CHAPTER 10 BUILDING THERMAL CONTROL

10.1 Laplace Transfer Functions for Building Thermal Control

Laplace transforms facilitate the solution of differential equations representing dynamic building thermal behavior by converting them into algebraic functions of the transform variable *s*.

Laplace (s-domain) transfer functions may be employed to model the input-output behavior of building envelope components, heating/cooling system components and the control system. Laplace transfer functions for the three building subsystems may then be combined by means of simple algebraic equations based on block-diagram algebra to obtain the overall system response in the Laplace domain. The system Laplace domain response may be employed for two types of analysis:

1. Transient thermal control studies such as room temperature response to a setpoint change or a heat source; the transient response may be determined by analytical Laplace transform inversion for simple cases or numerically.

2. Frequency domain analysis of the open-loop and closed-loop transfer function for stability analysis.

Definitions

The Laplace transform *f(s)* for a function *f(t)* is defined as

$$L(f(t)) = f(s) = \int_0^{\infty} f(t) \cdot exp(-st) \, \mathrm{d}t$$

The Laplace transform is a linear transform. Thus, the superposition principle can be applied to determine effects of various inputs on building response.

Useful Laplace Transforms

1. Ramp function: (often used in room setpoint profiles)

$$f(t) = at \cdot (t > 0) \qquad f(s) = \frac{a}{s^2}$$

2. Exponential:

$$f(t) = exp(-at) \cdot (t > 0) \qquad f(s) = \frac{1}{s+a}$$

3. Step function: (common in setpoint changes)

$$f(t) = A \cdot (t > 0) \qquad f(s) = \frac{A}{s}$$

4.

$$f(t) = sin(wt) \qquad f(s) = \frac{w}{s^2 + w^2}$$

5.

$$f(t) = cos(w \cdot t) \qquad f(s) = \frac{s}{s^2 + w^2}$$

6. Translation function -- a very common function to model a change (e.g. of setpoint) after a given time or a time delay:

Given

$$L(f(t)) = f(s)$$

If a function *f(t)* is shifted by time t_d

then $$L(f(t - t_d)) = exp(-s \cdot t_d) \cdot f(s)$$

For example, a duct or pipe transfer function may be modeled by a delay transfer function. If a duct is of length *L* and air flows through it with velocity *V*, then the time taken to travel through the duct = t_d = *L/V*.

Therefore,

$$\frac{T_{out}(s)}{T_{in}(s)} = exp(-s \cdot t_d)$$

7. Integral and Derivative

$$L\left(\int_0^\infty f(t)\,\mathrm{d}t\right) = \frac{1}{s}\cdot f(s)$$

$$L\left(\frac{df(t)}{dt}\right) = s\cdot f(s) - f(0)$$ f(0) = initial condition

Simple Models for Thermal Control Analysis

Simple first-order models are often used for quick analysis of the building or the heating/cooling system.

Simple room model for thermal control (shown in schematic below):

Consider a room with total interior thermal capacity *C*, determined by summing the thermal capacity of room interior layers and lightweight contents. Its total conductance between inside and outside is U_t
(thermal resistance $R = 1 / U_t$).

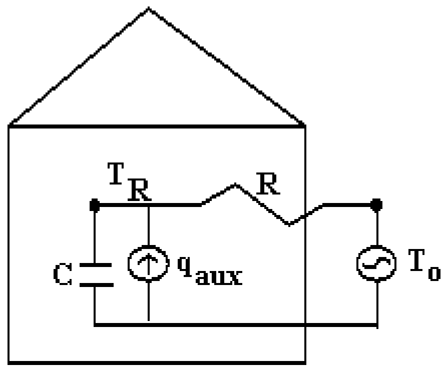
Let

T_R = room temperature
T_o = outside temperature
q_{aux} = auxiliary heating (positive) / or cooling (negative)

An energy balance for the room yields

energy stored + energy lost = heat input

$$C\cdot\frac{dT_R}{dt} + U_t\cdot\left(T_R - T_o\right) = q_{aux}$$



First order zone model

In the Laplace domain:

$$s \cdot C \cdot T_R(s) + U_t \cdot T_R(s) = U_t \cdot T_o(s) + q_{aux}(s)$$

Rearranging, we can show that

$$T_R(s) = \frac{U_t}{C \cdot s + U_t} \cdot T_o(s) + \frac{1}{C \cdot s + U_t} \cdot q_{aux}(s)$$

Transfer functions defined based on the above equation:

$$\frac{U_t}{C \cdot s + U_t} = \frac{1}{\tau \cdot s + 1}$$ transfer function for effect of T_o and T_R

$$\frac{1}{C \cdot S + U_t} = \frac{R}{\tau \cdot s + 1}$$ transfer function for effect of q_{aux} on T_R

$$\tau = R \cdot C$$ room time constant

Example

Investigate the effect of furnace cycling on room temperature for a house with total *U* value (including walls, infiltration, and windows) equal to 500 watt/degC and wall area 200 square meters. The interior lining of the walls is gypsum board (assume this to be the main thermal capacity element). Assume that the furnace is sized based on a minimum outside temperature of -5 degC.

We will determine the room temperature swing as a function of time constant.

First determine furnace size based on

$$U_t := 500 \cdot \frac{watt}{\Delta°C}$$

$$A := 200 \cdot m^2$$

$T_o := -5 \cdot \Delta°C$ outside temperature

$T_R := 20 \cdot \Delta°C$ room temperature

$$q_{max} := U_t \cdot \left(T_R - T_o\right)$$

$q_{max} = \left(1.25 \cdot 10^4\right)\ watt$ furnace capacity

Gypsum board:

$L := 0.013 \cdot m$ thickness

$c := 750 \cdot \frac{joule}{kg \cdot \Delta°C}$ specific heat

$\rho := 800 \cdot \frac{kg}{m^3}$ density

Heat capacity:

$C := A \cdot L \cdot c \cdot \rho$ $\quad C = \left(1.56 \cdot 10^6\right) \frac{joule}{\Delta°C}$

$\tau := \frac{C}{U_t}$ $\quad \tau = 0.867\ hr$ time constant

We will determine the effect of time constant magnitude:

$$\tau_{var} := \tau, 2 \cdot \tau .. 10 \cdot \tau$$

Period of cycling and frequency:

$$P := 1 \cdot hr \qquad w := \frac{2 \cdot \pi}{P}$$

Assume on/off cycling (between 0 and q_{max}). Then, we will consider the effect of the main harmonic of q_{aux} on room temperature swing.

$$q_{mean} := \frac{q_{max}}{2} \qquad q_{aux}(t) := \frac{q_{max}}{2} \cdot \sin(w \cdot t) + q_{mean}$$

The transfer function *G(s)* for the effect of a heat source on T_R was determined as

$$G(s) = \frac{R}{\tau \cdot s + 1} = \frac{1}{U_t \cdot (\tau \cdot s + 1)}$$

Laplace-domain response (harmonic term):

$$\Delta T_R(s) = G(s) \cdot q_{aux}(s)$$

$$\Delta T_R(s) = \frac{1}{U_t \cdot (\tau \cdot s + 1)} \cdot \frac{q_{max} \cdot w}{2\left(s^2 + w^2\right)}$$

It can be shown that

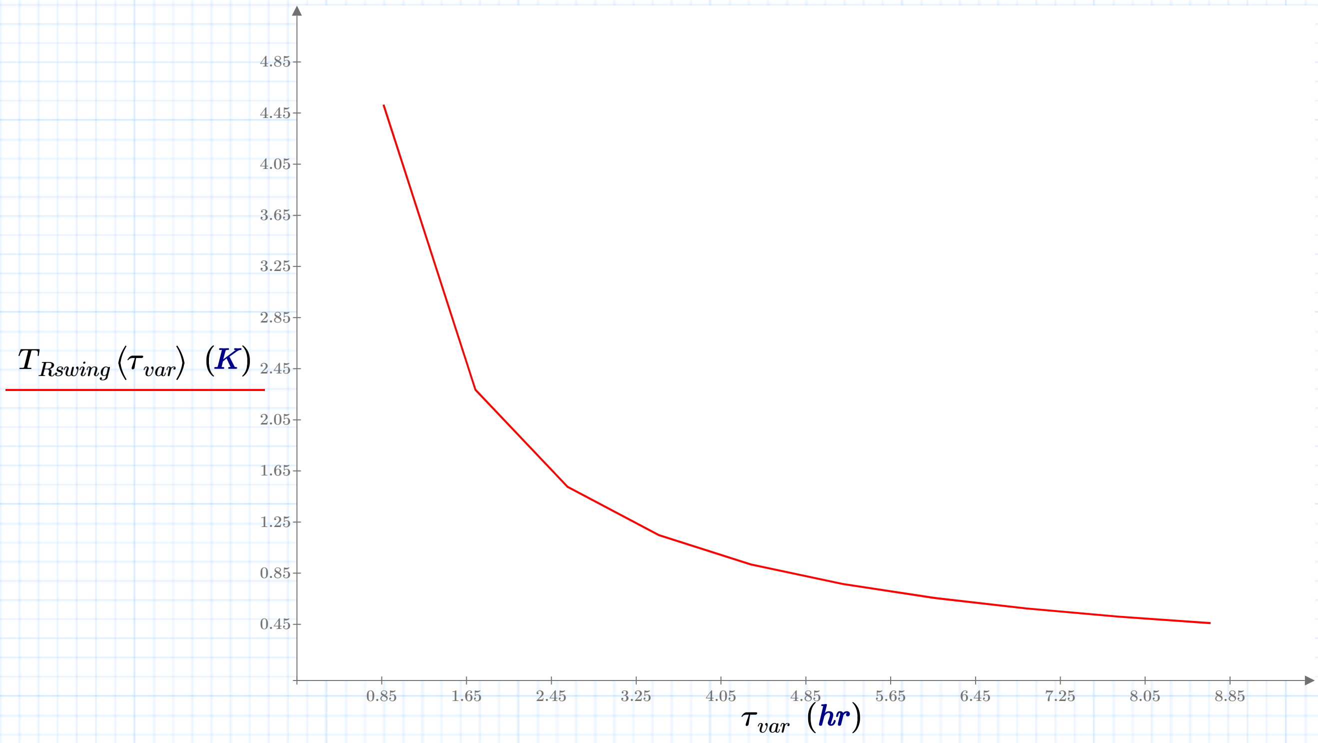
$$\Delta T_R(t) = \frac{q_{max}}{2 \cdot U_t \cdot \sqrt{{\tau_{var}}^2 \cdot w^2 + 1}} \cdot sin(w \cdot t + \phi)$$

where

$$\phi = atan(-w \cdot \tau)$$

$$T_{Rswing}(\tau_{var}) := \frac{q_{max}}{U_t \cdot \sqrt{{\tau_{var}}^2 \cdot w^2 + 1}}$$

As can be seen from the above equation, the room temperature swing decreases with decreasing time constant. This analysis was fairly simple. For a more accurate analysis, a detailed wall model may be used such as the admittance model described in **Section 4.3**.



First order transfer functions are also commonly used for heating coils and sensors (possibly multiplied by a time delay).