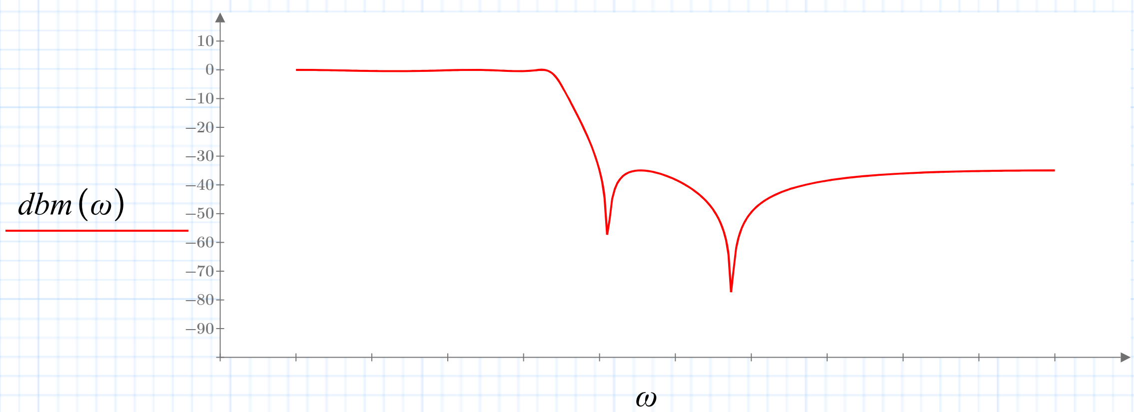

Fig. 13.2.3 Magnitude frequency response of filter in dB

Polynomial Expansion

Next expand the transfer function into **N**th order polynomials in the numerator and denominator. The arrays **num** and **den** hold the coefficients of the numerator and denominator polynomials. The integral used to calculate these coefficients by Cauchy's formula will return accurate results when the order **N** is modest (say, less than 15), but the calculation is time-consuming.

Numerator calculation:

$$NUM(s) := \prod_m \left(s - sr_m\right) \qquad u := 0..N - I + 1$$

$$num_u := A \cdot \frac{\int_0^1 NUM((\exp(2j \cdot \pi \cdot x))) \cdot \cos(u \cdot 2 \cdot \pi \cdot x)\, dx}{\textbf{if}(u = 0, 1, .5)}$$

Denominator calculation:

$$DEN(s) := \prod_l \left(s - sp_l\right) \qquad w := 0..N$$

$$den_w := \frac{\int_0^1 DEN((\exp(2j \cdot \pi \cdot x))) \cdot \cos(w \cdot 2 \cdot \pi \cdot x)\, dx}{\textbf{if}(w = 0, 1, .5)}$$

The numerator is

$$\sum_u \left(num_u \cdot s^u\right)$$

The denominator is

$$\sum_w \left(den_w \cdot s^w\right)$$

$$u = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad num_u = \begin{bmatrix} -0.3675 + 2.5967i \cdot 10^{-14} \\ 9.1918 \cdot 10^{-17} + 3.6148i \cdot 10^{-17} \\ -0.3654 - 1.5659i \cdot 10^{-10} \\ 1.0844 \cdot 10^{-16} - 4.0667i \cdot 10^{-17} \\ -0.0814 - 1.0316i \cdot 10^{-10} \end{bmatrix}$$

$$w = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \qquad den_w = \begin{bmatrix} 0.3675 - 2.6201i \cdot 10^{-14} \\ \vdots \end{bmatrix}$$

In some cases, it may be desirable to specify a filter order, and calculate other terms in the design parameters. For example, if the filter order above is not meeting constraints closely enough, try an elliptic filter of a larger order. To calculate the magnitude of the ripple given the other constraints, follow the procedure below to calculate the **k**'s. First, specify the filter order, **N**:

$$N := 6 \qquad k := \frac{\omega p_p}{\omega s_s} \qquad k' := \sqrt{1-k^2}$$

$$r := N \cdot \left(\frac{K(k')}{K(k)}\right) \qquad x := .005$$

$$k_1 := \mathbf{root}\left(\frac{K\left(\mathrm{Re}\left(\sqrt{1-x^2}\right)\right)}{K(x)} - r, x\right)$$

$$k'_1 := \sqrt{1-{k_1}^2}$$

$$check := k' < 1 - 10^{-9} = 1 \qquad check_1 := k'_1 < 1 - 10^{-9} = 1$$

Now if, for example, the stopband ripple is given, we can calculate the minimal passband ripple as follows:

$$\delta_2 = 0.05 \qquad \varepsilon := k_1 \cdot \sqrt{\frac{1}{{\delta_2}^2} - 1} \qquad \delta_1 := 1 - \frac{1}{\sqrt{1+\varepsilon^2}}$$

The minimal passband ripple is:

$$\delta_1 = 0.0005$$