

Phasors and Complex Numbers

1. Notation for instantaneous and phasor quantities:

- (a). Euler's Identity: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ and $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$
 $\text{Re}(e^{j\theta}) = \cos\theta$ and $\text{Im}(e^{j\theta}) = \sin\theta$
- (b). Instantaneous: $v(t) = V_m \cdot \cos(\omega t + \phi_v)$ and $i(t) = I_m \cdot \cos(\omega t + \phi_i)$
- (c). RMS Phasor Quantity: $\underline{V} = \left(\frac{V_m}{\sqrt{2}}\right) \angle \phi_v = V \angle \phi_v$ and $\underline{I} = \left(\frac{I_m}{\sqrt{2}}\right) \angle \phi_i = I \angle \phi_i$

2. Complex numbers:

By complex number in canonical (algebraical) form, we understand a number of the form $\underline{z} = x + jy$, where x and y are real numbers and j is imaginary unit defined by $j^2 = -1$. The real number x is called the real component or real part of the complex number \underline{z} , or $\text{Re}(\underline{z})$. The real number y is called the imaginary component or imaginary part of the complex number \underline{z} , or $\text{Im}(\underline{z})$. The complex number (i.e. phasor) can be geometrically represented on the complex plane, z-plane, or R-X plane.

The vector that represents \underline{z} number, has a length $r = |\underline{z}| = \sqrt{x^2 + y^2}$, and its direction angle defined for $\underline{z} \neq 0$, via $\cos(\phi) = \frac{x}{|\underline{z}|}$, $\sin(\phi) = \frac{y}{|\underline{z}|}$. The length is called also **absolute value** or **modulus** of \underline{z} (written $\text{mod}(\underline{z})$). The ϕ is called a argument of \underline{z} . Complex number has: main argument (i.e. $\text{arg}(\underline{z})$) when argument belongs to $(-\pi, \pi)$. Set of all arguments of the complex number \underline{z} , (i.e. $\text{Arg}(\underline{z})$) defined as:

$$\text{Arg}(\underline{z}) = \text{arg}(\underline{z}) + 2 \cdot \pi \cdot k, (k = +/-1, +/-2, +/-3, \dots).$$

Following are properties of the complex numbers in canonical form:

$$\underline{z} = |\underline{z}|(\cos(\phi) + j\sin(\phi))$$

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

$$(a + jb) \cdot (c + jd) = (a \cdot c - b \cdot d) + j(b \cdot c + a \cdot d)$$

$$\frac{(a + jb)}{(c + jd)} = \frac{a \cdot c + b \cdot d}{c^2 + d^2} + j \cdot \frac{b \cdot c - a \cdot d}{c^2 + d^2}$$

$$\underline{z}_1 \cdot \underline{z}_2 = [r_1(\cos(\phi_1) + j\sin(\phi_1))] \cdot r_2(\cos(\phi_2) + j\sin(\phi_2))$$

$$\underline{z}_1 \cdot \underline{z}_2 = (r_1 r_2) [\cos(\phi_1 + \phi_2) + j\sin(\phi_1 + \phi_2)]$$

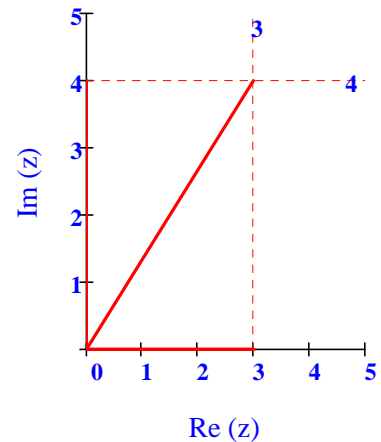
$$\frac{\underline{z}_1}{\underline{z}_2} = \frac{[r_1(\cos(\phi_1) + j\sin(\phi_1))]}{[r_2(\cos(\phi_2) + j\sin(\phi_2))]} = \frac{r_1}{r_2} \cdot (\cos(\phi_1 - \phi_2) + j\sin(\phi_1 - \phi_2))$$

$$\frac{1}{\underline{z}} = \frac{1}{r} \cdot (\cos(-\phi) + j\sin(-\phi)). \text{ Conjugate is } \bar{\underline{z}} = x - jy,$$

Following are properties of the complex numbers in polar form: $\underline{z} = r \cdot e^{j\phi}$

$$\underline{z}_1 \cdot \underline{z}_2 = (r_1 e^{j\phi_1}) \cdot (r_2 e^{j\phi_2}) = r_1 \cdot r_2 \cdot e^{j(\phi_1 + \phi_2)}$$

$$\frac{\underline{z}_1}{\underline{z}_2} = \frac{r_1 e^{j\phi_1}}{r_2 e^{j\phi_2}} = \frac{r_1}{r_2} \cdot e^{j(\phi_1 - \phi_2)}$$



Per Unit Calculations

DEFINITION OF A PER UNIT OF QUANTITY

$$(\text{per unit value}) = \left(\frac{\text{actual value}}{\text{base value}} \right)$$

BASE POWER EQUATIONS

$$(\text{base VA}_{3\text{ph}}) = 1000 \cdot (\text{base kVA}_{3\text{ph}}) = 10^6 \cdot (\text{base MVA}_{3\text{ph}})$$

BASE VOLTAGE EQUATION

$$(\text{base V}_{\text{LL}}) = 1000 \cdot (\text{base kV}_{\text{LL}})$$

BASE CURRENT EQUATIONS

$$(\text{base A}) = \frac{(\text{base VA}_{3\text{ph}})}{\sqrt{3} \cdot (\text{base V}_{\text{LL}})} = \frac{(\text{base kVA}_{3\text{ph}})}{\sqrt{3} \cdot (\text{base kV}_{\text{LL}})} = \frac{1000 (\text{base MVA}_{3\text{ph}})}{\sqrt{3} \cdot (\text{base kV}_{\text{LL}})}$$

BASE IMPEDANCE / ADMITTANCE EQUATIONS

$$(\text{base } \Omega) = \frac{(\text{base V}_{\text{LL}})}{\sqrt{3} \cdot (\text{base A})} = \frac{1000 \cdot (\text{base kV}_{\text{LL}})}{\sqrt{3} \cdot (\text{base A})} = \frac{1000 \cdot (\text{base kV}_{\text{LL}})^2}{(\text{base kVA}_{3\text{ph}})} = \frac{(\text{base kV}_{\text{LL}})^2}{(\text{base MVA}_{3\text{ph}})}$$

$$(\text{base } \Omega^{-1}) = (\text{base S}) = \frac{1}{(\text{base } \Omega)}$$

CHANGING THE BASE OF A PER UNIT QUANTITY

$$S_{\text{pu.OLD}} = S_{\text{pu.OLD}} \cdot \frac{(\text{base VA}_{\text{OLD}})}{(\text{base VA}_{\text{NEW}})}$$

$$V_{\text{pu.OLD}} = V_{\text{pu.OLD}} \cdot \frac{(\text{base V}_{\text{OLD}})}{(\text{base V}_{\text{NEW}})}$$

$$I_{\text{pu.OLD}} = I_{\text{pu.OLD}} \cdot \frac{(\text{base I}_{\text{OLD}})}{(\text{base I}_{\text{NEW}})} = I_{\text{pu.OLD}} \cdot \frac{(\text{base VA}_{\text{OLD}})}{(\text{base VA}_{\text{NEW}})} \cdot \frac{(\text{base V}_{\text{NEW}})}{(\text{base V}_{\text{OLD}})}$$

$$Z_{\text{pu.OLD}} = Z_{\text{pu.OLD}} \cdot \frac{(\text{base Z}_{\text{OLD}})}{(\text{base Z}_{\text{NEW}})} = Z_{\text{pu.OLD}} \cdot \frac{(\text{base VA}_{\text{NEW}})}{(\text{base VA}_{\text{OLD}})} \cdot \left[\frac{(\text{base V}_{\text{OLD}})}{(\text{base V}_{\text{NEW}})} \right]^2$$

Symmetrical Components

$$\begin{aligned} \underline{I}_A &= I_A \angle \delta_A & \underline{I}_{A0} &= I_0 \angle \delta_0 & \underline{I}_{A1} &= I_1 \angle \delta_1 & \underline{I}_{A2} &= I_2 \angle \delta_2 \\ \underline{I}_B &= I_B \angle \delta_B & \underline{I}_{B0} &= I_0 \angle \delta_0 & \underline{I}_{B1} &= I_1 \angle (\delta_1 - 120\text{deg}) & \underline{I}_{B2} &= I_2 \angle (\delta_2 + 120\text{deg}) \\ \underline{I}_C &= I_C \angle \delta_C & \underline{I}_{C0} &= I_0 \angle \delta_0 & \underline{I}_{C1} &= I_1 \angle (\delta_1 + 120\text{deg}) & \underline{I}_{C2} &= I_2 \angle (\delta_2 - 120\text{deg}) \end{aligned}$$

$\underline{I}_{A0}, \underline{I}_{B0}, \underline{I}_{C0}$ - are called the zero sequence components

$\underline{I}_{A1}, \underline{I}_{B1}, \underline{I}_{C1}$ - are called the positive sequence components

$\underline{I}_{A2}, \underline{I}_{B2}, \underline{I}_{C2}$ - are called the negative sequence components

Note:

- All three sets of symmetrical components are balanced (magnitudes equal)
- Only the positive and negative sequence components are symmetric (equally spaced over 0 - 360deg)
- The negative sequence components are in the reverse order from the phase sequence ABC

The unbalanced phasors are then defined to be linear combination of the symmetrical components:

$$\underline{I}_A = \underline{I}_{A0} + \underline{I}_{A1} + \underline{I}_{A2}$$

$$\underline{I}_B = \underline{I}_{B0} + \underline{I}_{B1} + \underline{I}_{B2}$$

$$\underline{I}_C = \underline{I}_{C0} + \underline{I}_{C1} + \underline{I}_{C2}$$

If the special complex constant a is defined, then the above equations can be simplified:

$$\text{If } a = 1 \angle (120\text{deg}) \text{ then } a^2 = [1 \angle (120\text{deg})] \cdot [1 \angle (120\text{deg})] = 1 \angle (240\text{deg}) = 1 \angle (-120\text{deg})$$

$$\begin{aligned} \underline{I}_{A0} &= I_0 \angle \delta_0 = \underline{I}_0 & \underline{I}_{A1} &= I_1 \angle \delta_1 = \underline{I}_1 & \underline{I}_{A2} &= I_2 \angle \delta_2 = \underline{I}_2 \\ \underline{I}_{B0} &= I_0 \angle \delta_0 = \underline{I}_0 & \underline{I}_{B1} &= I_1 \angle (\delta_1 - 120\text{deg}) = a^2 \cdot \underline{I}_1 & \underline{I}_{B2} &= I_2 \angle (\delta_2 + 120\text{deg}) = a \cdot \underline{I}_2 \\ \underline{I}_{C0} &= I_0 \angle \delta_0 = \underline{I}_0 & \underline{I}_{C1} &= I_1 \angle (\delta_1 + 120\text{deg}) = a \cdot \underline{I}_1 & \underline{I}_{C2} &= I_2 \angle (\delta_2 - 120\text{deg}) = a^2 \cdot \underline{I}_2 \end{aligned}$$

$$\underline{I}_A = \underline{I}_{A0} + \underline{I}_{A1} + \underline{I}_{A2} = \underline{I}_0 + \underline{I}_1 + \underline{I}_2$$

$$\underline{I}_B = \underline{I}_{B0} + \underline{I}_{B1} + \underline{I}_{B2} = \underline{I}_0 + a^2 \cdot \underline{I}_1 + a \cdot \underline{I}_2$$

$$\underline{I}_C = \underline{I}_{C0} + \underline{I}_{C1} + \underline{I}_{C2} = \underline{I}_0 + a \cdot \underline{I}_1 + a^2 \cdot \underline{I}_2$$

Putting these equations into matrix form will allow the calculation of the symmetrical components from the unbalanced set of phasors:

$$\begin{pmatrix} \underline{I}_A \\ \underline{I}_B \\ \underline{I}_C \end{pmatrix} = \frac{1}{h} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{pmatrix} \cdot \begin{pmatrix} \underline{I}_0 \\ \underline{I}_1 \\ \underline{I}_2 \end{pmatrix} \quad \begin{pmatrix} \underline{I}_0 \\ \underline{I}_1 \\ \underline{I}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \underline{I}_A \\ \underline{I}_B \\ \underline{I}_C \end{pmatrix} = \frac{h}{3} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{pmatrix} \cdot \begin{pmatrix} \underline{I}_A \\ \underline{I}_B \\ \underline{I}_C \end{pmatrix}$$

Note: $h = 1$ for the Fortescue transformation (this presentation)

$h = \sqrt{3}$ for the power invariant transformation

The same transformation equations are used for L-N voltages also.

In general, for n phase system, there are $i = n - 1$ symmetrical components and one "zero" component

operators. The component operator λ_i is defined as $\lambda_i = e^{j \cdot \frac{2 \cdot \pi}{n} \cdot (i-1)}$. For 3-phase system with $n = 3$,

there are $n - 1 = 2$ components (positive and negative) $\lambda_2 = e^{j \cdot \frac{2 \cdot \pi}{3} \cdot (2-1)} = e^{j \cdot \frac{2 \cdot \pi}{3}} = e^{j \cdot 120 \text{deg}} = a$

rectangular notation

$$\underline{a} := -\frac{1}{2} + j \cdot \frac{\sqrt{3}}{2} = -0.5 + 0.866j$$

$$\underline{a}^2 = -0.5 - 0.866j$$

$$\underline{a}^3 = 1$$

$$\underline{a}^4 = -0.5 + 0.866j$$

$$\underline{a}^5 = -0.5 - 0.866j$$

$$\underline{a} + \underline{a}^2 + 1 = 0$$

$$\underline{a} + \underline{a}^2 = -1$$

$$\underline{a} - \underline{a}^2 = 1.732j$$

$$\underline{a}^2 - \underline{a} = -1.732j$$

$$1 - \underline{a} = 1.5 - 0.866j$$

$$1 - \underline{a}^2 = 1.5 + 0.866j$$

$$\underline{a} - 1 = -1.5 + 0.866j$$

$$\underline{a}^2 - 1 = -1.5 - 0.866j$$

$$1 + \underline{a} = 0.5 + 0.866j$$

$$1 + \underline{a}^2 = 0.5 - 0.866j$$

polar notation

$$z2r\theta(\underline{a}) = "(1 \angle 120^\circ)"$$

$$z2r\theta(\underline{a}^2) = "(1 \angle -120^\circ)"$$

$$z2r\theta(\underline{a}^3) = "(1 \angle 0^\circ)"$$

$$z2r\theta(\underline{a}^4) = "(1 \angle 120^\circ)"$$

$$z2r\theta(\underline{a}^5) = "(1 \angle -120^\circ)"$$

$$z2r\theta(\underline{a} + \underline{a}^2) = "(1 \angle 180^\circ)"$$

$$z2r\theta(\underline{a} - \underline{a}^2) = "(1.732 \angle 90^\circ)"$$

$$z2r\theta(\underline{a}^2 - \underline{a}) = "(1.732 \angle -90^\circ)"$$

$$z2r\theta(1 - \underline{a}) = "(1.732 \angle -30^\circ)"$$

$$z2r\theta(1 - \underline{a}^2) = "(1.732 \angle 30^\circ)"$$

$$z2r\theta(\underline{a} - 1) = "(1.732 \angle 150^\circ)"$$

$$z2r\theta(\underline{a}^2 - 1) = "(1.732 \angle -150^\circ)"$$

$$z2r\theta(1 + \underline{a}) = "(1 \angle 60^\circ)"$$

$$z2r\theta(1 + \underline{a}^2) = "(1 \angle -60^\circ)"$$