

## APPLICATIONS IN FILTER DESIGN

### Section 15 Digital FIR Filter Design by Windowing

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This document defines five standard windowing functions used in digital filter design: Bartlett, Hanning, Hamming, Blackman, and Kaiser. Two design examples illustrate the application of these functions. You enter the following parameters:

- $\beta$ , the Kaiser parameter
- $M$ , the sampling length
- $\omega_c$ , the cutoff frequency

Example designs for a lowpass filter using a Hamming window and a Kaiser window are shown.

#### References

Example 8.2 follows the discussion in Section 7.4 of Oppenheim and Schaffer, *Discrete-Time Signal Processing*, Prentice Hall. See also T. W. Parks and C. S. Burrus, *Digital Filter Design*, John Wiley & Sons (New York, 1987).

#### Background

A linear-phase FIR filter can be obtained by windowing, or truncating, the impulse response of an ideal filter with one of several available windowing functions. The window truncation in time results in a convolution of the window transform with the filter transform in frequency.

#### Choosing a Window

The sampled window in time should be as short as possible, to reduce calculation time. The window should also be narrow in frequency, so that convolution with the impulse response results in minimal distortion. Since time and frequency are inversely and inextricably related, these two requirements are at odds.

A variety of window shapes in time can be used to minimize undesirable effects in the frequency domain. Each optimizes some feature of the resulting frequency response: some provide narrower passbands with less attenuation in the stopbands, some provide shallower transition regions with better attenuation in the stopband, etc.

#### Phase

Since filter design is concerned with phase distortion as well as amplitude filtering, it's important to know how windowing affects the phase of the original filter. Windows that are symmetric in time do not distort the phase of the system.

## Mathcad Implementation

This document defines some of the standard windowing functions used in FIR filter design and illustrates their application.

Below are definitions and plots of five common windowing functions: Bartlett, Hanning, Hamming, Blackman, and Kaiser. The first four are given as functions of two variables (the length and the term index). The Kaiser window is also a function of the parameter **b**; it is plotted below for **b** = 5.

### Rectangular window:

$$R(n, N) := (n \geq 0) \cdot (n \leq N - 1)$$

### Bartlett (triangular) window:

$$Ba(n, N) := \text{if} \left( n \leq \frac{N-1}{2}, \frac{2 \cdot n}{N-1}, 2 - \frac{2 \cdot n}{N-1} \right) \cdot R(n, N)$$

### Hanning window:

$$Hn(n, N) := \left( .5 - .5 \cdot \cos \left( \frac{2 \cdot \pi \cdot n}{N-1} \right) \right) \cdot R(n, N)$$

### Hamming window:

$$Hm(n, N) := \left( .54 - .46 \cdot \cos \left( \frac{2 \cdot \pi \cdot n}{N-1} \right) \right) \cdot R(n, N)$$

### Blackman window:

$$Bl(n, N) := \left( .42 - .5 \cdot \cos \left( \frac{2 \cdot \pi \cdot n}{N-1} \right) + .08 \cdot \cos \left( \frac{4 \cdot \pi \cdot n}{N-1} \right) \right) \cdot R(n, N)$$

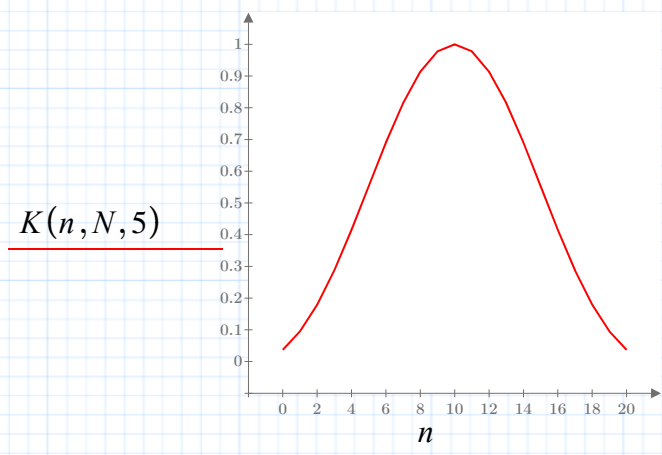
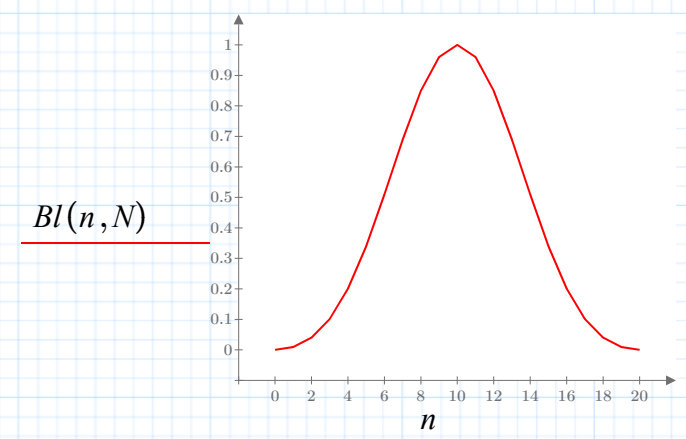
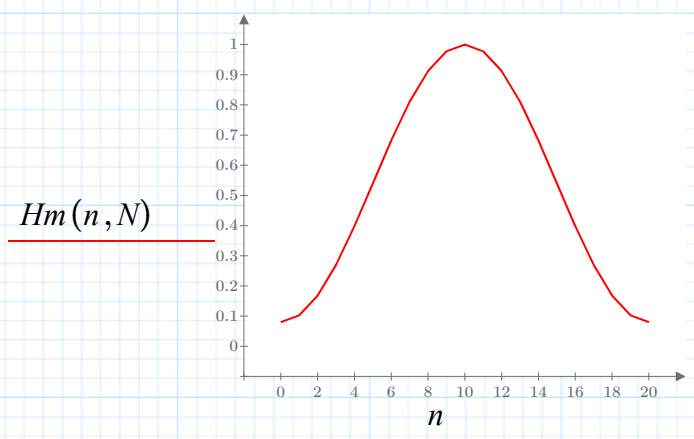
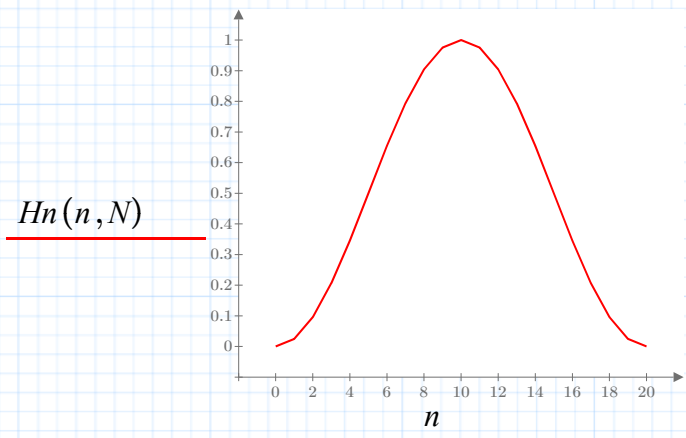
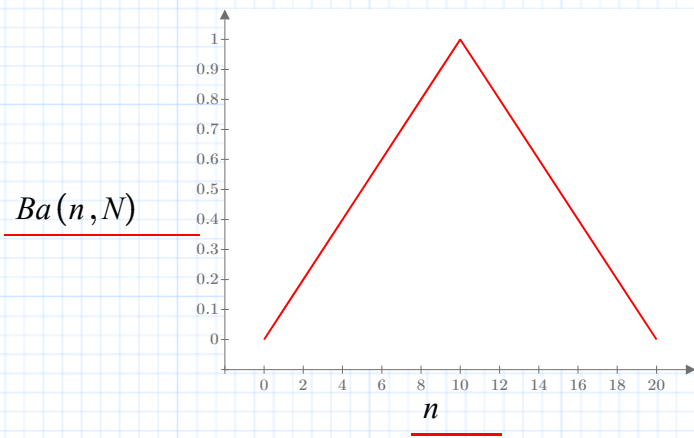
### Kaiser window with parameter b:

$$K(n, N, \beta) := \frac{\text{I0} \left( \beta \cdot \sqrt{1 - \left( \frac{N-1-2 \cdot n}{N-1} \right)^2} \right)}{\text{I0}(\beta)} \cdot R(n, N)$$

Define a length and a range variable to plot the functions:

$$N := 21$$

$$n := 0 \dots N - 1$$



**Fig. 15.1 Plots of Windowing Functions**

The following equations give the impulse response of an ideal lowpass filter with a specified length and cutoff frequency. First, specify a length and cutoff frequency to use in constructing the time-domain impulse train:

length:

$$M := 21 \qquad m := 0..M-1$$

cutoff frequency:

$$\omega_c := .5 \cdot \pi$$

The time response is given by:

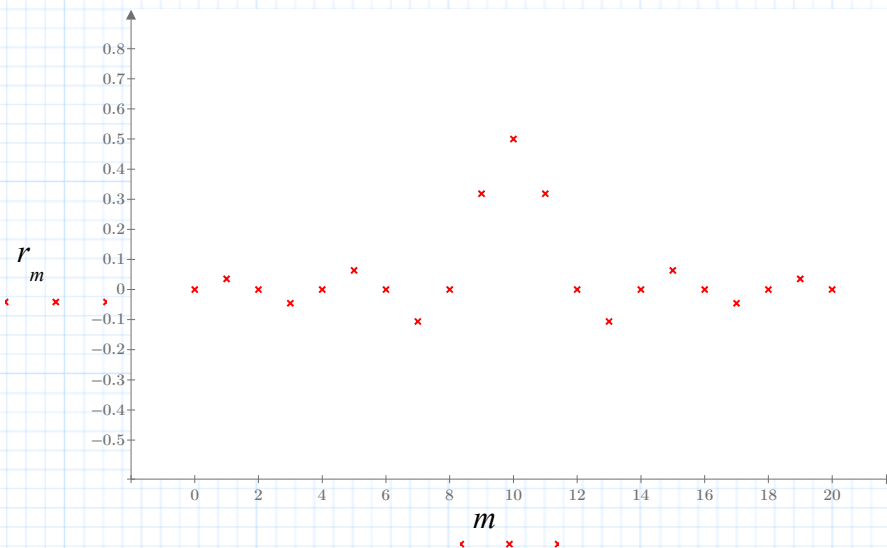
$$r_m := \frac{\sin\left(\omega_c \cdot \left(m - \frac{M-1}{2}\right)\right)}{\pi \cdot \left(m - \frac{M-1}{2}\right) + 10^{-6}}$$

Add a small offset to avoid a *divide by zero* when  $m = 10$

For odd  $M$ , these functions center the filter impulse response in time:

$$odd := \text{mod}(M, 2) \qquad center := \text{if}\left(odd, \frac{M-1}{2}, \frac{M}{2}\right)$$

$$r_{center} := \text{if}\left(odd, \frac{\omega_c}{\pi}, r_{center}\right)$$



**Fig. 15.2** The filter impulse response in time

## Linear Phase FIR Using a Hamming Window

This design example windows the impulse response of an ideal lowpass filter with a windowing function  $\mathbf{w}$ . This example uses a Hamming window. To see the effect of a different window, edit the function name on the right side of the definition for  $\mathbf{w}$ .

Choose a windowing function:

$$w_m := Hm(m, M)$$

Mathcad calculates the vector of coefficients for the windowed impulse response using the **vectorize** operator, which multiplies each element of  $\mathbf{r}$  by the corresponding element of  $\mathbf{w}$ .

The coefficients  $\mathbf{a}$  of the windowed impulse response are given by

$$a := \overrightarrow{r \cdot w}$$

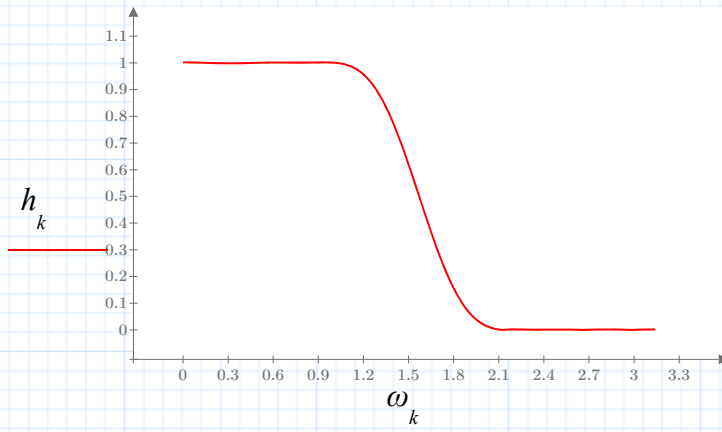
The corresponding frequency response is:

$$H(\omega) := \sum_m \left( a_m \cdot \exp(-1j \cdot m \cdot \omega) \right)$$

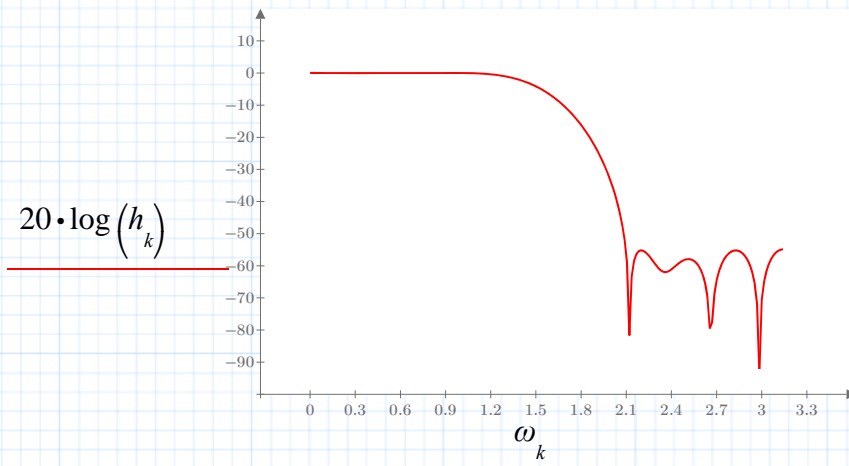
To plot the frequency response, define range variables for the frequency:

$$k := 0 .. 200$$

$$\omega_k := \frac{\pi \cdot k}{200} \quad h_k := \left| H(\omega_k) \right|$$

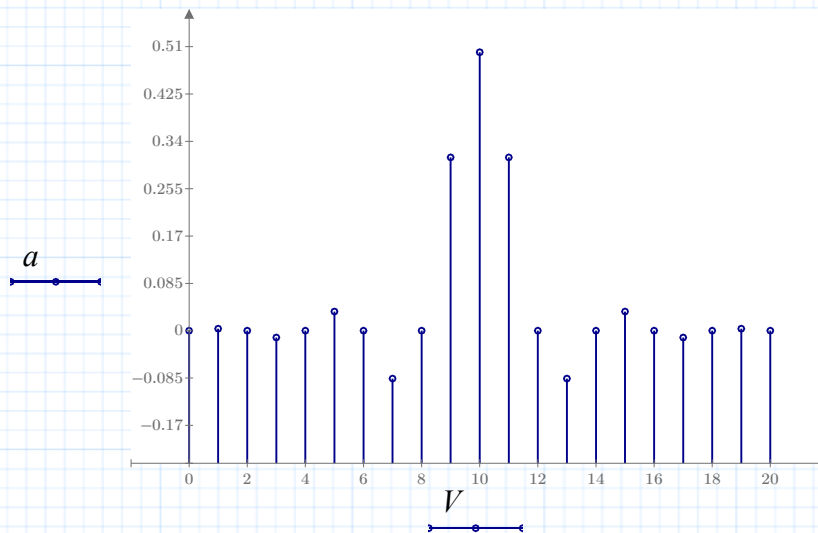


**Fig. 15.3 Filtered frequency response**



**Fig. 15.4 Filtered frequency response in dB**

$$V_m := m$$



**Fig. 15.5 Fourier coefficients**

## Linear Phase FIR Using a Kaiser Window

Now, use a Kaiser window to design a filter with a given approximation error,  $\delta$ , and given stopband and passband cutoff frequencies.

Enter values for these three design parameters.

$$\text{peak approximation error:} \quad \delta := .001$$

$$\text{passband cutoff:} \quad \omega_p := .4 \cdot \pi$$

$$\text{stopband cutoff:} \quad \omega_s := .6 \cdot \pi$$

First, set the cutoff frequency equal to the average of the passband and stopband cutoffs:

$$\omega_c := .5 \cdot (\omega_s + \omega_p) = 1.571$$

The following formulas calculate the necessary filter length and value of the parameter  $\mathbf{b}$  for the Kaiser window.

$$A := -20 \cdot \log(\delta)$$

$$\beta := \text{if}(A < 21, 0, \text{if}(A \leq 50, .5842 \cdot (A - 21)^4 + .07886 \cdot (A - 21), .1102 \cdot (A - 8.7)))$$

$$M := \text{ceil}\left(1 + \frac{A - 8}{2.285 \cdot (\omega_s - \omega_p)}\right)$$

For the parameters chosen above,

$$\beta = 5.653 \quad M = 38$$

Now calculate the filter coefficients:

$$M = 38 \quad m := 0..M - 1$$

$$r_m := \frac{\sin\left(\omega_c \cdot \left(m - \frac{M-1}{2}\right)\right)}{\pi \cdot \left(m - \frac{M-1}{2}\right)}$$

$$\text{odd} := \text{mod}(M, 2) \quad \text{center} := \text{if}\left(\text{odd}, \frac{M-1}{2}, \frac{M}{2}\right)$$

$$r_{\text{center}} := \text{if}\left(\text{odd}, \frac{\omega_c}{\pi}, r_{\text{center}}\right)$$

The filter coefficients are

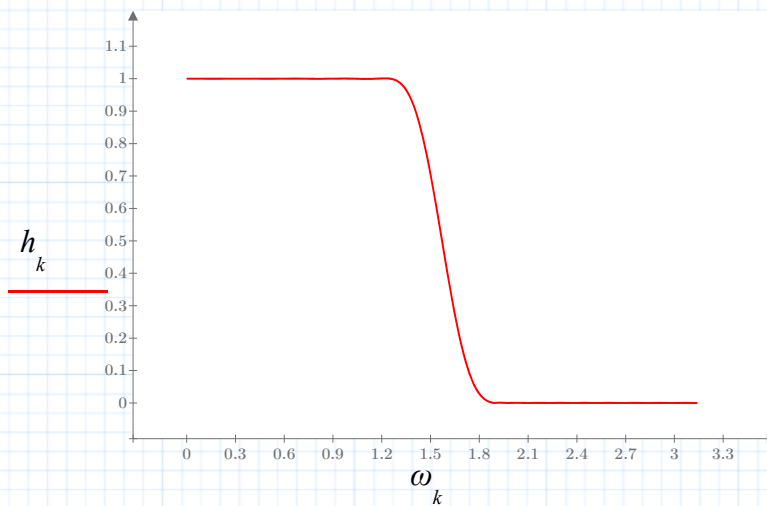
$$a_m := r_m \cdot K(m, M, \beta)$$

The frequency response is

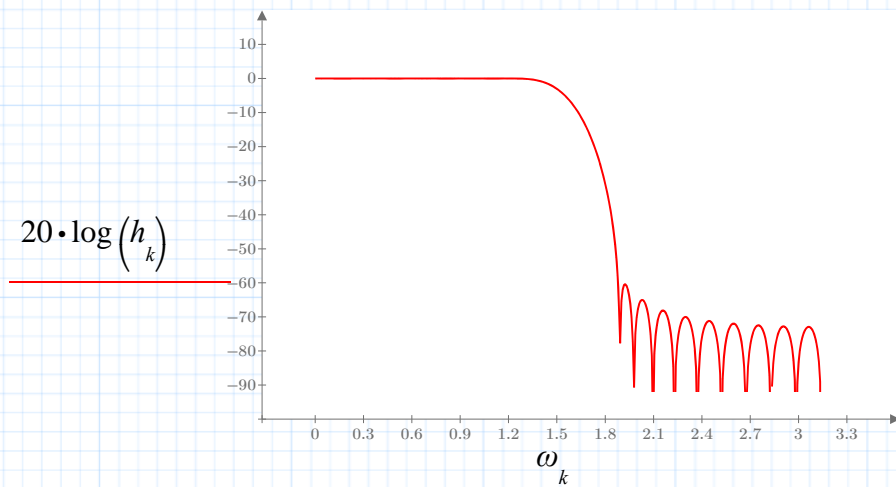
$$H(\omega) := \sum_m (a_m \cdot \exp(-1j \cdot m \cdot \omega))$$

Define the frequency range:

$$k := 0..400 \quad \omega_k := \frac{\pi \cdot k}{400} \quad h_k := |H(\omega_k)|$$



**Fig. 15.6 Filtered frequency response**



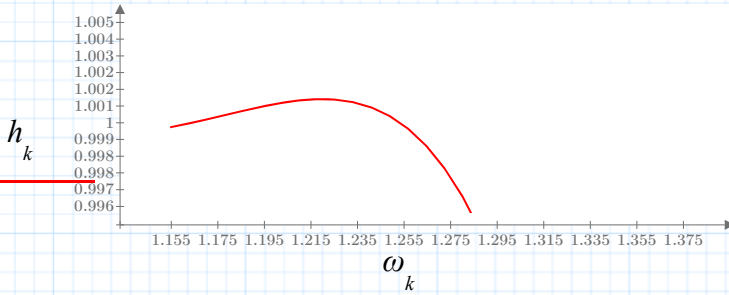
**Fig. 15.7 Filtered frequency response in dB**



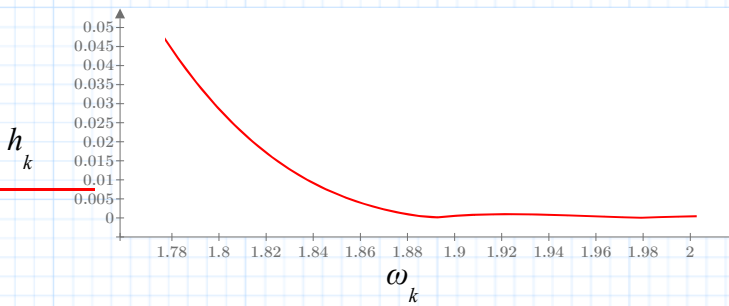


You may wish to magnify the regions around the passband and stopband cutoff frequencies, to show the real peak error. This is done below. First, select **Mathcad Ribbon: Plots** tab. Next, place your cursor on the argument (input variable) for the x axis ( $\omega$ ). Then, select **Tick Mark Values** from the ribbon to display the values on the x axis. Finally, change the first and last tick mark values along the x axis to display smaller section of x data in a magnified view.

Notice that the values specified earlier for peak error and cutoff frequencies may not match perfectly.



**Fig. 15.8 Filtered frequency response at passband cutoff**



**Fig. 15.9 Filtered frequency response at stopband cutoff**

In this example, the actual peak error is slightly larger than **d**. It can be calculated using the **root** and **derivative** operators:

$$x := \omega_p \quad (\text{guess for true error})$$

$$\left| H \left( \text{root} \left( \frac{d}{dx} |H(x)|, x \right) \right) \right| - 1 = 0.00113 \quad \text{actual peak error}$$

The actual passband and stopband cutoff frequencies corresponding to the design value of **d** can be calculated using the **root** function, using the original design for guess values:

$$x := \omega_p \quad y := \omega_s$$

$$\text{root}(|H(x)| - 1 + \delta, x) = 0.402 \pi \quad \text{actual passband cutoff}$$

$$\text{root}(|H(y)| - \delta, y) = 0.598 \pi \quad \text{actual stopband cutoff}$$

The pass band and stop band frequencies found in this way are, respectively, a bit larger than  $0.4p$  and a bit smaller than  $0.6p$ , which agrees with the original design values.