Academic hat math
or
Hybrid of symbol, number and graphic in an optimization problem

Valery Ochkov

National Research University «Moscow Power Engineering Institute (MPEI»

Higher School of Economics

"I am a master," he grew stern and took from the pocket a completely greasy black hat with the letter "M" embroidered on it in yellow silk.

Mikhail Bulgakov. The Master and Margarita (Translated from the russian by Richard Pevear and Larissa Volokhonsky[1])

Abstract: The article describes how we can design, tailor and stitch an academic hat for the head of mathematician and how to optimize it using computer math programs

Keywords: Constrained optimization, symbolic and numerical computer mathematics, derivative, Mathcad

At the beginning of each summer young people run out of the doors of educational institutions in gowns and *academic hats*. It is time to get their bachelors and masters degrees.

According to legend, the academic hat originated in the walls of a madrasah. Graduates of this Muslim theological educational institution fastened a square edition of the Koran on top of their fez / yarmulke [1].

In the author's day these events were more modest — the university's badge was just hung on the lapel of the jacket, — a "float", an academic badge. It was a blue rhombus with a white edging and the emblem of the USSR in the middle. Speaking abbreviations of names, some elite universities had special academic signs. On the alma mater of the author of this article, for example, in the middle of the rhombus there was the university abbreviation MPEI (МЭИ, Московский энергетический институт), and a reduced coat of arms was placed at the upper corner of the rhombus — see Fig. 0. This sign was also called the float because higher education helped one to stay "afloat".

[1] https://web.archive.org/web/20110606093139/http://lib.ru/BULGAKOW/master97_engl.txt

Fig. 0. Copyright academic mark

Some universities also produce special academic hats. Let's create a special hat for mathematicians here. Mathematics is considered the queen of sciences. Immanuel Kant is credited with the following saying: "In every natural science there is as much truth as there is applied mathematics in it."[2] This gives rise to a completely “natural” desire to create a special academic hat for graduates of the mathematical departments of universities, different from the hats of non-mathematicians. Unfortunately, it is not sensible to embroider the letter “M” on it — see the epigraph[3]. However, the letter π will be quite appropriate there. Not only because of the form, but also because of the content.

Let's cut and sew not just an academic hat for mathematicians, but *an optimized hat*. The search for optimal solutions is one of the most interesting and complicated sections of mathematics, which has important practical applications. We will design a hat in the form of a straight circular cylinder, covered with a top in the form of a square with side length equal to the diameter of the cylinder (Fig. 1), and with... a minimum surface area. So, we can save material on sewing hats, at the same time as solving a simple and beautiful optimization problem!

---

[2] This is extracted from Metaphysical Foundations of Natural Science (Metaphysische Anfangsgründe der Naturwissenschaft, 1786), original in German: "Ich behaupte aber, daß in jeder besonderen Naturlehre nur so viel eigentliche Wissenschaft angetroffen werden könne, als darin Mathematik anzutreffen ist."

[3] A hat with the letter M can be offered to graduates of the literary and philological departments of universities who want to become masters!

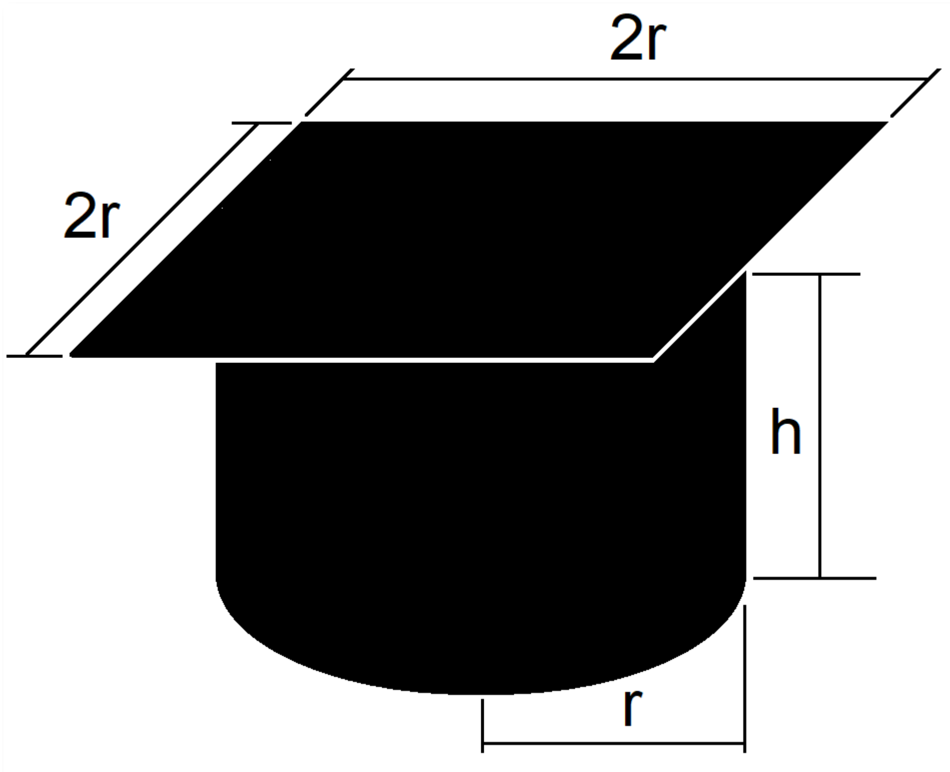


Fig. 1. To the problem of the mathematician academic hat

Figure 2 shows a Mathcad document with the solution to this optimization problem: the derivative with respect to the variable r (the radius of the cylinder) is taken from the expression for the total surface area, S, of the cylinder and the square (a function with argument r and parameter constant V, volume). The derivative is searched for zeros [2][4] (for a smooth continuous function, the derivative at the minimum point is equal to zero). This zero (one of the three zeros is real, not complex) is substituted into the ratio r / h (h is the height of the cylinder), which gives the answer π / 4. This expression could be embroidered on such an optimized academic hat as a mathematician, but it can be “optimized” to π, as suggested below.

[4] The computer can be helped by transforming the derivative into a simple fraction and looking for zero not in the whole fraction, but only in part of its numerator, while checking that the denominator is not equal to zero (see Fig. 5).

$$V = \pi r^2 h \text{ solve}, h \rightarrow \frac{V}{\pi r^2}$$

$$S = 2\pi r h + (2r)^2 \text{ substitute}, h = \frac{V}{\pi r^2} \rightarrow S = \frac{2\left(2r^3 + V\right)}{r}$$

$$\frac{d}{dr}\frac{2\left(2r^3 + V\right)}{r} \rightarrow 12r - \frac{4r^3 + 2V}{r^2}$$

$$12r - \frac{4r^3 + 2V}{r^2} = 0 \text{ solve}, r \rightarrow \begin{bmatrix} \left(\frac{V}{4}\right)^{\frac{1}{3}} \\ -\frac{\left(\frac{V}{4}\right)^{\frac{1}{3}}}{2} + \frac{\sqrt{3}\left(\frac{V}{4}\right)^{\frac{1}{3}} i}{2} \\ -\frac{\left(\frac{V}{4}\right)^{\frac{1}{3}}}{2} - \frac{\sqrt{3}\left(\frac{V}{4}\right)^{\frac{1}{3}} i}{2} \end{bmatrix}$$

$$\frac{r}{h} = \frac{r}{\frac{V}{\pi (r)^2}} \text{ substitute}, r = \left(\frac{V}{4}\right)^{\frac{1}{3}} \rightarrow \frac{r}{h} = \frac{\pi}{4} \qquad \frac{\pi}{4} = 0.785$$

Fig. 2. The analytical solution to the problem of the academic hat mathematician (Mathcad)

First of all, the author “tried on” the designed academic hat on himself: he calculated its dimensions for the case when the head has a fifty-eighth (author's) size. The reader will laugh, but it turned out that with a head circumference of 58 cm, the volume of the hat turned out to be... *three and fourteen hundredths* (3.14) of a liter (see Figs. 3 and 8). This reassured and pleased the author. The fact is that the author is not a Mathematician by education (he is a thermal power engineer[5]). The letter π on the "facade" of the hat indicates the size (or volume) of the head, and not its content.

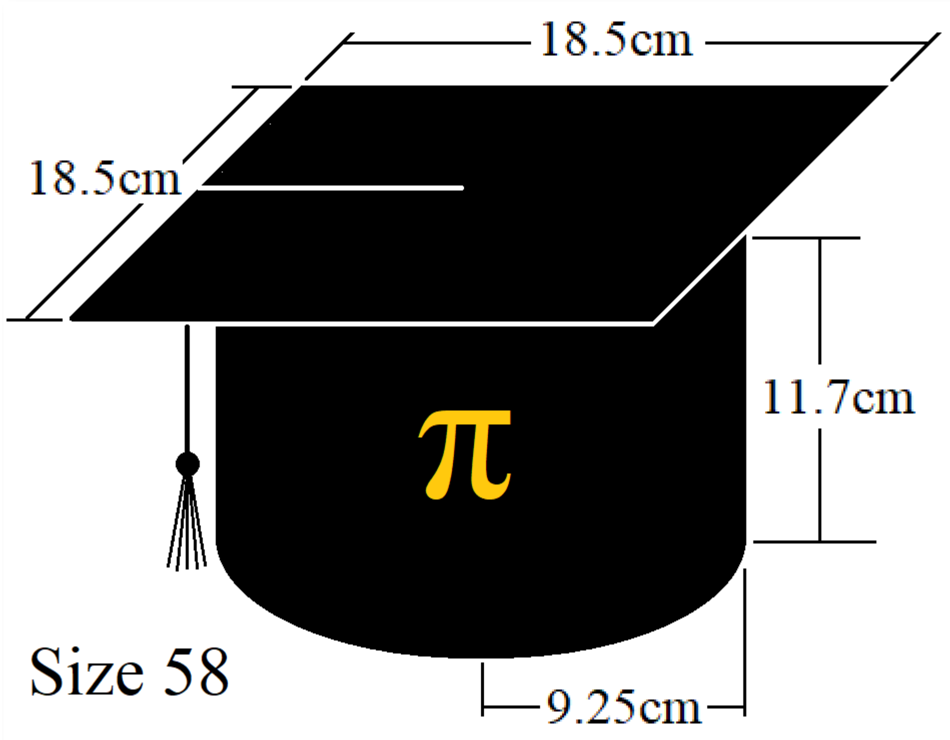


Fig. 3. Sketch of the math academic hat

The author at one of his lectures on computer science [3] asks students to write on pieces of paper their height, weight and head size. The author then uses these data in a lecture about regression analysis as a kind of typical statistical sample [4]. It turns out that the average size of the head circumference of students is 57 cm. This is understandable: the professor should be somewhat “smarter” than his students! ☺ But we should expect that the heads of students during their studies at the university will grow to size 58. So this size (58) can be considered the most suitable for hats.

It is possible to continue optimizing the tailoring of academic hats – to calculate how much a piece of fabric will be required with the optimal cutting of squares and rectangles going to tailor the hats for the entire university graduation so that there is a minimum of trimmings. This optimization problem belongs

[5] In the author’s second alma mater — at the University of Stuttgart, where the author underwent a scientific internship in the field of heat power engineering, a person who wore a doctoral dissertation was put on an academic hat with an entire thermal power station at the top. There was a boiler and a steam turbine; it all spun and whistled, generating electricity. The newly minted doctor was put on a special trolley and transported around the university campus.

to the class of linear programming problems. "The term" programming "must be understood in the sense of" planning "(one of the translations of English. Programming). It was proposed in the mid-1940s by George Danzig, one of the founders of linear programming, even before computers were used to solve linear optimization problems" [5]. A great contribution to the theory of linear programming was made by the Soviet scientist Leonid Kantorovich, who published the work "Mathematical Methods of Organization and Production Planning". The Nobel Memorial Prize, which he shared with Tjalling Koopmans, was given "for their contributions to the theory of optimum allocation of resources." (https://en.wikipedia.org/wiki/Leonid_Kantorovich)

And now let's talk about the second title of the article — about numerical (approximate), symbolic (analytical) and graphical computer tools for solving ~~this~~ optimization problems, based on a more complex example. It is necessary to find the parameters of a straight circular cone, covered with a hemisphere, at which the outer surface of such a composite geometric body will be minimal for a given volume. The problem "can also have a purely practical application" — a cone with ice cream (see Fig. 4) at its minimum outer surface will slowly melt in the hands of a mathematician who dresses in a heavy academic robe in the hot summer, puts a hat on his head and decides to chill with ice cream.

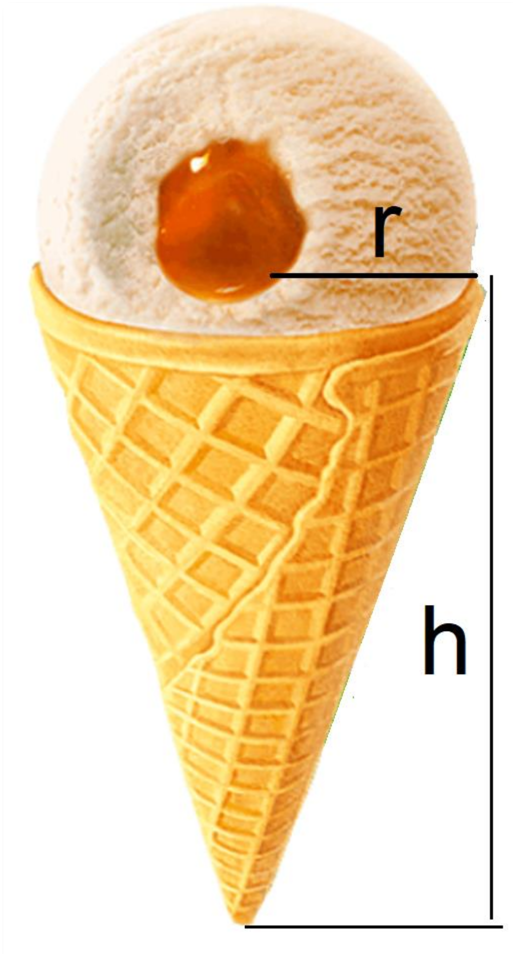


Fig. 4. A cone with ice cream

Figure 5 shows the solution to this optimization problem by the method already shown in Fig. 2: taking the derivative and finding zeros from it. But the attempt to find zeros in the numerator of the derivative of the surface function was not entirely successful — see the comment (error message) in the penultimate operator in Fig. 5. The Mathcad package could not show the expression itself — I had to be content with its approximate value 0.888...

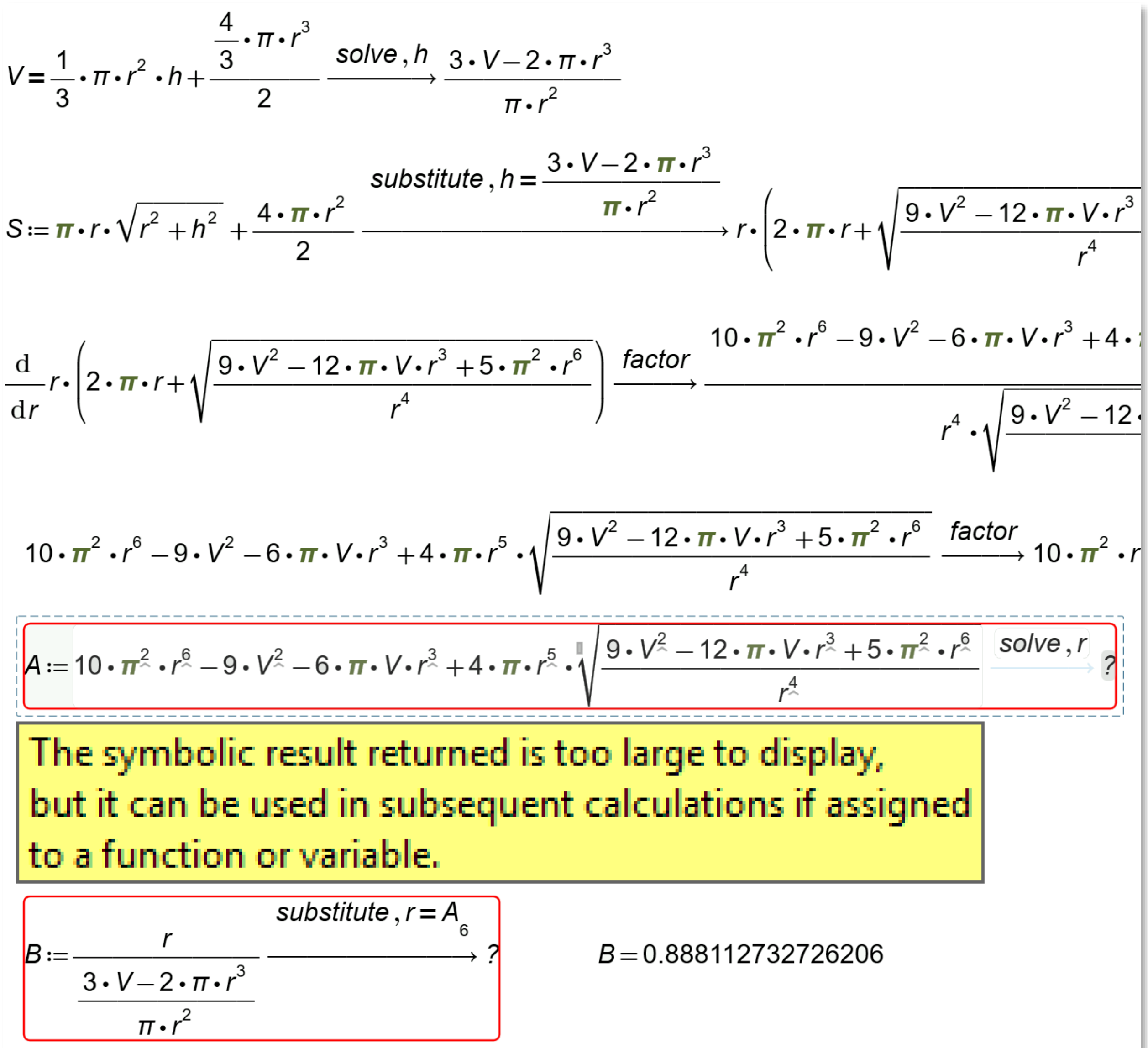


Fig. 5. An attempt to analytically solve the ice cream cone problem

In the last statement in Figure 5, by manually changing the index (0, 1, 2, etc., to 6), various zeros were *substituted* until the seventh zero gave the desired numerical answer 0.888... But this "absolutely accurate and absolutely useless"[6] expression could not be displayed. This expression could only be seen in the Maple environment — see Fig. 6.

---

[6] A balloon burst from the clouds. The flying people saw a man on Earth and shouted to him: "Where are we ?!" "You are in the basket of a balloon," was the answer. This answer was given by a mathematician. Only from a mathematician it is possible to hear a completely accurate and at the same time completely useless answer.

$$+\Bigg(-\Bigg(3\Bigg(5\sqrt{\frac{5\left(18\,\mathrm{I}\sqrt{503}+253\right)^{2/3}+107\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}+305}{\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}}}\left(18\,\mathrm{I}\sqrt{503}+253\right)^{2/3}$$

$$+1032\sqrt{3}\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}-214\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}$$

$$\sqrt{\frac{5\left(18\,\mathrm{I}\sqrt{503}+253\right)^{2/3}+107\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}+305}{\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}}}$$

$$+305\sqrt{\frac{5\left(18\,\mathrm{I}\sqrt{503}+253\right)^{2/3}+107\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}+305}{\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}}}\Bigg)\Bigg)\Bigg/\Bigg(\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}$$

$$\sqrt{\frac{5\left(18\,\mathrm{I}\sqrt{503}+253\right)^{2/3}+107\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}+305}{\left(18\,\mathrm{I}\sqrt{503}+253\right)^{1/3}}}\Bigg)\Bigg)^{1/2}-9\Bigg)\pi^{2}\Bigg)^{1/3}\Bigg),$$

Fig. 6. Figure academic mantle math (Maple)

Figure 6 shows about one hundredth of the expression for the zeros of the derivative generated by the Maple package when solving the problem of the optimal ice cream cone. This full expression would be appropriate on the mantle of a bachelor or master, if embroidered with gold on a black background. And it would be even better if the mantles of the individual were embroidered with fragments of their final work: with formulas, tables and graphs.

Yes, a purely symbolic solution to a problem often does not give an answer or gives an “absolutely accurate and absolutely useless answer” — see fig. 6. Therefore, it is necessary to move on to numerical (approximate) mathematics. The second reason for the rejection of symbolic mathematics is the rapid development of computer technology.

However, the best solution here would be a hybrid technology: a combination of symbolic and numerical mathematics, together with the graphics of modern software computing systems.

Figure 7 shows a hybrid solution to the problem of an optimal ice cream cone using symbolic and numerical mathematics of the Mathcad package, as well as its graphics: the functional dependence of the surface area of the cone with the ice cream volume of 200 cm^3 is obtained symbolically in terms of the radius of the hemisphere, whose minimum is obtained with the help of the "digital ”built-in Minimize function. This particular point is shown in the graph for clarity.

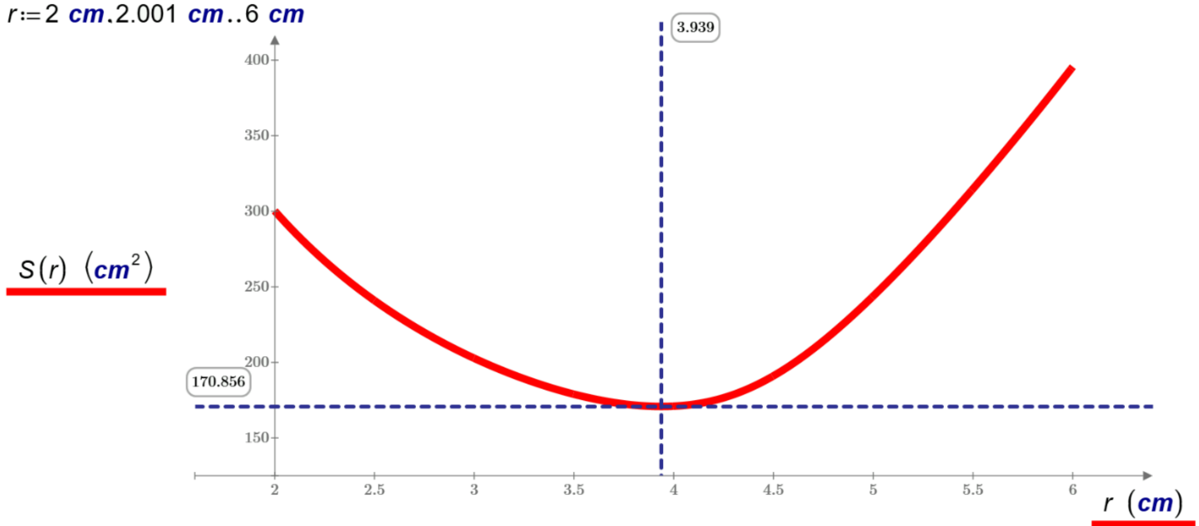
$$V = \frac{1}{3}\cdot\pi\cdot r^2\cdot h + \frac{\frac{4}{3}\cdot\pi\cdot r^3}{2} \xrightarrow{solve,\,h} \frac{3\cdot V - 2\cdot\pi\cdot r^3}{\pi\cdot r^2}$$

$$V := 200\ cm^3 \qquad \mathbf{clear_{sym}}(V)$$

$$S(r) := \pi\cdot r\cdot\sqrt{r^2 + h^2} + \frac{4\cdot\pi\cdot r^2}{2} \xrightarrow{substitute,\, h = \frac{3\cdot V - 2\cdot\pi\cdot r^3}{\pi\cdot r^2}} r\cdot\left(2\cdot\pi\cdot r + \sqrt{\frac{9\cdot V^2 - 12\cdot\pi\cdot V\cdot r^3 + 5\cdot\pi^2\cdot r^6}{r^4}}\right)$$

$$r_{opt} := 3\ cm \qquad r_{opt} := \mathbf{Minimize}\left(S, r_{opt}\right) = 3.939\ cm \qquad S_{min} := S\left(r_{opt}\right) = 170.856\ cm^2$$

$$r := 2\ cm, 2.001\ cm .. 6\ cm$$



$$h_{opt} := \frac{3\cdot V - 2\cdot\pi\cdot r_{opt}^{\ 3}}{\pi\cdot r_{opt}^{\ 2}} = 4.435\ cm \qquad \frac{r_{opt}}{h_{opt}} = 0.888112562583671$$

Fig. 7. Hybrid solution of the ice cream cone problem

Rounding off our narrative — moving on to the first title of the article, it can be noted that the problem of an optimized mathematician hat can be solved purely numerically — without "academic sophistication". Such a calculation is a control one made to verify symbolic calculations in which errors are also possible [7].

Figure 8 shows the "control" numerical solution to the problem of the optimal academic hat of a mathematician with a head size of 58 cm (sorry, 57.971 cm).

$$V(r,h) := \pi \cdot r^2 \cdot h \qquad S(r,h) := 2\,\pi \cdot r \cdot h + (2\,r)^2$$

Solve

Guess values

$$r := 10\ cm \qquad h := 15\ cm$$

Constraints

$$V(r,h) = \pi \cdot L$$

Solver

$$\begin{bmatrix} r \\ h \end{bmatrix} := \text{Minimize}(S, r, h) = \begin{bmatrix} 9.226 \\ 11.747 \end{bmatrix} cm$$

$$V(r,h) = 3.142\ L \qquad S(r,h) = 0.102\ m^2$$

$$d := 2\,r = 18.453\ cm \qquad Size := \pi \cdot d = 57.971\ cm$$

Fig. 8. Numerical solution of the problem of the academic hat mathematician

Final Remarks

This mathematical study is included in a book that is being prepared for publication by CRC/Tailor & Francis with the title “2^5 Problems for STEM Education”[7]. The book introduces a new and emerging course for undergraduate STEM programs called “Physical-Mathematical Informatics”, following a new direction in education called STE(A)M (Science, Technology, Engineering, [Art] and Mathematics). The book is for undergraduate students (and high school students), teachers of mathematics, physics,

[7] https://www.crcpress.com/25-Problems-for-STEM-Education/Ochkov/p/book/%209780367345259

chemistry and other disciplines (humanities), and readers who have a basic understanding of mathematics and math software.

Bibliography

1. https://en.wikipedia.org/wiki/Square_academic_hat
2. Valery Ochkov, Yulia Chudova, Alexey Dolgushev. Solving a problem on a computer: number, graph, symbol // Computer science at school. No. 3. 2019.S. 55-63 (http://twt.mpei.ac.ru/ochkov/Cylinder.pdf)
3. Ochkov VF, Bogomolova EP, Ivanov DA. Physics and mathematics studies with Mathcad and Internet. St. Petersburg: Publishing House "Lan", 2018. — 560 p. (http://twt.mpei.ac.ru/ochkov/T-2018/PhysMathStudies.pdf)
4. Ochkov VF, Bogomolova EP Interpolation, extrapolation, approximation or Lies, blatant lies and statistics) // Cloud of Science. T. 2, № 1. 2015. C. 61-88 (http://twt.mpei.ac.ru/ochkov/CoS_2_1.pdf)
5. https://en.wikipedia.org/wiki/Linear_programming
6. Ochkov VF, Bobryakov AV, Khorkov SN Hybrid problem solving on a computer // Cloud of Science. Value 4. # 2. 2017. pp. 5-26 (https://cloudofscience.ru/sites/default/files/pdf/CoS_14_168.pdf)
7. Ochkov VF, Fedorov YuS, Voronova YuS, Moiseev AD. Submarine Nautilus, and new educational technologies) // Cloud of Science. Том 5 № 1.2018. C. 5-39 (https://cloudofscience.ru/sites/default/files/pdf/CoS_5_005.pdf)