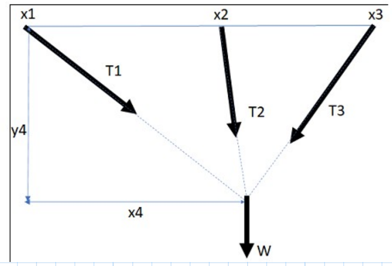
Statics by vectors $\quad$ **ORIGIN** $:= 1$




Let's define a 3D coordinate system, the origin is at x1 in the sketch.

Three unit vectors (for x, y, and z) are $i := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $j := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $k := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. x is positive to the right, y is positive up, and z is positive out of the page. (The sketch is looking at the XY plane.)

$ii := 1..3 \qquad ij := 1..4$

$$XY := \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 7 & 0 & 0 \\ 5 & -3 & 0 \end{bmatrix}$$

$Wm := 1000$ $\qquad Crd_{ij} := \left(XY^{T}\right)^{\langle ij \rangle}$ $\quad$ 3D coordinates for the three T forces and the weight W

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad Crd^{T} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} \end{bmatrix}$$

We'll define direction unit vectors for the three tensions, and another for the weight W. The direction of the tension vectors is such that a positive value is positive in the coordinate system.

$$dr_{ii} := \frac{Crd_{ii} - Crd_{4}}{\left|Crd_{ii} - Crd_{4}\right|} \qquad dr^{T} = \begin{bmatrix} \begin{bmatrix} -0.857 \\ 0.514 \\ 0 \end{bmatrix} & \begin{bmatrix} -0.555 \\ 0.832 \\ 0 \end{bmatrix} & \begin{bmatrix} 0.555 \\ 0.832 \\ 0 \end{bmatrix} \end{bmatrix} \qquad dr_{w} := \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

The force vectors (for the tensions and the weight) are now going to be the magnitude of the force times the direction vector: $\left(Wm \cdot dr_{w}\right)^{T} = \begin{bmatrix} 0 & -1000 & 0 \end{bmatrix}$. We can "*normalize*" the problem by dividing all of the force vectors by the magnitude of the weight. Assume (for discussion) that the magnitudes of the tensions are $Tm := \begin{bmatrix} 00 & 605 & 605 \end{bmatrix}^{T}$.

Sum of forces: $\sum T + W = 0$

$$\sum_{f=1}^{3} Tm_{f} \cdot dr_{f} + Wm \cdot dr_{w} = \begin{bmatrix} 0 \\ 6.781 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \text{“Force in X direction”} \\ \text{“Force in Y direction”} \\ \text{“Force in Z direction”} \end{bmatrix}$$

Using vector notation allows us to calculate moments by vector cross-product: $\mathbf{Mom = arm \times force}$ The moment vector reflects the "right-hand rule" for torque--point your right thumb along the vector, and your fingers will curl in the direction of the torque.

$$Crd_{4} \times \left(Wm \cdot dr_{w}\right) = \begin{bmatrix} 0 \\ 0 \\ -5000 \end{bmatrix} \qquad Crd_{4} \times dr_{w} = \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$$

Moment about origin due to weight

$$MT_{ii} := Crd_{ii} \times \left(Tm_{ii} \cdot dr_{ii}\right) \qquad MT^{T} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1510.171 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 3523.733 \end{bmatrix} \end{bmatrix}$$

Note that the first cable (anchored at the origin) produces no moment.

The "sum of moments" equation now becomes

$$\sum_{f=1}^{3} Crd_{f} \times \left(Tm_{f} \cdot dr_{f}\right) + Crd_{4} \times \left(Wm \cdot dr_{w}\right) = \begin{bmatrix} 0 \\ 0 \\ 33.904 \end{bmatrix}$$

If we "normalize" these equations:

$$\sum_{f=1}^{3} dr_{f} + dr_{w} = 0 \qquad \sum_{f=1}^{3} Crd_{f} \times dr_{f} + Crd_{4} \times dr_{w} = 0$$

Now we build the solution matrix:

sum of forces (X and Y) $M_{1,ii} := \left(dr_{ii}\right)_1$ $M_{2,ii} := \left(dr_{ii}\right)_2$ $B_1 := dr_{w_1}$ $B_2 := dr_{w_2}$

sum of moments (Z) $M_{3,ii} := \left(Crd_{ii} \times dr_{ii}\right)_3$ $B_3 := \left(Crd_4 \times dr_w\right)_3$ $B := -B$

$$M = \begin{bmatrix} -0.857 & -0.555 & 0.555 \\ 0.514 & 0.832 & 0.832 \\ 0 & 2.496 & 5.824 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} \qquad \det(M) = 0$$

As Werner said, "Three linear equations in three variables only have one single unique solution if the determinant A of the matrix of coefficients is different from zero."

**<u>Okay, with three cables the problem is statically indeterminate.</u>**

Let's eliminate one cable:

*Remove cable 1:*

$$M_1 := \text{submatrix}(M, 1, 3, 2, 3) = \begin{bmatrix} -0.555 & 0.555 \\ 0.832 & 0.832 \\ 2.496 & 5.824 \end{bmatrix}$$

$AP_1 := {M_1}^{T} \cdot M_1$ $\qquad AB_1 := {M_1}^{T} \cdot B$

$t\_w_1 := {AP_1}^{-1} \cdot AB_1$ $\qquad t\_w_1 = \begin{bmatrix} 0.601 \\ 0.601 \end{bmatrix}$

$Tm_1 := 0$ $\qquad Tm_2 := t\_w_{1_1}$ $\qquad Tm_3 := t\_w_{1_2}$

$Tm := Tm \cdot Wm$

$$\sum_{f=1}^{3} Tm_f \cdot dr_f + Wm \cdot dr_w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \sum_{f=1}^{3} Crd_f \times \left(Tm_f \cdot dr_f\right) + Crd_4 \times \left(Wm \cdot dr_w\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*Remove cable 2:*

$$M_2 := \text{augment}\left(M^{\langle 1 \rangle}, M^{\langle 3 \rangle}\right) = \begin{bmatrix} -0.857 & 0.555 \\ 0.514 & 0.832 \\ 0 & 5.824 \end{bmatrix}$$

$AP_2 := {M_2}^{T} \cdot M_2$ $\qquad AB_2 := {M_2}^{T} \cdot B$

$t\_w_2 := {AP_2}^{-1} \cdot AB_2$ $\qquad t\_w_2 = \begin{bmatrix} 0.555 \\ 0.858 \end{bmatrix}$

$Tm_1 := t\_w_{2_1}$ $\qquad Tm_2 := 0$ $\qquad Tm_3 := t\_w_{2_2}$

$Tm := Tm \cdot Wm$

$$\sum_{f=1}^{3} Tm_f \cdot dr_f + Wm \cdot dr_w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \sum_{f=1}^{3} Crd_f \times \left(Tm_f \cdot dr_f\right) + Crd_4 \times \left(Wm \cdot dr_w\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*Remove cable 3:*

$$M_3 := \text{augment}\left(M^{\langle 1 \rangle}, M^{\langle 2 \rangle}\right) = \begin{bmatrix} -0.857 & -0.555 \\ 0.514 & 0.832 \\ 0 & 2.496 \end{bmatrix}$$

$AP_3 := {M_3}^{T} \cdot M_3$ $\qquad AB_3 := {M_3}^{T} \cdot B$

$t\_w_3 := {AP_3}^{-1} \cdot AB_3$ $\qquad t\_w_3 = \begin{bmatrix} -1.296 \\ 2.003 \end{bmatrix}$

$Tm_1 := t\_w_{3_1}$ $\qquad Tm_2 := t\_w_{3_2}$ $\qquad Tm_3 := 0$

$Tm := Tm \cdot Wm$

$$\sum_{f=1}^{3} Tm_f \cdot dr_f + Wm \cdot dr_w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \sum_{f=1}^{3} Crd_f \times \left(Tm_f \cdot dr_f\right) + Crd_4 \times \left(Wm \cdot dr_w\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**Summary:**

The case of three supporting cables is statically indeterminate. We can get a closed solution for any combination of two supports.

Supports 2 and 3 eliminate 1: $Wm \cdot t\_w_1 = \begin{bmatrix} 600.925 \\ 600.925 \end{bmatrix}$

Supports 1 and 3 eliminate 2: $Wm \cdot t\_w_2 = \begin{bmatrix} 555.329 \\ 858.465 \end{bmatrix}$

Supports 1 and 2 eliminate 3: $Wm \cdot t\_w_3 = \begin{bmatrix} -1295.767 \\ 2003.084 \end{bmatrix}$

*Note that support 1 is in compression. If these were cables, this would not be feasible.*