Chapter 1

BASIC EQUATIONS OF A SUSPENDED WIRE

Hence no force, however great,
Can stretch a cord, however fine,
Into an horizontal line
That is accurately straight...

William Whewell (1794-1866)

1.1 Basic assumptions.

When a perfectly flexible wire, of uniform mass per unit of length, m, is suspended between two fixed points at a tension T, it assumes a curve known as a CATENARY (from Latin catena = chain). The theoretical equation for the rise of this curve at a distance x from its lowest point is,

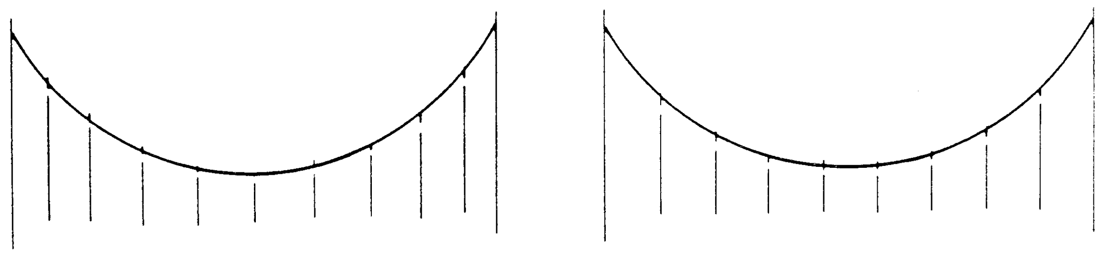
$$y = C.[\cosh(x/C) - 1]$$

where $C = T/(m.g)$

In overhead electrification work, however, this formula is not generally used. With spans and tensions of the order usually encountered in this kind of work (the spans being generally much shorter than those used in overhead transmission lines), the difference between this formula and the simpler one that we shall derive below is negligible. In fact, the above formula, if not used with care, can give serious errors.

> The reader may try the exercise of calculating y at various values of x, taking m=0.1 kg/m, g=9.807 and T=10000 N. If this is done, first by using four-figure tables of hyperbolic cosines, and then by using a calculator giving the "cosh" function to eight or nine significant figures, the danger will become clear.

We therefore make the assumption that the mass per unit of span, and not per unit of length of the wire itself, is uniform. In Figs. 1.1a and 1.1b the sag has been exaggerated to show this difference clearly. (The reader should note that the term "mass per unit length", or "specific mass", as used in overhead electrification work, may be taken without significant error to mean "mass per unit of span". To avoid confusion, the latter term is used throughout this work.) We also make the assumption already mentioned: that the wire is perfectly flexible. This may surprise anyone who takes a short piece of solid contact wire in his hand, but it can be shown that over a span of 10 m or more, the stiffness of the wire has negligible effect on its sag.



(a) Uniform mass per unit of LENGTH

(b) Uniform mass per unit of SPAN

Fig. 1.1

1.2 Derivation of sag equations.

Let us now consider a wire suspended between two fixed points, A and B, not necessarily at the same height (Fig. 1.2). The horizontal distance, L, between A and B is called the SPAN LENGTH, or simply SPAN.

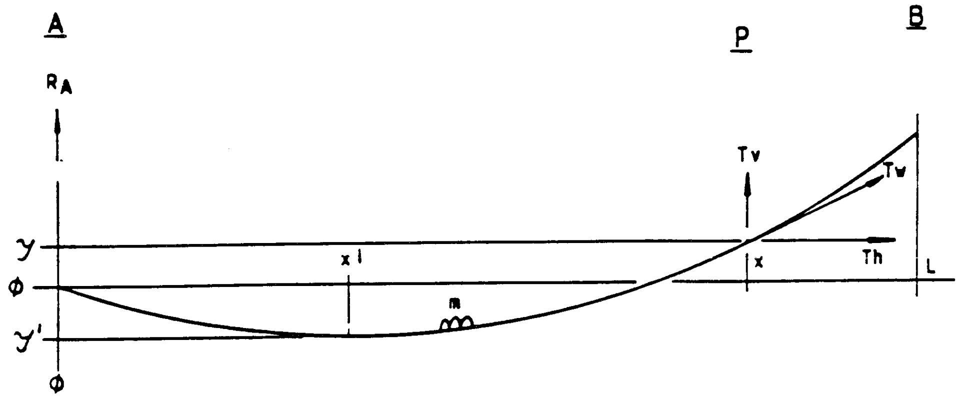


Fig. 1.2

Since the wire is perfectly flexible, it is free to take up a position such that there is no bending moment at any point in it. The tension at any point can therefore be represented by a vector Tw, tangential to the wire at that point. This vector can be resolved into horizontal and vertical components, Th and Tv respectively, related by the equation,

$$Tw = sqr(Th^2 + Tv^2)$$

When the wire, being free to take up any position as described, has come to a state of rest, the sum of all the forces acting at any point must be zero. From this we deduce:

1. The horizontal component of tension, Th, is constant at all points throughout the span.

2. The vertical component, Tv, at any point in the span is equal to the weight of the wire lying between that point and the lowest point in the span. (This follows from the fact that at the lowest point, the tangential tension is horizontal, and hence there is no vertical component of force at that point).

3. The sum of the moments acting about any point is zero, and, further,

the separate sums of the moments acting in the wire on either side of that point are both equal to zero.

From 1 and 2 it follows that Tv, and hence Tw, is at a maximum at the ends of the span, while at the lowest point Tv=0 and Tw=Th. Using the third deduction, we can now find the vertical position of any point in the wire.

The vertical reaction at A is equal to the weight of the length of wire between A and the low point:

$$Ra = m.g.x0$$

If we take any point P, at a horizontal distance of x and a vertical distance of y from A, then there are three moments acting in the wire to the left of that point. Taking clockwise moments as positive, these are:

1. The vertical reaction, Ra, acting through the distance x, giving rise to a moment of Ra.x = m.g.x0.x

2. The weight of the wire between A and P, which is equal to m.g.x, and can be treated as a point load acting midway between A and P, so that the moment is -m.g.x^2/2

3. The horizontal component of the tension, Th, acting though the distance y, giving a moment of Th.y (Note that y is negative if below the origin, as in Fig. 1.2).

The sum of these moments can then be equated to zero:

$$m.g.x0.x - m.g.x^2/2 + Th.y = 0$$

from which

$$y = - \frac{m.g.(x0.x - x^2/2)}{Th} \qquad ...(1.1)$$

We can now note two important special cases:

1. By making x equal to x0 in equation 1.1, we find that the vertical distance between A and the low point is

$$y0 = - \frac{m.g.x0^2}{2\ Th} \qquad ...(1.2)$$

The curve represented by this equation is a PARABOLA.

Since the resultant force is tangential to the wire at all points including the support point, this equation gives the rise from the low point to ANY point in the wire, where x0 is the distance between the low point and the point in question.

2. If A and B are at the same height, it is obvious that,

$$x0= L/2$$

$$Ra = m.g.L/2$$

$$y = -\frac{m.g.L.x}{2\ Th} + \frac{m.g.x^2}{2\ Th} = -\frac{m.g.x.(L-x)}{2\ Th} \qquad \ldots(1.3)$$

$$y0 = -\frac{m.g.L^2}{8\ Th} \qquad \ldots(1.4)$$

We can now define the SAG in this symmetrical span as the vertical distance from the supports to the low point, $S = -y0$

Since it is the horizontal component of the tension, Th, that is always the critical parameter in all equations defining the position of the wire, we will, from here onwards, define TENSION (unless otherwise stated) as the horizontal component of the tension in the wire, and designate it by the symbol T, without suffix.

Equation 1.4 thus becomes, for the sag in a symmetrical span,

$$S = \frac{m.g.L^2}{8\ T} \qquad \ldots(1.5)$$

In the derivation of y and y0 we have observed the usual sign convention, by which forces in the downward sense, and dimensions below the origin, are negative. We regard Sag, however, as a simple dimension and eliminate the negative sign.

Going back to the general case where A and B are not at equal heights, we can now find the horizontal position of the low point.

From Equation 1.2, the vertical distances from A and B respectively to the low point are:

$$ya = -\frac{m.g.x0^2}{2\ T} \quad \text{and} \quad yb = -\frac{m.g.(L-x0)^2}{2\ T}$$

Let the difference in height between A and B be $H = ya - yb$

Then,

$$H = \frac{m.g}{2\ T}\left[(L-x0)^2 - x0^2\right]$$

from which

$$x0 = \frac{L}{2} - \frac{T.H}{m.g.L} \qquad \ldots(1.6)$$

The reader may verify that, if moments acting on the right-hand side of the point P had been considered, the identical results would have been obtained.

From Equation 1.6, we can now write,

$$Ra = m.g.x0 = \frac{m.g.L}{2} - \frac{m.g.T.H}{L}$$

and substituting this in the derivation of equation 1.1 gives an alternative equation for y:

$$y = -\frac{m.g.x.(L-x)}{2T} + \frac{H.x}{L} \qquad \ldots(1.3a)$$

Comparing this with equation 1.3 shows that the two equations differ only by the term H.x/L. Now this term clearly represents the rise at distance x from point A of a line joining A and B (Fig. 1.3). In other words, the vertical distance to the wire from a line joining the two support points is independent of the gradient of that line.

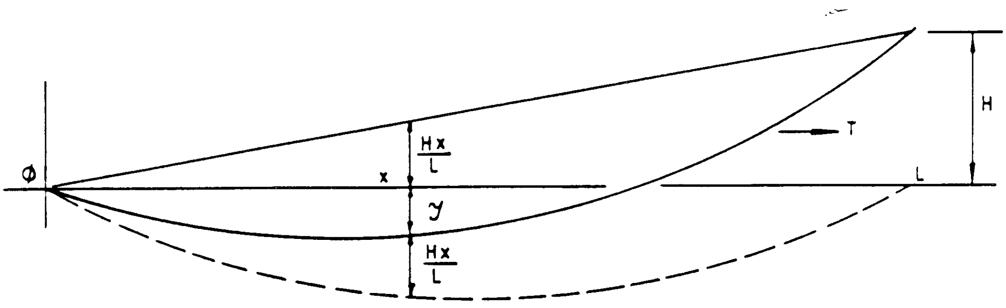


Fig. 1.3

This means that we can represent the asymmetrical span by superimposing onto a corresponding symmetrical span a straight line having the slope H/L. The practical advantage of this is that we can relate all support heights to ground level (or track level) and ignore any gradient of the ground or track. The "sag" of the wire can be measured from the line joining the support points, whether that line is horizontal or not.

1.3 Point loads.

So far we have considered a wire of uniform mass per unit of span. We now examine a wire having a load of mass M, concentrated at a point P, at a distance of Xp from support A (Fig. 1.4). First of all, we will imagine that the wire itself has no mass.

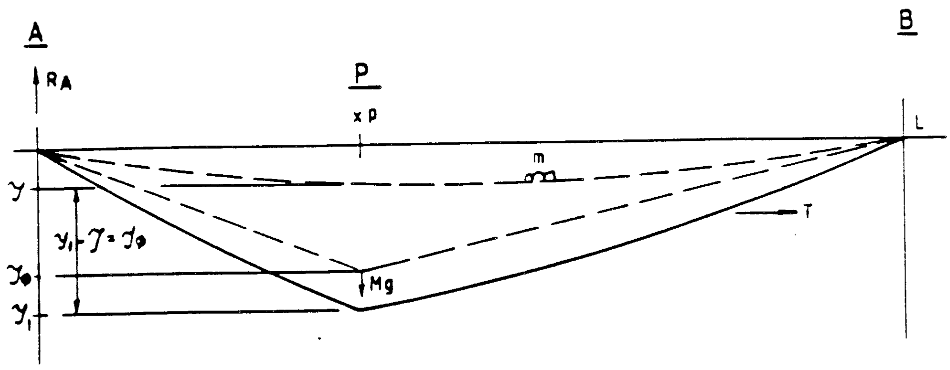


Fig. 1.4

By equating the sum of moments about B to zero, we obtain the vertical reaction at A:

$$Ra = \frac{M.g.(L-Xp)}{L}$$

Then, from the sum of moments at P, the vertical distance from A to P is,

$$y0 = -\frac{Ra.Xp}{T} = -\frac{M.g.Xp.(L-Xp)}{T.L} \qquad \ldots(1.7)$$

Now, if we take a wire of uniform mass m, with a point load M at P, by the same process we get,

$$Ra = \frac{m.g.L}{2} + \frac{M.g.(L-Xp)}{L}$$

from which the vertical distance from A to P is

$$y1 = -\frac{Ra.Xp - m.g.Xp^2/2}{T} = -\frac{m.g.Xp.(L-Xp)}{2\,T} - \frac{M.g.Xp.(L-Xp)}{T.L} \qquad \ldots(1.8)$$

Comparing this with equations 1.7 and 1.3 we see that

$$y1 = y + y0$$

In other words, the FALL of the wire from the support to any to any point, due to the combined effects of the uniform mass and the point load, is equal to that which would occur with the uniform mass alone, plus that which would occur with the point load alone.

The ADDITIONAL sag of the wire at P, due to the point load, is thus

$$Yp = y1 - y = y0$$

Using this principle, we can find the fall due to the effect of a point load at P, combined with the uniform mass, at some other point Q, where Q is less than P (Fig. 1.5).

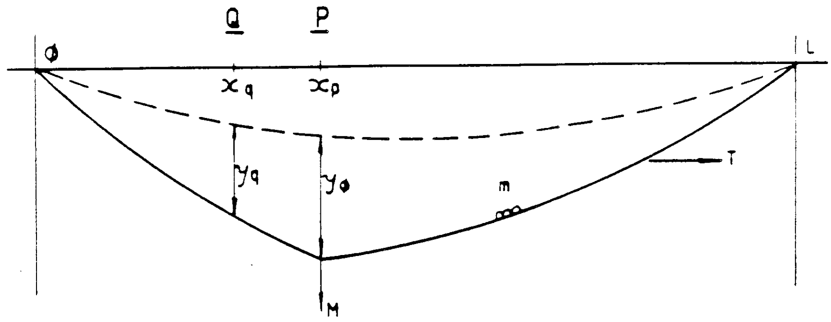


Fig. 1.5

Since the imagined massless wire would lie in a straight line between A and P, the ADDITIONAL fall at P1 due to the point load is clearly,

$$Yq = -\frac{M.g.Xp.(L-Xp)}{T.L} \cdot \frac{Xq}{Xp} = \frac{M.g.Xq.(L-Xp)}{T.L} \quad ...(1.9a)$$

In the same way we find that, where Q is greater than P, the additional fall at Q due to the point load is,

$$Yq = -\frac{M.g.Xp.(L-Xq)}{T.L} \quad ...(1.9b)$$

This principle can be extended to any number of point loads in the same span.

1.4 Wind loading and blow-off.

Wind blowing at right-angles to the wire will apply a uniform force, w, per unit of span. How this force is evaluated will be dealt with later (Chapter 8, section 8.2.2). The combination of this force and the vertical load m.g will produce a resultant load fr, where,

fr = sqr[(m.g)^2 + w^2]

The wire, being flexible, will swing into a plane parallel with the direction of this resultant force. The distance, measured in this plane, from the supports to the wire at mid-span, we will call the RESULTANT SAG, and is,

$$Sr = \frac{fr.L^2}{8\ T}$$

Note: It is assumed for the present that the value of T is known. The effect of the wind force on the tension will be considered later.

The resultant sag Sr can be resolved into a SAG in the vertical plane, S and a BLOW-OFF in the horizontal plane, B, related by the equation,

$$Sr = \mathrm{sqr}(S^2 + B^2)$$

Since the plane in which the wire lies is parallel with the resultant force fr, it follows that,

$$S = \frac{m.g.L^2}{8\ T} \qquad \text{...(1.10a}$$

and,

$$B = \frac{w.L^2}{8\ T} \qquad \text{...(1.10b)}$$

Equation 1.10a is identical with equation 1.5, and equation 1.10b is of the same form. We can thus say that the SAG (defined as the deflection of th wire in the vertical plane) and the BLOW-OFF (defined as the deflection o the wire in the horizontal plane) are due to the vertical and horizontal forces respectively, and can be considered independently of one another.

This point should be especially noted, because the terms "SAG" and what we have called here "RESULTANT SAG" are sometimes confused. Throughout this work, "SAG" is to be understood as being in the vertical plane.

1.5 Ice loading.

Ice is usually assumed to form a coating on the wire, of uniform radia' thickness. It thus forms a cylinder co-axial with the wire itself. Con sequently its mass is uniformly distributed over the span. In all the equations derived above, therefore, we can simply add the mass of the ice, mi, to the mass of the wire, m. So, for example, equation 1.5 becomes,

$$S = \frac{(m+mi).g.L^2}{8\ T} \qquad \text{...(1.11)}$$

1.6 Variation of wire tension.

We will now study the behaviour of a suspended wire under changes o temperature and loading.

First, we have to find the length of the wire, C, in a span of length L.

Fig. 1.6 shows a span of wire, of length L, and with its supports at a differential height of H.

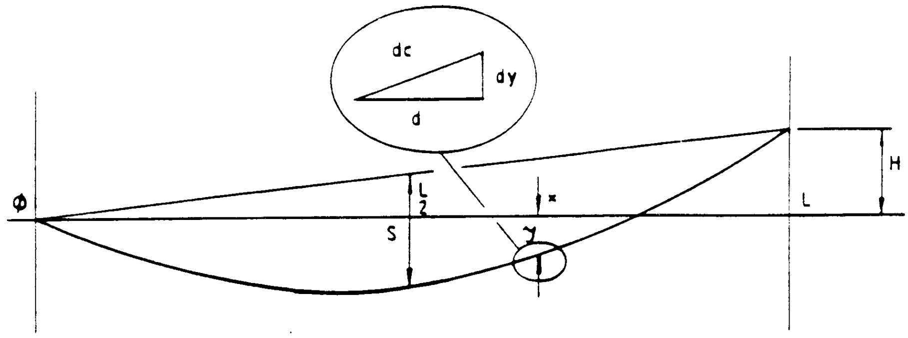


Fig. 1.6

A small element of the wire, of length dC, forms the hypotenuse of a right-angled triangle with the corresponding elements dx and dy, so that,

$$dC = \sqrt{(dx)^2 + (dy)^2}$$

or,

$$dC = \sqrt{1 + (dy/dx)^2}\,.dx$$

Since dy is small in relation to dx, we can make the approximation,

$$dC = [1 + (dy/dx)^2/2].dx \qquad ...(1.12)$$

The derivation of this approximation is as follows:

$$[a + b^2/(2a)]^2 = a^2 + b^2 + b^4/(4a)^2$$

If $b<a$, the last term can be neglected, giving

$$a + b^2/(2a) = \text{(approx.)} \sqrt{a^2 + b^2}$$

If $b<a/4$, the error will be less than 0.05%

Now we know that the equation of the curve of the wire is

$$y = -\frac{m.g.x.(L-x)}{2\,T} + \frac{H.x}{L} \qquad \text{(see equation 1.3a)}$$

and so, by differentiation,

$$\frac{dy}{dx} = -\frac{m.g.(L-2x)}{2\,T} + \frac{H}{L}$$

Substituting this in equation 1.12 and integrating with respect to x between the limits 0 and L gives

$$C = L + \frac{(m.g)^2.L^3}{24\,T^2} + \frac{H^2}{2\,L} \qquad ...(1.13)$$

C can also be expressed in terms of the sag, by substituting equation 1.5 in equation 1.13:

$$C = L + \frac{8\ S^2}{3\ L} + \frac{H^2}{2\ L} \qquad \ldots(1.14)$$

Now let a length of wire, C0, be suspended in a symmetrical span of lengt L, at a known tension of T0.

From equation 1.13,

$$C0 = L + \frac{f0^2.L^3}{24\ T0^2} + \frac{H^2}{2\ L}$$

where f0 is the initial resultant load, being

$$f0 = sqr[((m+mi0).g)^2 + w0^2]$$

m, mi0 and w0 being the mass, initial ice mass and initial wind load respectively. (i0, and w0 may, of course, be zero.)

Now let the ice and wind loads change to mi1 and w1. Since the wire is elastic, its length will change to:

$$C1 = L + \frac{f1^2.L^3}{24\ T1^2} + \frac{H^2}{2\ L}$$

where f1 = sqr[((m+mi1).g)^2 + w1^2] and T1 is the new tension, which we have to find.

Likewise, if the temperature changes, the length of the wire will be altered by expansion or contraction, and this will also affect the tension which in turn will bring about a further elastic change in the length o the wire.

The change in length due to elastic elongation is,

$$dC(e) = \frac{C0}{E.A}\ (T1-T0)$$

where E is the Modulus of Elasticity (or Young's Modulus) of the wire and A is the cross-sectional area of the wire.

The change in length due to temperature change is,

$$dC(t) = a.C.(t1-t0)$$

where a is the coefficient of thermal expansion,
and t0 and t1 are the initial and new temperatures respectively.

The total change in wire length is thus,

$$C1 - C0 = dC(e) + dC(t)$$

$$= \frac{C0}{E.A}\ (T1 - T0) + a.C0(t1-t0)$$

Since the difference between C0 and L is very small, we can approximate:

$$C1 - C0 = \frac{L}{E.A}(T1 - T0) + a.L.(t1 - t0)$$

from which,

$$T1 - T0 = - E.A.(t1-t0) + \frac{E.A}{L}(C1 - C0)$$

Substituting for C1 and C0 gives the CHANGE-OF-STATE EQUATION:

$$T1 = T0 - E.A.a.(t1-t0) - \frac{E.A.f0^2\ L^2}{24\ T0^2} + \frac{E.A.f1^2\ L^2}{24\ T1^2} \qquad ...(1.15)$$

All the terms in this equation, except T1, are known, and so it can be reduced to,

$$T1 = Q + \frac{R}{T1^2} \qquad ...(1.15a)$$

where Q and R are constants that can be evaluated.

This equation can be solved in either of two ways.

1. Using a calculator.

Estimate an initial value of T1 (you will at least know whether it is greater or less than T0) and substitute this on both sides of equation 1.15a.

If the two sides are not then equal, the true value of T1 must lie between the estimated value on the left, and the calculated value on the right. Taking a new value between the two, repeat the process, and so on until the two sides of the equation balance. With experience, it is usually possible to arrive at the solution in three or four attempts.

EXAMPLE: An aluminium wire of 157.6 sq.mm cross-sectional area is suspended in a span of 60 m, at a tension of 5000 N at 10 C without wind or ice. What is its tension at -15 C with a wind load of 7.47 N/m and an ice load of 0.755 kg/m? The Modulus of Elasticity may be taken as 56000 N/sq.mm, the Coefficient of Expansion as 0.000023 per deg.C, and the mass of the wire as 0.434 kg/m.

E=56000 N/sq.mm	a=157.6 sq.mm	a=000023 /C	L=60 m
m=0.434 kg/m	t0=10 C	t1=-15 C	T0=5000 N

f0 = 0.434x9.807 = 4.256 N/m
f1 = sqr[((0.434+0.755)x9.807)^2 + 7.47^2] = 13.848 N/m

$$T1 = 5000 - 56000\times157.6\times0.000023\times(-15-10) - \frac{56000\times157.6\times4.256^2\times60^2}{24\times5000^2}$$

$$+ \frac{56000 \times 157.6 \times 13.848^2 \times 60^2}{24\ T1^2}$$

$$= 5000 + 5075 - 959 + \frac{2.539\ E11}{T1^2}$$

$$= 9116 + \frac{2.539\ E11}{T1^2}$$

Let the initial estimate of T1 be 12000 N

$$9116 + \frac{2.539\ E11}{12000^2} = 10879$$

This is lower than the estimated value, so we make the second estimat the approximate mean of the two values, say 11400 N

$$9116 + \frac{2.539\ E11}{11400^2} = 11070$$

The third estimate must lie between 11070 and 11400, and we deduce fro the results above that it will be nearer to the lower figure; say 11155.

$$9116 + \frac{2.539\ E11}{11155^2} = 11156$$

Since this is within 1 N, it can be accepted as the solution.

So the tension T1 is 11155 N

2. By the Newton-Raphson Method (more suitable for computer programs).

The full description of this method can be found in mathematical textbooks. We give here a subroutine, suitable for incorporation in a computer program. It is written in BASIC, but could easily be translated into any othe programming language.

```
100 REM SUBROUTINE: SOLUTION OF CUBIC EQUATION
110 REM
120 REM This subroutine solves cubic equations of the form
130 REM T1=Q+R/T1^2 by the Newton Raphson Method.
140 REM
150 T1=R**(1/3)+ABS(Q)       REM initial guess
160 ICOUNT=0                 REM iteration counter zeroed
170 T0=(R-T1^3+Q*T1^2)/(3*T1^2-2*Q*T1)
180 T1=T1+T0 : ICOUNT=ICOUNT+1
190 IF ABS(T0)<1 THEN 230    REM within 1 unit of solution
200 IF ICOUNT<20 THEN 170    REM traps if no convergence in 20
                                                     iterations
210 PRINT "NO CONVERGENCE" : STOP
220 REM
230 RETURN                   REM end of subroutine
```

1.7 Equivalent span.

It will have been noticed that the span length is a parameter in the variation of tension with temperature and loading. Now, in a length of overhead equipment, the spans will not usually all be equal, and so, even though the tension is made constant throughout the length under some defined condition, there will be differential tensions between the adjacent spans under all other conditions. Spans can rarely be considered in isolation from one another, since the supports usually incorporate some form of link that allows movement of wire between one span and another. In the case of overhead contact systems mounted on cantilevers, or carried over pulleys, complete equalisation of tension between adjacent spans takes place. It is therefore necessary to find what the final tension will be, under any given condition.

Consider two spans, of lengths L1 and L2 (Fig. 1.7). At the support between them, the wire is free to move. (For practical purposes it does not matter whether wire moves from one span into the other, as over a pulley, or the span lengths themselves change slightly, as with swinging cantilevers. Since the movement is very small in relation to the span lengths, the latter can be treated as constant.) The supports at the outer ends of the two spans are fixed.

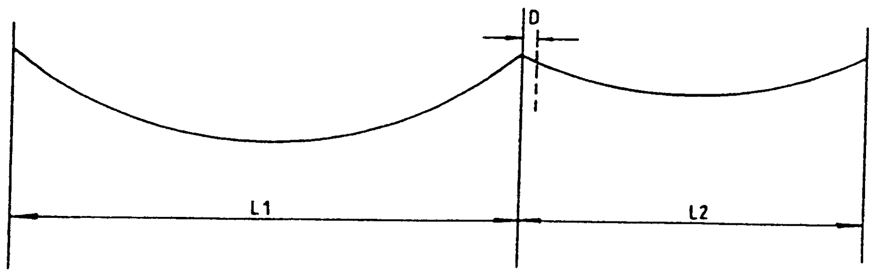


Fig. 1.7

Initially, at temperature t0 and loading f0, both spans are at the same tension, T0. If we first suppose the wire to be fixed at the intermediate support, then at some other condition (t1,f1) the wires in the spans L1 and L2 will reach different tensions T1 and T2 respectively. These tensions are given by the equations:

$$T1 = T0 - E.A.a.(t1-t0) - \frac{E.A.f0^2.L1^2}{24\ T0^2} + \frac{E.A.f1^2.L1^2}{24\ T1^2} \qquad ...(1.16a)$$

$$T2 = T0 - E.A.a.(t1-t0) - \frac{E.A.f0^2.L2^2}{24\ T0^2} + \frac{E.A.f1^2.L2^2}{24\ T1^2} \qquad ...(1.16b)$$

If we now suppose the intermediate support to be released, a length of wire, dL, will move from one span to the other. This introduces a new term into the Change-of State Equation. Taking the movement as positive if it is from the span L1 into the span L2, the wire in both spans will now reach a new tension T3, given , in the span L1, by:

$$T3 = T1 + \frac{E.A.dL}{L1} - \frac{E.A.f1^2.L1^2}{24\ T1^2} + \frac{E.A.f1^2.L1^2}{24\ T3^2}$$

Substituting for T1 from equation 1.16a:

$$T3 = T0 - E.A.a.(t1-t0) + \frac{E.A.dL}{L1} - \frac{E.A.f0^2.L1^2}{24\ T0^2} + \frac{E.A.f1^2.L1^2}{24\ T3^2} \quad ...(1.17a)$$

Likewise, in the span L2:

$$T3 = T0 - E.A.a.(t1-t0) - \frac{E.A.dL}{L2} - \frac{E.A.f0^2.L2^2}{24\ T0^2} + \frac{E.A.f1^2.L2^2}{24\ T3^2} \quad ...(1.17b)$$

To eliminate the unknown dL, we multiply equations 1.18a and 1.18b by L1 and L2 respectively: add the resulting equations together and divide by (L1 + L2). The result is a single equation:

$$T3 = T0 - E.A.a.(t1-t0) - \frac{E.A.f0^2.Le^2}{24\ T0^2} + \frac{E.A.f1^2.Le^2}{24\ T3^2} \quad ...(1.18)$$

where $Le = sqr[(L1^3 + L2^3)/(L1 + L2)]$

This parameter Le is called the EQUIVALENT SPAN. It is defined as the length of a single span which, subjected to the same change of conditions, would undergo the same change in tension as the two spans L1 and L2.

It can be shown that this process can be extended to any number of spans, so that the general formula for the equivalent span of a series of spans L1, L2, L3... etc. is,

$$Le = sqr[(L1^3 + L2^3 + L3^3\ ...\ etc.)/(L1 + L2 + L3\ ...\ etc.)] \quad ...(1.19)$$

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