Multiple Degrees of Freedom Structural Dynamics with MATHCAD (examples)

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MULTIPLE DEGREES OF FREEDOM STRUCTURAL DYNAMICS [1] WITH MATHCAD

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[1] Luis E. García and Mete A. Sozen (2003). “Multiple degrees of freedom structural dynamics.” Purdue University, Indiana, USA.

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1. REVIEW OF REPORT [1]

The report “Multiple Degrees of Freedom Structural Dynamics” by Luis E. Garcia and Mete Sozen presents the principles of structural dynamics applied to systems having several degrees of freedom. It is organized in seven sections with a corresponding example.

Modal Analysis & Mass Normalization

On Section 1, the classical method for solving the equations of motion of multi degree of freedom systems (MDOF) is presented. For free vibration we have the following system of n differential simultaneous equilibrium equation:

$$[M]\{\ddot{U}\}+[K]\{U\}=\{0\} \tag{1}$$

Where, $[M]$ and $[K]$ correspond to the mass and stiffness matrixes, respectively. Then, we propose the following solution of the simultaneous differential equations:

$$\{U_i(t)\}=\{\Phi^{(i)}\}f_i(t) \tag{2}$$

This is a solution that is separable into an amplitude vector, $\{\Phi^{(i)}\}$ and a time dependent function, $f_i(t)$. Substituting (2) in (1) we obtain.

$$[M]\{\Phi^{(i)}\}\ddot{f}_i+[K]\{\Phi^{(i)}\}f_i(t)=\{0\} \tag{3}$$

By using classical differential equation solution of separation of variable, equation (3) can be converted into two equation: one of them being dependent on time and the other dependent of $\{\Phi^{(i)}\}$; and both, in turn, equal to the constant ω_i^2 (natural frequency). The values that ω_i can take are obtained from:

$$\big[[K]-\omega_i{}^2[M]\big]\{\Phi^{(i)}\}=\{0\} \tag{4}$$

We solve equation (4) by taking the determinant of the coefficient matrix equal to zero. The n roots are the natural frequencies of the system, or eigenvalues; and the smaller frequency, ω_1 is called fundamental frequency. Now, by replacing the values of ω_i^2 in equation (4), we get n systems of simultaneous equation of the type:

$$\big[[K]-\omega_r{}^2[M]\big]\{\Phi^{(r)}\}=\{0\} \qquad r=1,2,\dots,n \tag{5}$$

Where $\{\Phi^{(r)}\}$ is the characteristic vector, vibration mode or "eigenvector". Each vector has a definite shape, but arbitrary amplitude. So we can normalize the eigenvector in different ways, but it is convenient to normalize the modes with respect to the mass matrix $[M]$, as follows:

$$\{\Phi^{(r)}\}^T[M]\{\Phi^{(r)}\}=1 \tag{6}$$

The different modes are collected in a single matrix, called modal matrix, $[\Phi]$ having dimensions of $n\ x\ n$, and in which each column corresponds to a mode.

Uncoupling of the Dynamic Equilibrium Equation

On this section, the author shows how to uncouple the dynamic MDOF system. Given the orthogonality property of the mass normalized eigenvectors. the total response can be described using a set of new degrees of freedom, $[\eta_i]$.

$$\{U(t)\}=[\Phi]\{\eta(t)\} \tag{7}$$

Through this procedure, the MDOF system is transformed into the summation of n independent single-degree of freedom.

$$\{U\} = [\Phi]\{\eta\} = \sum_{i=1}^{n}(\{\Phi^{(i)}\}\eta_i(t)) = \{\Phi^{(1)}\}\eta_1(t) + \cdots + \{\Phi^{(n)}\}\eta_n(t) \qquad (8)$$

Free Vibration

For free vibration, each of the terms of vector $\{\eta(t)\}$ have the following form:

$$\eta_i(t) = A_i\, sin(\omega_i t) + B_i\, cos(\omega_i t) \qquad (9)$$

Equation (10) presents the solution of the MDOF system with initial conditions using a superposition of the response of the uncoupled degrees of freedom.

$$\{U(t)\} = \sum_{i=1}^{n}(\Phi^{(i)}\, A_i\, sin(\omega_i t)) + \sum_{i=1}^{n}(\Phi^{(i)}\, B_i\, sin(\omega_i t)) \qquad (10)$$

We also can get the response of the structure in an easier way by doing the superposition of the individual contribution from each mode. To do that, first, we have to obtain the response in time of each one of the generalized degrees of freedom, η_{i}, and replace the values on equation (8).

Damped Modal Analysis and Forced Vibration

So far, the inherent damping of the system have not been considered in the equations. On this section, the damped modal analysis is presented, as follows:

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = \{0\} \qquad (11)$$

Then, the equations to uncouple a MDOF system with viscous damping is derived based on previous sections, as follows:

$$\ddot{\eta}_i + 2\xi_i\omega_i\dot{\eta}_i + {\omega_i}^2\eta_i = 0 \qquad (12)$$

Where ξ_{i} is the viscous damping associated with mode i. This type of damping in which the damping matrix is uncoupled by the vibration modes obtained only from mass and stiffness matrices is known as classic damping.

In addition, in this section we deal with MDOF systems subjected to forced vibration, which can be described in the following manner:

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = \{P(t)\} \qquad (13)$$

Uncoupling the problem by using the modes and frequencies of the structure obtained for free vibration and the transformation presented in equation (7), we obtain:

$$\ddot{\eta}_i + 2\xi_i\omega_i\dot{\eta}_i + {\omega_i}^2\eta_i = \sum_{j=1}^{n}(\,{\Phi_i}^{(i)}\, p_j(t)) \qquad (14)$$

With this, it is easy to derive the solutions for harmonic and transient forced vibration, which are presented in detail in the reference and illustrated through examples later on Section 3.4.

Base Excitation

We also study the base excitation of a MDOF system, which can be expressed in the following manner (assuming damping):

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = -[M][\gamma]\{\ddot{x}_0\} \qquad (15)$$

The equations to solve this problem are derived by using the uncoupling procedure explained previously, then we get:

$$\ddot{\eta_i} + 2\xi_i\omega_i\dot{\eta_i} + {\omega_i}^2\eta_i = -\{\propto_i\}\{\ddot{x}_0\} \tag{16}$$

Where $\{\propto_i\}$ is the participation coefficient and corresponds to row i of matrix $[\propto]$ obtained from:

$$[\propto] = [\Phi]^T[M][\gamma] \tag{17}$$

Knowing that the solution for displacement $[U]$ can be calculated with equation (7), now we can get the forces imposed by the ground motion for each mode by multiplying the displacements caused by each mode by the stiffness matrix of the structure:

$$\{F^{(i)}\} = [K]\{U^{(i)}\} \tag{18}$$

Likewise, the base shear, V_i and overturning moment, M_i of mode i at instant t are:

$$V_i = \{1\}^T\{F^{(i)}\} \tag{19}$$

$$M_i = \{h\}^T\{F^{(i)}\} \tag{20}$$

Where: $\{1\}$ is a column vector with n rows with unitary value; and $\{h\}$ is column vector that contains the height of the n stories measured from the base of the structure.

Modal Spectral Analysis

The author presents the modal spectral analysis as a practical alternative to get the response of the MDOF system subjected to an earthquake. First, we have to develop the displacement response spectrum, $S_d(T, \xi)$, which is the collection of maximum displacements obtained by single degree of freedom systems having period T and damping coefficient ξ, when subjected to the ground motion record.

Then, the maximum displacement that an uncoupled degree of freedom of the structure can have can be obtained as follows:

$$(\eta_i)_{max} = |\propto_i . S_d(T_i, \xi_i)| \tag{21}$$

Substituting (21) in (7) we obtain the values of maximum displacements that the structure can have for each individual mode. Similarly, substituting (22) in (18), we can get the maximum lateral forces for each individual mode i.

$$\{{U_{mod}}^{(i)}\} = \{\Phi^{(i)}\}.(\eta_i)_{max} = \{\Phi^{(i)}\}.|\propto_i . S_d(T_i, \xi_i)| \tag{22}$$

$$\{{F_{mod}}^{(i)}\} = [K]\{{U_{mod}}^{(i)}\} = [K]\{\Phi^{(i)}\}.|\propto_i . S_d(T_i, \xi_i)| \tag{23}$$

Modal Combination (SRSS)

It is important to notice that the parameters calculated by the response spectral analysis do not occur at the same time. Then we have to come up with a method to combine the contribution of each uncoupled mode. The most widely known method of modal spectral combination is called Square Root of the Sum of the Squares (SRSS), and it can be calculated, for the response parameter r_i, with the following formula:

$$\bar{r} \approx \sqrt{\sum_{i=1}^{n} {r_i}^2} \tag{24}$$

We use this technique to estimate the maximum response of a MDOF system in terms of lateral displacement, base shear, overturning moment, story drift, etc. As a conclusion, the author notes that this procedure is a reasonable good estimation if we compare with the results obtained from the time-history analysis (step-by-step procedure).

2. STRUCTURAL MODEL

In this section, we present a structural model developed in order to conduct the same type of dynamic analyses described in examples 1 to 7 of the reference report [1].

2.1. Building data

In this study, we select a 5-story reinforced concrete moment resisting frame building. The frame has two spans of 5.5 m and a total height of 16.0m. The height of the first floor is 4.0m while of the other floors are 3.0m. Damping of the structure is estimated to be $\xi = 5\%$ of critical. All girders have width $\mathrm{b} = 0.40\ \mathrm{m}$ and depth $\mathrm{h} = 0.50\ \mathrm{m}$. All columns are square with a section side dimension of $\mathrm{h} = 0.40\ \mathrm{m}$. The modulus of elasticity of the concrete is $\mathrm{E} = 25\ \mathrm{GPa}$. The building has loads only due to its self-weight.

- 5-story concrete moment-resisting frame building
- Typical beam span: 5.5m
- Typical Story height: 3.0m
- Total mass: 42.9. ton
- Fixed supports at base

We are going to neglect the contribution of axial deformation in the calculation of stiffness, and consider that beams are very rigid elements. To do that, we apply a very high stiffness modifiers to the Shear Area in 2 direction, and to the Cross-section (axial) Area.

Figure 1. ETABS 2D structural model.

2.2. Modal Analysis

By running a modal analysis on ETABS, we get the modal shapes of the structure, as well as the dynamic properties presented in Table 1. We evidence that we reach an effective mass participation of 100.0% (minimum recommended equals 90%). We also find that the fundamental period of the building, T_1 equals 0.359s.

Table 1. Periods of Vibration and Mode Shapes

Vibration Mode	Frequency (Hz)	Period T (s)	Effective Mass (%)
1	2.786	0.359	91.06
2	8.746	0.114	7.10
3	15.62	0.064	1.43
4	23.26	0.043	0.34
5	30.24	0.033	0.06
			100.00

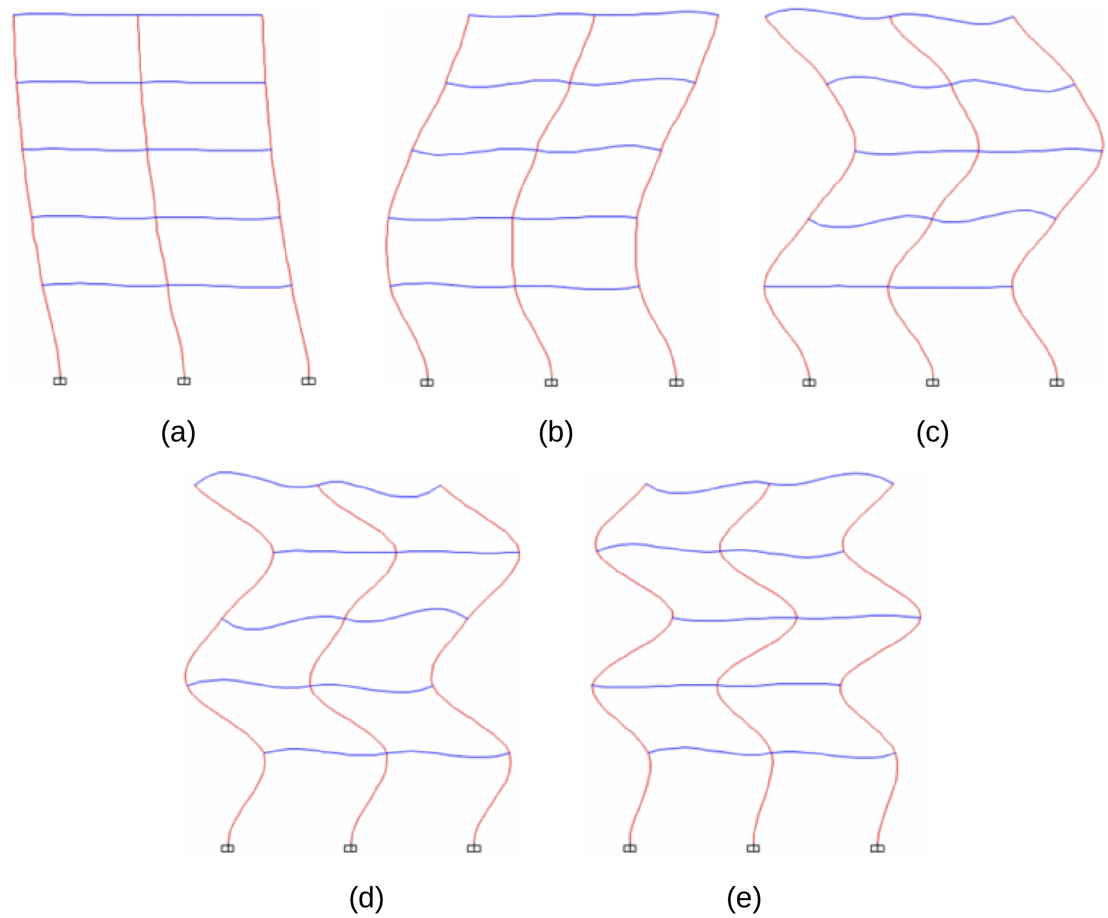


Figure 2. Modal Shapes obtained from ETABS.

The dynamic properties are also calculated by solving the Eigenvalue problem (Equation 4) with MATLAB. In Table 2, we compare the modal analysis results from ETABS with the hand calculations results (MATLAB). Note that the values are pretty close due to the assumptions made on the model. The little differences might be because of the approximation of the stiffness matrix calculation (Muto Method).

Table 2. Periods of Vibration and Mode Shapes

Vibration Mode	Period (s)	
	ETABS	MATHCAD
1	0.359	0.354
2	0.114	0.123
3	0.064	0.080
4	0.043	0.063
5	0.033	0.056

3. GROUND MOTION (NONLIN)

The seismic record: 'NGA1787_HectorMine_Hector_00' at California, in October 16 of 1999 is selected from NONLIN [2] database. Specifically, in Sections 4.5, 4.6 and 4.7, we study the response of the building subjected to this earthquake.

Table 3. Selected Ground Motion (Source: NONLIN).

ID	Earthquake Event	Comp.	PGA (g)	Time Step (s)	Duration (s)
NGA1787	Hector Mine (California, USA)	E-W	0.266	0.01	45.31

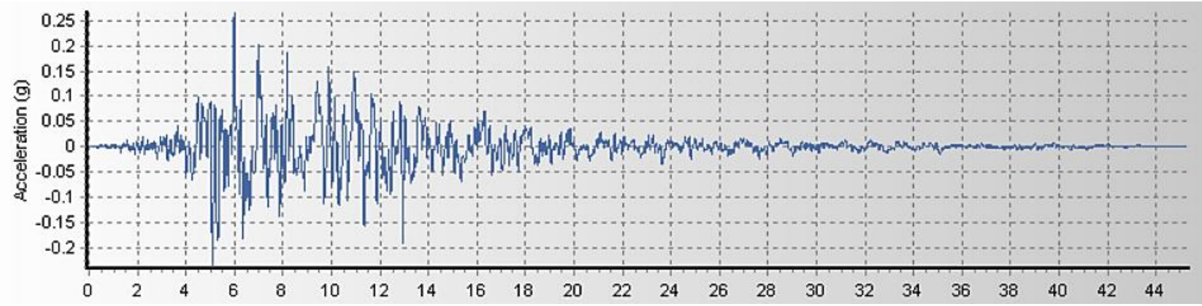


Figure 3. Hector-Mine accelerograms (Source: SeismoSignal).

4. DYNAMIC RESPONSE OF SIMPLIFIED FE MODEL

Using the model developed in Section 2, we conduct the same type of analyses described in examples 1 to 7 of the reference report. To do that, we must first find the stiffness characteristics of the building. The frame stiffness is obtained using Muto Method. We estimate the stiffness of each floor and construct the 5x5 stiffness matrix as follows:

$$k1 = 4382.37\ ton/m \qquad k2 = k3 = k4 = k5 = 2926.89\ ton/m$$

$$K := \begin{pmatrix} k5 & -k5 & 0 & 0 & 0 \\ -k5 & k4+k5 & -k4 & 0 & 0 \\ 0 & -k4 & k3+k4 & -k3 & 0 \\ 0 & 0 & -k3 & k2+k3 & -k2 \\ 0 & 0 & 0 & -k2 & k1+k2 \end{pmatrix} = \begin{pmatrix} 28.70 & -28.70 & 0.00 & 0.00 & 0.00 \\ -28.70 & 57.41 & -28.70 & 0.00 & 0.00 \\ 0.00 & -28.70 & 57.41 & -28.70 & 0.00 \\ 0.00 & 0.00 & -28.70 & 57.41 & -28.70 \\ 0.00 & 0.00 & 0.00 & -28.70 & 71.68 \end{pmatrix} \cdot 10^{3} \cdot \frac{kN}{mt}$$

Given the concrete density $\gamma_c = 2.4\ t/m^3$, we calculate the mass for each floor. Then, the mass matrix of the building is:

$$M := \begin{pmatrix} m5 & 0 & 0 & 0 & 0 \\ 0 & m4 & 0 & 0 & 0 \\ 0 & 0 & m3 & 0 & 0 \\ 0 & 0 & 0 & m2 & 0 \\ 0 & 0 & 0 & 0 & m1 \end{pmatrix} \cdot m = \begin{pmatrix} 8.352 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 8.352 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 8.352 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 8.352 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 9.504 \end{pmatrix} 10^{3} \cdot kg$$

Now, with the mass and stiffness matrix as an input data we can replicate the examples of the reference report. For the calculations, we will use MATHCAD 15 software. For example 5, we will use the “NGA1787_HectorMine_Hector_00” record from database of ground motions included in the NONLIN software.

4.1. Example 1: Modal Analysis & Mass Normalization

For the building presented in Section 2, we solve the eigenvalue problem by using MATHCAD spreadsheet. The mode shapes are shown in Figure 4. We also notice that the periods are pretty close to the ones calculated with ETABS (See Table 2). For instance the fundamental period, T_1 equals 0.354s.

Dynamic Matrix: $D := M^{-1} \cdot K \qquad \lambda := \mathrm{sort}(\mathrm{eigenvals}(D)) \qquad X^{\langle j \rangle} := \mathrm{eigenvec}(D, \lambda_j)$

Normalized Mode Shapes: $z_{1,j} := 0 \qquad i := 1..N+1 \qquad \psi^{\langle j \rangle} := \dfrac{X^{\langle j \rangle}}{\overrightarrow{\max\left(\left|X^{\langle j \rangle}\right|\right)}}$

Frequencies and Eigenvalues: $f := \overrightarrow{\dfrac{1}{2 \cdot \pi} \cdot \sqrt{\lambda}}$

$$\omega^2 = \begin{pmatrix} 314.2 \\ 2605.7 \\ 6189.3 \\ 9837.7 \\ 12651.7 \end{pmatrix} s^{-2.0} \qquad \omega = \begin{pmatrix} 17.7 \\ 51.0 \\ 78.7 \\ 99.2 \\ 112.5 \end{pmatrix} s^{-1.0}$$

Modal Matrix:

$$\psi = \begin{pmatrix} 1.000 & -0.900 & 0.786 & -0.537 & 0.285 \\ 0.909 & -0.218 & -0.629 & 1.000 & -0.764 \\ 0.734 & 0.630 & -0.911 & -0.326 & 1.000 \\ 0.492 & 1.000 & 0.448 & -0.719 & -0.917 \\ 0.206 & 0.612 & 1.000 & 0.946 & 0.542 \end{pmatrix}$$

Corresponding, graphically, to:

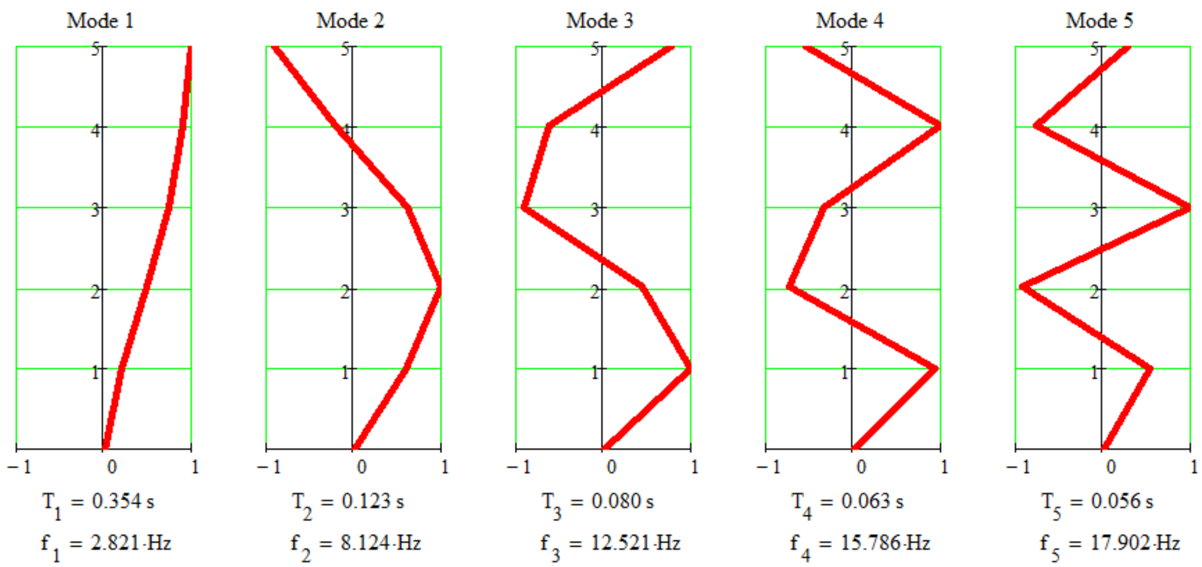


Figure 4. Modal Shapes obtained (MATHCAD 15).

Finally, we mass normalize the modal matrix as follows:

$$i := 1..N \qquad \Phi^{\langle i \rangle} := \frac{\psi^{\langle i \rangle}}{\sqrt{\psi^{\langle i \rangle T} \cdot M \cdot \psi^{\langle i \rangle}}}$$

Then:

$$\Phi = \begin{pmatrix} 6.715469 & -6.016821 & 4.819509 & -3.432287 & 1.850373 \\ 6.101420 & -1.454833 & -3.860190 & 6.392929 & -4.961586 \\ 4.929470 & 4.210218 & -5.587877 & -2.082162 & 6.492028 \\ 3.306779 & 6.683058 & 2.747927 & -4.596886 & -5.954099 \\ 1.381723 & 4.088765 & 6.134849 & 6.047370 & 3.519182 \end{pmatrix} \cdot 10^{-3} \cdot kg^{-0.5}$$

4.2. Example 2: Uncoupling of the Dynamic Equilibrium Equation

Uncouple the dynamic system of Example 1 using the modal matrix [Φ].

$$\Phi^T \cdot M \cdot \Phi = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{pmatrix}$$

$$\Phi^T \cdot K \cdot \Phi = \begin{pmatrix} 0.314 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 2.606 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 6.189 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 9.838 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 12.652 \end{pmatrix} \cdot 10^3 s^{-2.000} \quad => \quad \lambda = \begin{pmatrix} 0.314 \\ 2.606 \\ 6.189 \\ 9.838 \\ 12.652 \end{pmatrix} \cdot 10^3 s^{-2.000}$$

Then, assuming that there is no damping, the uncouple equations are:

$$\begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{pmatrix} \cdot \begin{pmatrix} \eta''_1 \\ \eta''_2 \\ \eta''_3 \\ \eta''_4 \\ \eta''_5 \end{pmatrix} + \begin{pmatrix} 0.314 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 2.606 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 6.189 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 9.838 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 12.652 \end{pmatrix} \cdot \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{pmatrix} \cdot 10^{-3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Or seen as three independent differential equations:

$$\eta''_1 + 0.31410^{-3} \cdot \eta_1 = 0$$

$$\eta''_2 + 2.60610^{-3} \cdot \eta_2 = 0$$

$$\eta''_3 + 6.18910^{-3} \cdot \eta_3 = 0$$

$$\eta''_4 + 9.83810^{-3} \cdot \eta_4 = 0$$

$$\eta''_5 + 12.65210^{-3} \cdot \eta_5 = 0$$

4.3. Example 3: Free Vibration

Case (a) – Find the free vibration response given a unit displacement at each story of the building at time = 0, without any initial velocity.

The initial displacement vector is: $U_0 := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Constants **bi** are obtained from:

$$B := \Phi^T \cdot M \cdot U_0 = \begin{pmatrix} 188.968 \\ 67.437 \\ 42.599 \\ 26.418 \\ 11.954 \end{pmatrix} kg^{0.500} \qquad w := \sqrt{\lambda} = \begin{pmatrix} 17.727 \\ 51.046 \\ 78.672 \\ 99.185 \\ 112.480 \end{pmatrix} s^{-1.000}$$

$$\Phi^{\langle 1 \rangle} \cdot B_1 = \begin{pmatrix} 1.269 \\ 1.153 \\ 0.932 \\ 0.625 \\ 0.261 \end{pmatrix} \quad \Phi^{\langle 2 \rangle} \cdot B_2 = \begin{pmatrix} -0.406 \\ -0.098 \\ 0.284 \\ 0.451 \\ 0.276 \end{pmatrix} \quad \Phi^{\langle 3 \rangle} \cdot B_3 = \begin{pmatrix} 0.205 \\ -0.164 \\ -0.238 \\ 0.117 \\ 0.261 \end{pmatrix} \quad \Phi^{\langle 4 \rangle} \cdot B_4 = \begin{pmatrix} -0.091 \\ 0.169 \\ -0.055 \\ -0.121 \\ 0.160 \end{pmatrix} \quad \Phi^{\langle 5 \rangle} \cdot B_5 = \begin{pmatrix} 0.022 \\ -0.059 \\ 0.078 \\ -0.071 \\ 0.042 \end{pmatrix}$$

Then, the response of the system is described by the following equation:

$$\begin{pmatrix} U_5 \\ U_4 \\ U_3 \\ U_2 \\ U_1 \end{pmatrix} = \begin{pmatrix} 1.269 \\ 1.153 \\ 0.932 \\ 0.625 \\ 0.261 \end{pmatrix} \cdot \cos(w_1 \cdot t) + \begin{pmatrix} -0.406 \\ -0.098 \\ 0.284 \\ 0.451 \\ 0.276 \end{pmatrix} \cdot \cos(w_2 \cdot t) + \begin{pmatrix} 0.205 \\ -0.164 \\ -0.238 \\ 0.117 \\ 0.261 \end{pmatrix} \cdot \cos(w_3 \cdot t) + \begin{pmatrix} -0.091 \\ 0.169 \\ -0.055 \\ -0.121 \\ 0.160 \end{pmatrix} \cdot \cos(w_4 \cdot t) + \begin{pmatrix} 0.022 \\ -0.059 \\ 0.078 \\ -0.071 \\ 0.042 \end{pmatrix} \cdot \cos(w_5 \cdot t)$$

Mode 1 response: $f(t) := \Phi^{\langle 1 \rangle} \cdot B_1 \cdot \cos(w_1 \cdot t)$

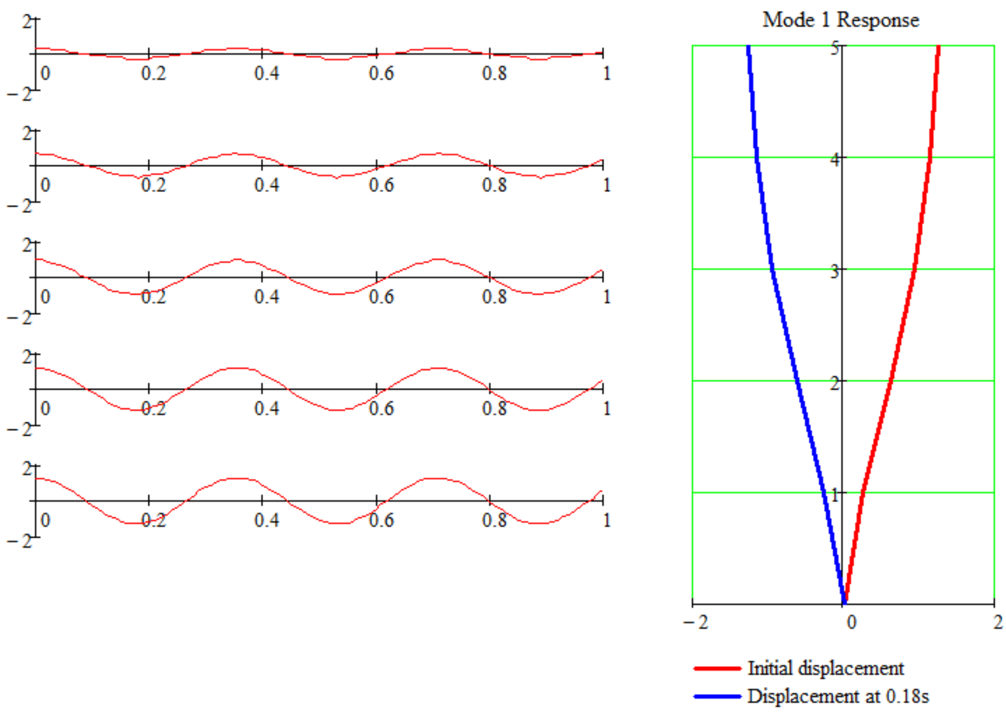


Figure 5. Mode 1 Response to initial displacement conditions. Case (a).

Mode 2 response: $f(t) := \Phi^{\langle 2\rangle} \cdot B_2 \cdot \cos(w_2 \cdot t)$

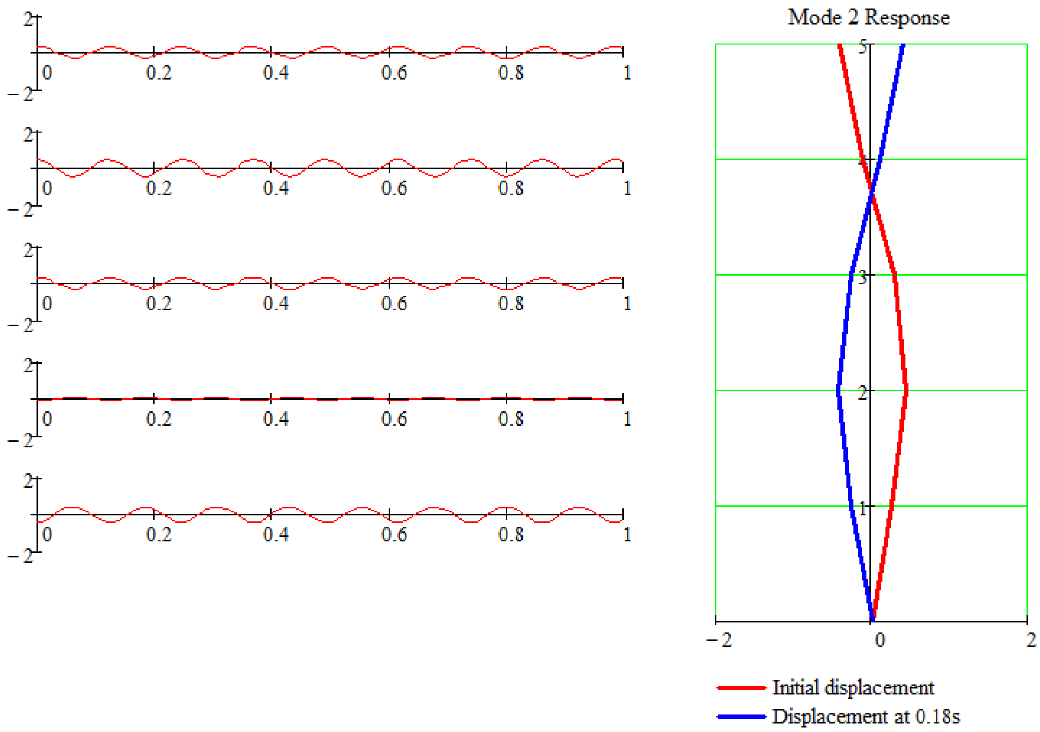


Figure 6. Mode 2 Response to initial displacement conditions. Case (a).

Mode 3 response: $f(t) := \Phi^{\langle 3\rangle} \cdot B_3 \cdot \cos(w_3 \cdot t)$

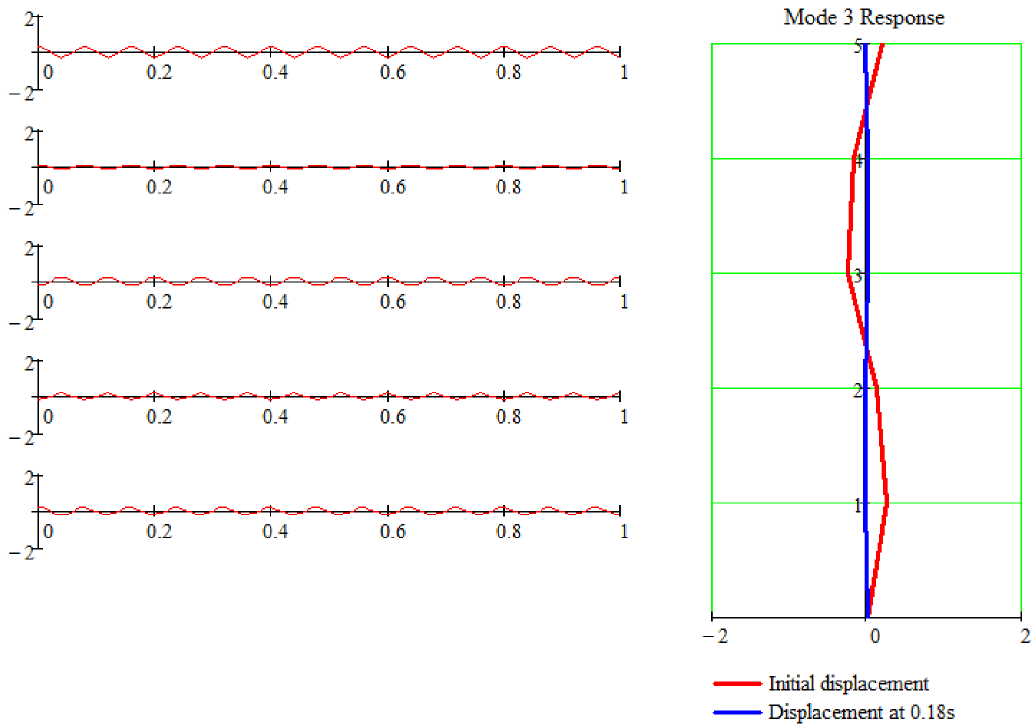


Figure 7. Mode 3 Response to initial displacement conditions. Case (a).

Mode 4 response: $f(t) := \Phi^{\langle 4 \rangle} \cdot B_4 \cdot \cos(w_4 \cdot t)$

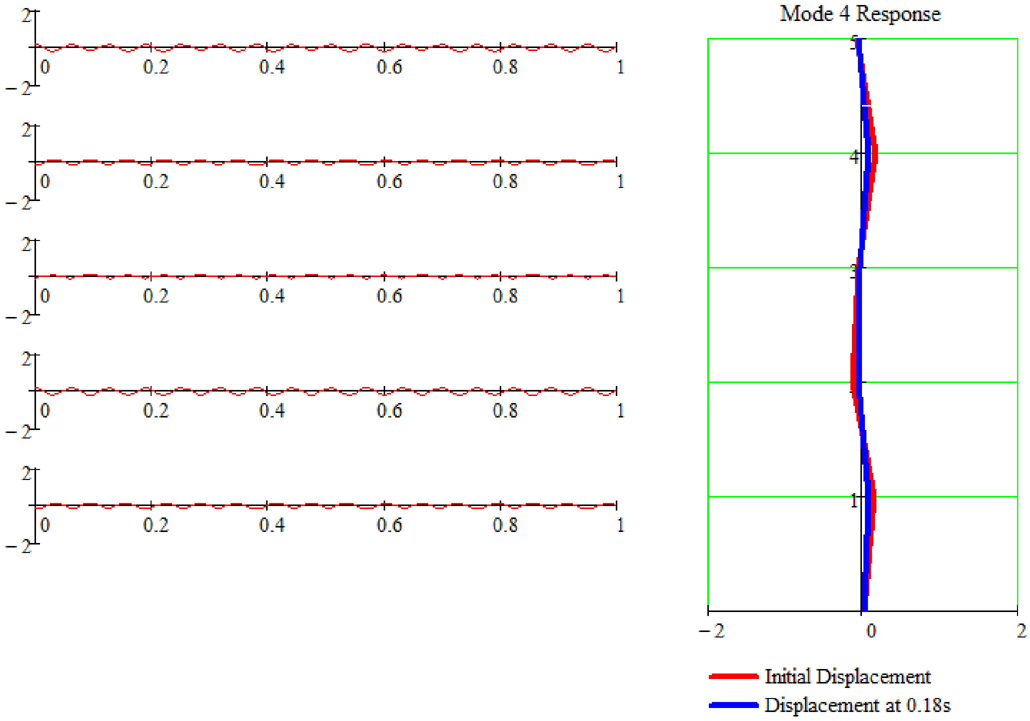


Figure 8. Mode 4 Response to initial displacement conditions. Case (a).

Mode 5 response: $f(t) := \Phi^{\langle 5 \rangle} \cdot B_5 \cdot \cos(w_5 \cdot t)$

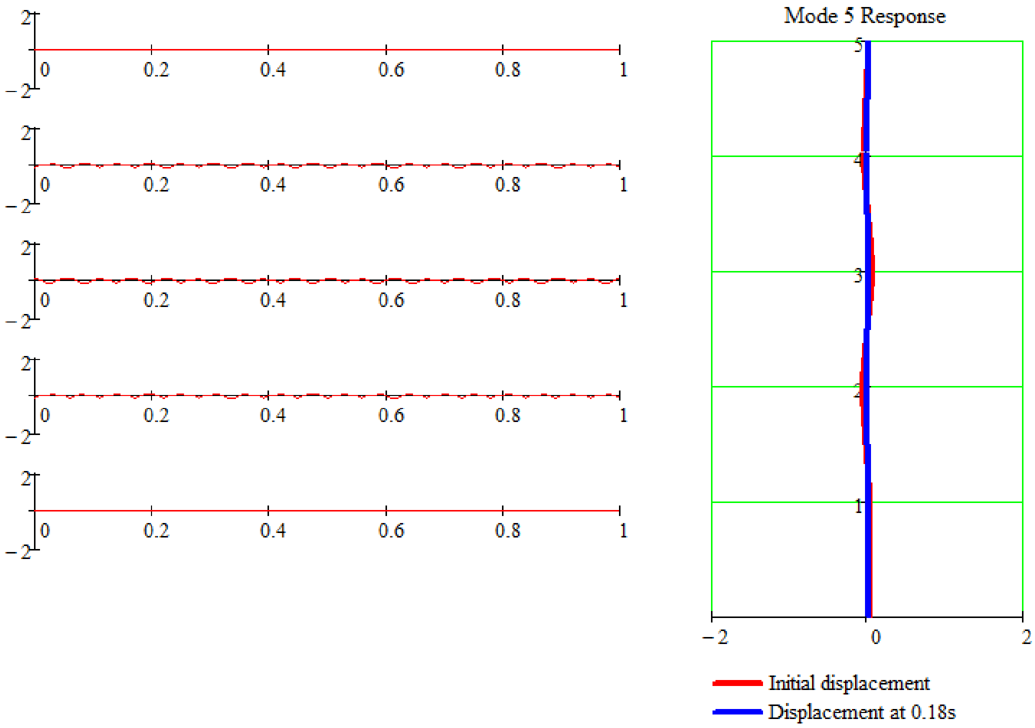


Figure 9. Mode 5 Response to initial displacement conditions. Case (a).

Total Response:

$$f(t) := \Phi^{\langle 1 \rangle} \cdot B_1 \cdot \cos(w_1 \cdot t) + \Phi^{\langle 2 \rangle} \cdot B_2 \cdot \cos(w_2 \cdot t) + \Phi^{\langle 3 \rangle} \cdot B_3 \cdot \cos(w_3 \cdot t) + \Phi^{\langle 4 \rangle} \cdot B_4 \cdot \cos(w_4 \cdot t) + \Phi^{\langle 5 \rangle} \cdot B_5 \cdot \cos(w_5 \cdot t)$$

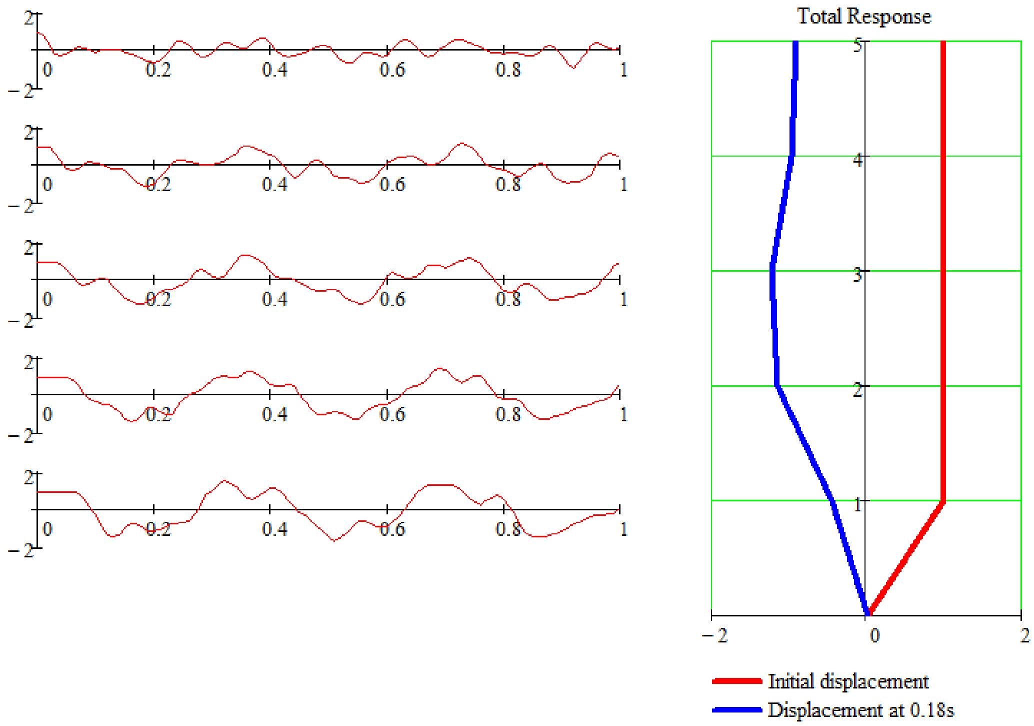


Figure 10. Total Response to initial displacement conditions. Case (a).

Figures 5 to 9, and 10 show the response for each mode and the total response of the building, respectively. We notice that the response of the system corresponds to the superposition of the individual responses from each mode. Supposing that at some instant in time the five responses are in phase, 83% would be contributed by the first mode, 11% by the second, 4% by the third, and 2% by the rest.

Case (b) – Find the free vibration response given a displacement condition in the shape of the first mode, without any initial velocity.

The initial displacement vector is:

$$U_0 := \psi^{\langle 1 \rangle} = \begin{pmatrix} 1.000 \\ 0.909 \\ 0.734 \\ 0.492 \\ 0.206 \end{pmatrix}$$

Constants **bi** are obtained from:

$$B := \Phi^T \cdot M \cdot U_0 = \begin{pmatrix} 148.910 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} kg^{0.500} \qquad w := \sqrt{\lambda} = \begin{pmatrix} 17.727 \\ 51.046 \\ 78.672 \\ 99.185 \\ 112.480 \end{pmatrix} s^{-1.000}$$

$$\Phi^{\langle 1 \rangle}\cdot B_1 = \begin{pmatrix} 1.000 \\ 0.909 \\ 0.734 \\ 0.492 \\ 0.206 \end{pmatrix} \quad \Phi^{\langle 2 \rangle}\cdot B_2 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \quad \Phi^{\langle 3 \rangle}\cdot B_3 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \quad \Phi^{\langle 4 \rangle}\cdot B_4 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \quad \Phi^{\langle 5 \rangle}\cdot B_5 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix}$$

Then, the response of the system is described by the following equation:

$$\begin{pmatrix} U_5 \\ U_4 \\ U_3 \\ U_2 \\ U_1 \end{pmatrix} = \begin{pmatrix} 1.269 \\ 1.153 \\ 0.932 \\ 0.625 \\ 0.261 \end{pmatrix}\cdot\cos\left(w_1\cdot t\right)$$

Therefore, the total response is dominated by the response of the first mode:

$$f(t) := \Phi^{\langle 1 \rangle}\cdot B_1\cdot\cos\left(w_1\cdot t\right)$$

Case (c) – Find the free vibration response given a displacement condition in the shape of the second mode, without any initial velocity.

The initial displacement vector is: $U_0 := \psi^{\langle 2 \rangle} = \begin{pmatrix} -0.900 \\ -0.218 \\ 0.630 \\ 1.000 \\ 0.612 \end{pmatrix}$

Constants **bi** are obtained from:

$$B := \Phi^{T}\cdot M\cdot U_0 = \begin{pmatrix} 0.000 \\ 149.632 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} kg^{0.500} \qquad w := \sqrt{\lambda} = \begin{pmatrix} 17.727 \\ 51.046 \\ 78.672 \\ 99.185 \\ 112.480 \end{pmatrix} s^{-1.000}$$

$$\Phi^{\langle 1 \rangle}\cdot B_1 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \quad \Phi^{\langle 2 \rangle}\cdot B_2 = \begin{pmatrix} -0.900 \\ -0.218 \\ 0.630 \\ 1.000 \\ 0.612 \end{pmatrix} \quad \Phi^{\langle 3 \rangle}\cdot B_3 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \quad \Phi^{\langle 4 \rangle}\cdot B_4 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix} \quad \Phi^{\langle 5 \rangle}\cdot B_5 = \begin{pmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{pmatrix}$$

Then, the response of the system is described by the following equation:

$$\begin{pmatrix} U_5 \\ U_4 \\ U_3 \\ U_2 \\ U_1 \end{pmatrix} = \begin{pmatrix} -0.900 \\ -0.218 \\ 0.630 \\ 1.000 \\ 0.612 \end{pmatrix} \cdot \cos\left(w_2 \cdot t\right)$$

Therefore, the total response is dominated by the response of the second mode:

$$f(t) := \Phi^{\langle 2 \rangle} \cdot B_2 \cdot \cos\left(w_2 \cdot t\right)$$

Conclusion: We can continue trying an initial displacement that follows the n^{th} mode shape, and will get that only the n^{th} mode contributes with a 100% of the response.

4.4. Example 4: Forced Vibration (Impulse)

In this example, the building shown in Section 2 is subjected to an explosion. The air pressure wave caused by the explosion varies in the form shown in Figure 11. Damping of the structure is estimated to be ξ = 5% of critical.

$$t := 0, 0.1s .. 0.8s$$

$$q(t) := \begin{cases} 50t & \text{if } t < 0.1s \\ 7s - 20t & \text{if } 0.1s \le t \le 0.4s \\ 5t - 3s & \text{if } 0.4s < t \le 0.6s \\ 0 & \text{if } t > 0.6s \end{cases}$$

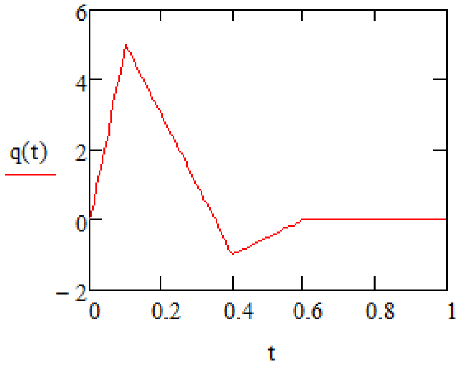


Figure 11. Impact Load (Pressure in kPa).

The explosion occurred far away, therefore we can assume that the pressure applied to the building doesn't vary with height and is applied uniformly to the building façade. Then the forces at each level can be determined as follows:

$$\text{force5}(t) := (5.5 \cdot 2) \cdot (1.5) \, q(t)$$
$$\text{force4}(t) := (5.5 \cdot 2) \cdot (3) \, q(t)$$
$$\text{force3}(t) := (5.5 \cdot 2) \cdot (3) \, q(t)$$
$$\text{force2}(t) := (5.5 \cdot 2) \cdot (3) \, q(t)$$
$$\text{force1}(t) := (5.5 \cdot 2) \cdot (3.5) \, q(t)$$

$$\text{force} := \Phi^T \cdot \begin{pmatrix} 16.5 \\ 33 \\ 33 \\ 33 \\ 38.5 \end{pmatrix} \cdot kg^{0.5} = \begin{pmatrix} 0.637 \\ 0.370 \\ 0.095 \\ 0.167 \\ 0.020 \end{pmatrix} * q(t)$$

Then, the uncouple equations are obtained as follows:

$$\eta''_1 + 2\cdot\xi\cdot\omega_1\cdot\eta'_1 + \omega_1^{\,2}\cdot\eta_1 = force_1\cdot q(t)$$

$$\eta''_2 + 2\cdot\xi\cdot\omega_2\cdot\eta'_2 + \omega_2^{\,2}\cdot\eta_2 = force_2\cdot q(t)$$

$$\eta''_3 + 2\cdot\xi\cdot\omega_3\cdot\eta'_3 + \omega_3^{\,2}\cdot\eta_3 = force_3\cdot q(t)$$

$$\eta''_4 + 2\cdot\xi\cdot\omega_4\cdot\eta'_4 + \omega_4^{\,2}\cdot\eta_4 = force_4\cdot q(t)$$

$$\eta''_5 + 2\cdot\xi\cdot\omega_5\cdot\eta'_5 + \omega_5^{\,2}\cdot\eta_5 = force_1\cdot q(t)$$

$yf := 2.5$

Given

$$\frac{d^2}{dy^2}\eta(y) + 2\cdot\xi\cdot\omega_1\cdot\left(\frac{d}{dy}\eta(y)\right) + \omega_1^{\,2}\cdot\eta(y) = force_1\cdot function(y)$$

$\eta(0) = 0$ $\eta'(0) = 0$

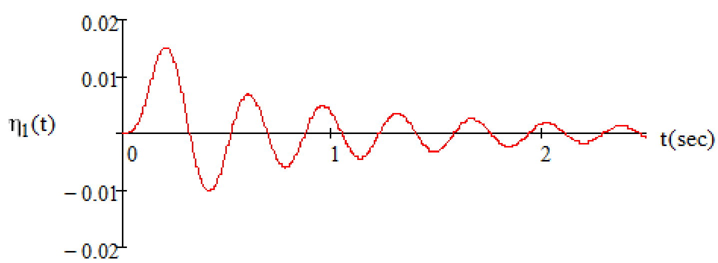
$$function(y) := \begin{cases} 50y & \text{if } y < 0.1 \\ 7 - 20y & \text{if } 0.1 \le y \le 0.4 \\ 5y - 3 & \text{if } 0.4 < y \le 0.6 \\ 0 & \text{if } y > 0.6 \end{cases}$$

$\eta_1 := Odesolve(y, yf)$

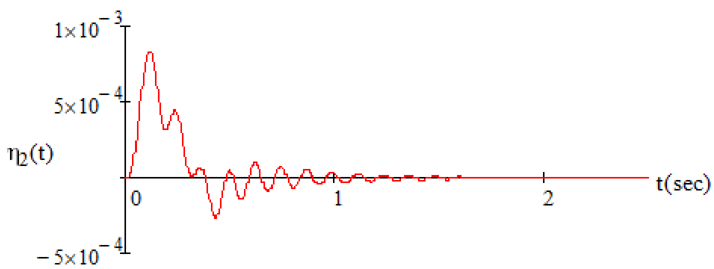
Figure 12.Mathcad Algorithm to solve Example 4.

In these five equations ξ = 0.05. The response of each of the uncoupled equations was obtained employing Ordinary Differential Equation Solver of Mathcad (Figure 12). The first 2.5s of response for each mode are shown in Figure 13:

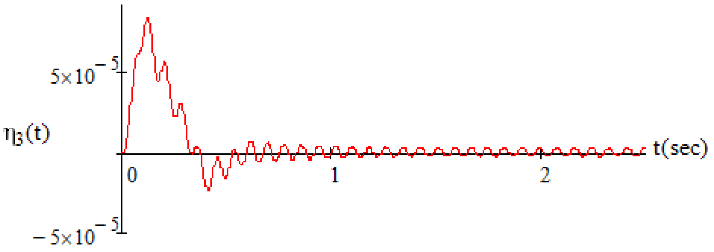
$\eta_1(t) - T_1 = 0.354$ s



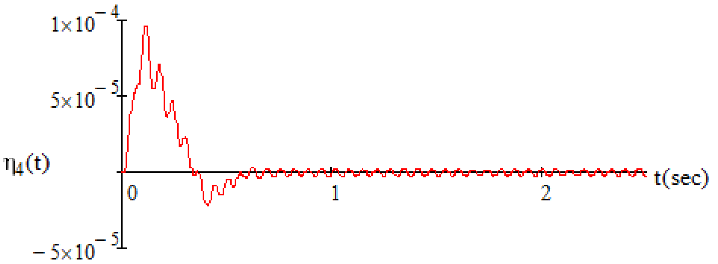
$\eta_2(t) - T_2 = 0.123$ s



$\eta_3(t) - T_3 = 0.080\ \mathrm{s}$



$\eta_4(t) - T_4 = 0.063\ \mathrm{s}$



$\eta_5(t) - T_5 = 0.056\ \mathrm{s}$

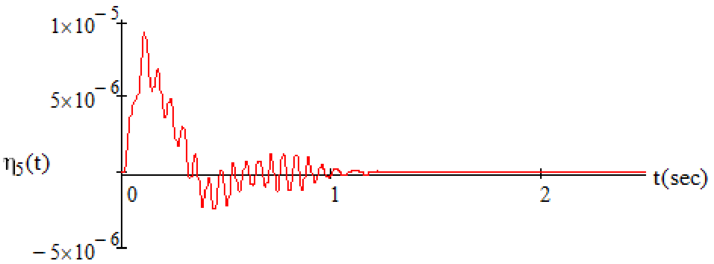


Figure 13. Response in time for the uncoupled degrees of freedom.

The response at some instants are presented below:

$t := 0, 0.025 .. 0.4$

$t =$	$\eta_1(t) =$	$\eta_2(t) =$	$\eta_3(t) =$	$\eta_4(t) =$	$\eta_5(t) =$
0.000	$0.00\cdot10^{0}$	$0.00\cdot10^{0}$	$0.00\cdot10^{0}$	$0.00\cdot10^{0}$	$0.00\cdot10^{0}$
0.025	$8.15\cdot10^{-5}$	$4.30\cdot10^{-5}$	$0.00\cdot10^{0}$	$1.49\cdot10^{-5}$	$0.00\cdot10^{0}$
0.050	$6.24\cdot10^{-4}$	$2.62\cdot10^{-4}$	$4.23\cdot10^{-5}$	$4.83\cdot10^{-5}$	$0.00\cdot10^{0}$
0.075	$1.98\cdot10^{-3}$	$5.82\cdot10^{-4}$	$5.99\cdot10^{-5}$	$5.73\cdot10^{-5}$	$0.00\cdot10^{0}$
0.100	$4.35\cdot10^{-3}$	$8.00\cdot10^{-4}$	$6.85\cdot10^{-5}$	$8.61\cdot10^{-5}$	$0.00\cdot10^{0}$
0.125	$7.59\cdot10^{-3}$	$8.13\cdot10^{-4}$	$8.30\cdot10^{-5}$	$8.53\cdot10^{-5}$	$0.00\cdot10^{0}$
0.150	$1.10\cdot10^{-2}$	$5.91\cdot10^{-4}$	$5.90\cdot10^{-5}$	$5.49\cdot10^{-5}$	$0.00\cdot10^{0}$
0.175	$1.37\cdot10^{-2}$	$3.61\cdot10^{-4}$	$4.37\cdot10^{-5}$	$7.13\cdot10^{-5}$	$0.00\cdot10^{0}$
0.200	$1.50\cdot10^{-2}$	$3.38\cdot10^{-4}$	$5.57\cdot10^{-5}$	$4.55\cdot10^{-5}$	$0.00\cdot10^{0}$
0.225	$1.46\cdot10^{-2}$	$4.33\cdot10^{-4}$	$3.99\cdot10^{-5}$	$4.14\cdot10^{-5}$	$0.00\cdot10^{0}$
0.250	$1.24\cdot10^{-2}$	$4.10\cdot10^{-4}$	$2.18\cdot10^{-5}$	$4.05\cdot10^{-5}$	$0.00\cdot10^{0}$
0.275	$8.73\cdot10^{-3}$	$2.20\cdot10^{-4}$	$2.92\cdot10^{-5}$	$1.80\cdot10^{-5}$	$0.00\cdot10^{0}$
0.300	$4.13\cdot10^{-3}$	$4.27\cdot10^{-5}$	$1.92\cdot10^{-5}$	$2.33\cdot10^{-5}$	$0.00\cdot10^{0}$
0.325	$-6.69\cdot10^{-4}$	$1.77\cdot10^{-5}$	$0.00\cdot10^{0}$	$0.00\cdot10^{0}$	$0.00\cdot10^{0}$
0.350	$-4.94\cdot10^{-3}$	$6.49\cdot10^{-5}$	$0.00\cdot10^{0}$	$0.00\cdot10^{0}$	$0.00\cdot10^{0}$
0.375	$-8.10\cdot10^{-3}$	$1.78\cdot10^{-5}$	$0.00\cdot10^{0}$	$0.00\cdot10^{0}$	$0.00\cdot10^{0}$
0.400	$-9.78\cdot10^{-3}$	$-1.43\cdot10^{-4}$	$-2.04\cdot10^{-5}$	$-2.06\cdot10^{-5}$	$0.00\cdot10^{0}$

The structure displacements are obtained from: $\{U\} = [\Phi]\{\eta\}$

For example, for instant *t=0.2873 s*, displacements in m for each mode and total values are:

$ti := 0.287$ $\quad n_1 := \eta_1(ti)\cdot mt \quad n_2 := \eta_2(ti)\cdot mt \quad n_3 := \eta_3(ti)\cdot mt \quad n_4 := \eta_4(ti)\cdot mt \quad n_5 := \eta_5(ti)\cdot mt$

$$\Phi = \begin{pmatrix} 6.715 & -6.017 & 4.820 & -3.432 & 1.850 \\ 6.101 & -1.455 & -3.860 & 6.393 & -4.962 \\ 4.929 & 4.210 & -5.588 & -2.082 & 6.492 \\ 3.307 & 6.683 & 2.748 & -4.597 & -5.954 \\ 1.382 & 4.089 & 6.135 & 6.047 & 3.519 \end{pmatrix} 10^{-3} \qquad n = \begin{pmatrix} 6.537947 \\ 0.117964 \\ 0.028124 \\ 0.020149 \\ 0.002970 \end{pmatrix} 10^{-3}\cdot mt$$

$$U := \Phi\cdot n = \begin{pmatrix} 4.327 \\ 3.972 \\ 3.255 \\ 2.237 \\ 0.982 \end{pmatrix} 10^{-5}\cdot mt \quad \text{or} \quad U_{sum} := \Phi^{\langle 1\rangle}\cdot n_1 + \Phi^{\langle 2\rangle}\cdot n_2 + \Phi^{\langle 3\rangle}\cdot n_3 + \Phi^{\langle 4\rangle}\cdot n_4 + \Phi^{\langle 5\rangle}\cdot n_5 = \begin{pmatrix} 4.327 \\ 3.972 \\ 3.255 \\ 2.237 \\ 0.982 \end{pmatrix} 10^{-5}\cdot mt$$

To obtain the forces caused by the explosion at the same instant for all the structure, the structure stiffness matrix is multiplied by the displacements obtained: $\{F\} = [K_E]\{U\}$

$$F := K\cdot U = \begin{pmatrix} 101.690 \\ 104.374 \\ 85.861 \\ 68.414 \\ 61.722 \end{pmatrix} 10^{-3}\cdot kN \qquad K = \begin{pmatrix} 28.703 & -28.703 & 0.000 & 0.000 & 0.000 \\ -28.703 & 57.406 & -28.703 & 0.000 & 0.000 \\ 0.000 & -28.703 & 57.406 & -28.703 & 0.000 \\ 0.000 & 0.000 & -28.703 & 57.406 & -28.703 \\ 0.000 & 0.000 & 0.000 & -28.703 & 71.679 \end{pmatrix} 10^{3}\cdot\frac{kN}{mt}$$

This operation can be made for each mode independently in order to obtain displacements of the structure for each mode:

$U_{mod}^{\langle 1\rangle} := \Phi^{\langle 1\rangle}\cdot n_1 \quad U_{mod}^{\langle 2\rangle} := \Phi^{\langle 2\rangle}\cdot n_2 \quad U_{mod}^{\langle 3\rangle} := \Phi^{\langle 3\rangle}\cdot n_3 \quad U_{mod}^{\langle 4\rangle} := \Phi^{\langle 4\rangle}\cdot n_4 \quad U_{mod}^{\langle 5\rangle} := \Phi^{\langle 5\rangle}\cdot n_5$

$$U_{mod} = \begin{pmatrix} 4.3905379 & -0.0709768 & 0.0135543 & -0.0069157 & 0.0005496 \\ 3.9890760 & -0.0171618 & -0.0108563 & 0.0128810 & -0.0014738 \\ 3.2228612 & 0.0496654 & -0.0157152 & -0.0041953 & 0.0019284 \\ 2.1619546 & 0.0788360 & 0.0077282 & -0.0092622 & -0.0017686 \\ 0.9033632 & 0.0482327 & 0.0172535 & 0.0121847 & 0.0010453 \end{pmatrix}\cdot 10^{-5}\cdot mt$$

The contribution to the applied force caused by each mode, in kN, at instant *t=0.2873 s*, is:

$$F_{mod} := K\cdot U_{mod} = \begin{pmatrix} 115.231 & -15.447 & 7.007 & -5.682 & 0.581 \\ 104.695 & -3.735 & -5.612 & 10.584 & -1.557 \\ 84.585 & 10.809 & -8.124 & -3.447 & 2.038 \\ 56.741 & 17.157 & 3.995 & -7.610 & -1.869 \\ 26.979 & 11.945 & 10.149 & 11.392 & 1.257 \end{pmatrix}\cdot 10^{-3}\cdot kN$$

4.5. Example 5: Base Excitation

In this section, we study the response of the building to the recorded accelerations 'NGA1787_HectorMine_Hector_00' at California, in October 16 of 1999 (See Section 3 for details). We are going to use mass and stiffness matrixes calculated before, as well as the resulting matrixes due to the mass normalization.

Base dynamic excitation equilibrium equations have the following form:

$$[M]\{\ddot{U}\}+[K]\{U\}=-[M][\gamma]\{\ddot{x}_o\}$$

The modal participation factors are obtained from:

$$\alpha := \left(\Phi^T \cdot M \cdot r\right) \cdot kg^{-0.500} = \begin{pmatrix} -188.968 \\ -67.437 \\ -42.599 \\ -26.418 \\ -11.954 \end{pmatrix}$$

The total effective mass is computed as $\alpha^2{}_i$

$$\Gamma_j := \frac{\left(\alpha^2\right)_j}{\sum \alpha^2} \qquad \Gamma = \begin{pmatrix} 0.8321 \\ 0.1060 \\ 0.0423 \\ 0.0163 \\ 0.0033 \end{pmatrix} \qquad \Gamma cum = \begin{pmatrix} -1.2690 \\ -1.7197 \\ -1.9810 \\ -2.1499 \\ -2.2275 \end{pmatrix}$$

Then:

Mode	α_i	α_i^2	$\%M_{tot}$	$\%M_{tot}$ accumulated
1	188.968	35708.793	0.8321	0.8321
2	67.437	4547.749	0.1060	0.9381
3	42.599	1814.640	0.0423	0.9804
4	26.418	697.915	0.0163	0.9967
5	11.954	142.904	0.0033	1.0000

Figure 14. Mass Participation Ratios per mode.

Now we modify the dynamic equilibrium equations by pre-multiplying by $[\Phi]^T$ as we have done in the former examples. Then, the uncoupled vibration equations are:

$$\eta''_1 + 2\cdot\xi\cdot\omega_1\cdot\eta'_1 + \omega_1^2\cdot\eta_1 = -188.968x'(t)$$

$$\eta''_2 + 2\cdot\xi\cdot\omega_2\cdot\eta'_2 + \omega_2^2\cdot\eta_2 = -67.437x'(t)$$

$$\eta''_3 + 2\cdot\xi\cdot\omega_3\cdot\eta'_3 + \omega_3^2\cdot\eta_3 = -42.599x'(t)$$

$$\eta''_4 + 2\cdot\xi\cdot\omega_4\cdot\eta'_4 + \omega_4^2\cdot\eta_4 = -26.418x'(t)$$

$$\eta''_5 + 2\cdot\xi\cdot\omega_5\cdot\eta'_5 + \omega_5^2\cdot\eta_5 = -11.954x'(t)$$

$$\omega := \sqrt{\lambda_{value}} = \begin{pmatrix} 17.727 \\ 51.046 \\ 78.672 \\ 99.185 \\ 112.480 \end{pmatrix}$$

Built time and force vector from Excel Spreadsheet of “NGA1787_HectorMine” record.

$$\begin{pmatrix} \mathrm{Time} \\ \mathrm{Acc} \end{pmatrix} :=$$

Worksheet

$$\Delta t := \mathrm{Time}_2 = 0.010$$

$$N_{times} := \frac{\max(\mathrm{Time})}{\Delta t} = 4531$$

$$i := 1 .. N_{times}$$

Set Newmark-Beta method parameters:

$$\gamma := \frac{1}{2} \qquad \beta := \frac{1}{4}$$

$$a_0 := \frac{1}{\beta \cdot \Delta t^2} \qquad a_1 := \frac{\gamma}{\beta \cdot \Delta t} \qquad a_2 := \frac{1}{\beta \cdot \Delta t} \qquad a_3 := \frac{1}{2\beta} - 1$$

$$a_4 := \frac{\gamma}{\beta} - 1 \qquad a_5 := \Delta t \cdot \left(\frac{\gamma}{2\beta} - 1\right) \qquad a_6 := \Delta t \cdot (1 - \gamma) \qquad a_7 := \Delta t \cdot \gamma$$

For mode 1:

$$m := 1 \qquad c := 2 \cdot \xi \cdot \omega_1 \qquad k := \omega_1^{\,2} \qquad \mathrm{Force} := -\alpha_1 \cdot \mathrm{gravity} \cdot \mathrm{Acc}$$

$$k_{eff} := k + a_0 \cdot m + a_1 \cdot c = 40668.778$$

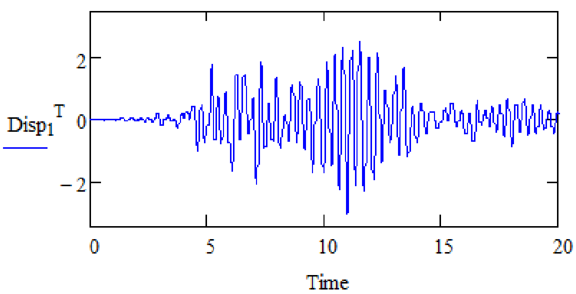
$$\eta_0 := 0 \qquad \eta'_0 := 0 \qquad \eta''_0 := -m^{-1} \cdot (k \cdot \eta_0 + c \cdot \eta'_0 - 0) = 0.000$$

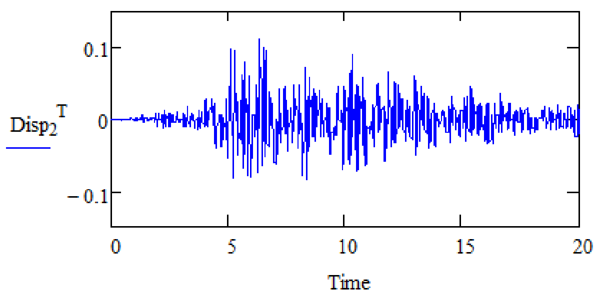
Step by step numerical integration:

$$\begin{aligned}
\mathrm{Disp}_1 := \; & \eta^{\langle 1 \rangle} \leftarrow \eta_0 \\
& \eta'^{\langle 1 \rangle} \leftarrow \eta'_0 \\
& \eta''^{\langle 1 \rangle} \leftarrow \eta''_0 \\
& \text{for } i \in 2 .. N_{times} + 1 \\
& \quad \mathrm{Feff} \leftarrow \mathrm{Force}_i + m \cdot \left(a_0 \cdot \eta^{\langle i-1 \rangle} + a_2 \cdot \eta'^{\langle i-1 \rangle} + a_3 \cdot \eta''^{\langle i-1 \rangle}\right) + c \cdot \left(a_1 \cdot \eta^{\langle i-1 \rangle} + a_4 \cdot \eta'^{\langle i-1 \rangle} + a_5 \cdot \eta''^{\langle i-1 \rangle}\right) \\
& \quad \eta^{\langle i \rangle} \leftarrow k_{eff}^{-1} \cdot \mathrm{Feff} \\
& \quad \eta''^{\langle i \rangle} \leftarrow a_0 \cdot \left(\eta^{\langle i \rangle} - \eta^{\langle i-1 \rangle}\right) - a_2 \cdot \eta'^{\langle i-1 \rangle} - a_3 \cdot \eta''^{\langle i-1 \rangle} \\
& \quad \eta'^{\langle i \rangle} \leftarrow \eta'^{\langle i-1 \rangle} + a_6 \cdot \eta''^{\langle i-1 \rangle} + a_7 \cdot \eta''^{\langle i \rangle} \\
& \text{return } \eta
\end{aligned}$$

Figure 15. Newmark-Beta Method – Mathcad Algorithm for example 5.

In the five equations shown before, ξ = 0.05. The response of each of the uncoupled equations was obtained employing the Newmark-Beta Method (Figure 15). Then, the first 20 seconds of response are shown in the following graphs.





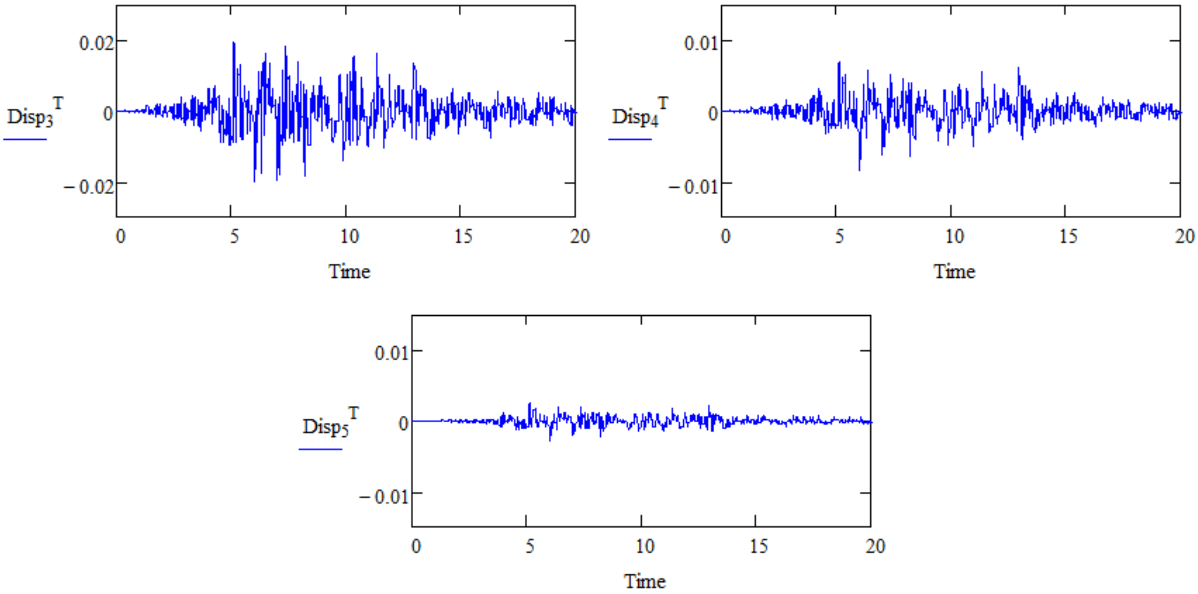


Figure 16. Response of the uncoupled coordinates.

The following table contains the response at selected instants, and the extreme values obtained for each uncoupled degree of freedom during the first 20 s of response.

Table 4. Maximum and Minimum Displacement Response of the uncouple coordinates

t (s)	n1 (m)	n2 (m)	n3 (m)	n4 (m)	n5 (m)
5.09	-0.1357	-0.0338	-0.0193	-0.0068	-0.0023
5.10	-0.2800	-0.0545	-0.0189	-0.0069	-0.0024
5.50	-0.5484	-0.0190	0.0079	0.0020	0.0009
5.99	1.1679	0.0695	0.0200	0.0079	0.0027
6.00	1.2599	0.0628	0.0192	0.0084	0.0025
6.37	0.5746	-0.1127	-0.0085	-0.0057	-0.0019
6.50	-0.6482	-0.0977	-0.0107	-0.0021	-0.0008
7.00	-0.0166	0.0544	0.0195	0.0061	0.0021
7.50	0.6831	-0.0034	-0.0035	-0.0011	-0.0005
8.00	-0.8049	0.0309	0.0034	0.0012	0.0003
8.34	-0.1372	0.0860	0.0021	0.0028	0.0001
8.50	0.6018	-0.0386	-0.0033	-0.0012	0.0003
9.00	-1.1298	0.0138	-0.0031	-0.0001	0.0001
9.50	1.7025	0.0263	0.0032	0.0019	0.0004
10.00	1.3255	-0.0361	0.0028	-0.0001	-0.0001
10.99	3.1146	0.0116	0.0053	0.0021	0.0008
11.51	-2.5363	-0.0323	-0.0007	-0.0013	-0.0005
max	3.1146	0.0860	0.0200	0.0084	0.0027
min	-2.5363	-0.1127	-0.0193	-0.0069	-0.0024
t(max)	10.99 s	8.34 s	5.99 s	6.00 s	5.99 s
t(min)	11.51 s	6.37 s	5.09 s	5.10 s	5.10 s

The structure displacements are obtained from: $\{U\} = [\Phi]\{\eta\}$

$$\Phi = \begin{pmatrix} 0.00672 & -0.00602 & 0.00482 & -0.00343 & 0.00185 \\ 0.00610 & -0.00145 & -0.00386 & 0.00639 & -0.00496 \\ 0.00493 & 0.00421 & -0.00559 & -0.00208 & 0.00649 \\ 0.00331 & 0.00668 & 0.00275 & -0.00460 & -0.00595 \\ 0.00138 & 0.00409 & 0.00613 & 0.00605 & 0.00352 \end{pmatrix}$$

$t := 1..N_{times}$

$mod1_t := \Phi^{\langle 1\rangle}\cdot\left(Disp_1{}^T\right)_t \qquad mod2_t := \Phi^{\langle 2\rangle}\cdot\left(Disp_2{}^T\right)_t \qquad mod3_t := \Phi^{\langle 3\rangle}\cdot\left(Disp_3{}^T\right)_t$

$mod4_t := \Phi^{\langle 4\rangle}\cdot\left(Disp_4{}^T\right)_t \qquad mod5_t := \Phi^{\langle 5\rangle}\cdot\left(Disp_5{}^T\right)_t$

$U := (mod1 + mod2 + mod3 + mod4 + mod5)\cdot m$

For example, for instant **t = 3.08 s (step=308)**, displacements in meters contributed by each mode are:

$$U_{step} = \begin{pmatrix} -0.00135 \\ -0.00121 \\ -0.00096 \\ -0.00064 \\ -0.00028 \end{pmatrix} m$$

Also, forces are calculated as follows: $F_t := K\cdot U_t$

This operation can be made for each mode independently, thus obtaining the contribution of the total internal forces caused by each one:

$\left(F_{mod1}\right)_t := K\cdot mod1_t\cdot m \qquad \left(F_{mod2}\right)_t := K\cdot mod2_t\cdot m \qquad \left(F_{mod3}\right)_t := K\cdot mod3_t\cdot m$

$\left(F_{mod4}\right)_t := K\cdot mod4_t\cdot m \qquad \left(F_{mod5}\right)_t := K\cdot mod5_t\cdot m$

Then, the force contribution in kN for each mode at instant **t = 3.08 s**, is:

$$Fmod_{308} = \begin{pmatrix} -3.499 & -0.304 & -0.494 & 0.197 & -0.059 \\ -3.179 & -0.074 & 0.396 & -0.366 & 0.158 \\ -2.568 & 0.213 & 0.573 & 0.119 & -0.207 \\ -1.723 & 0.338 & -0.282 & 0.263 & 0.189 \\ -0.819 & 0.235 & -0.716 & -0.394 & -0.127 \end{pmatrix}\cdot kN$$

Total forces in kN for instant t = 3.08 s, are:

$$F_{308} = \begin{pmatrix} -4.159 \\ -3.064 \\ -1.869 \\ -1.214 \\ -1.822 \end{pmatrix}\cdot kN$$

Base shear contributed by each mode, also in kN, at instant **t = 3.08 s**, is obtained from:

$$\{V\} = \{1\}^T \{F^{mod}\}$$

$one_{1,j} := 1 \qquad V_t := one \cdot F_t$

$V_{308} := one \cdot Fmod_{308} = (-11.787 \;\; 0.408 \;\; -0.523 \;\; -0.181 \;\; -0.046) \cdot kN$

The total base shear in kN at instant **t = 3.08 s**, is obtained as:

$V_{308} = -12.129 kN$

Likewise, the overturning moment contributed by each mode, is obtained from:

$$\{M\} = \{h\}^T \{F^{mod}\}$$

$h := (16 \;\; 13 \;\; 10 \;\; 7 \;\; 4) \cdot mt \qquad Moment_t := h \cdot F_t$

$Moment_{308} := h \cdot Fmod_{308} = (-138.3 \;\; -0.4 \;\; -1.9 \;\; -0.2 \;\; -0.1) \cdot kN \cdot mt$

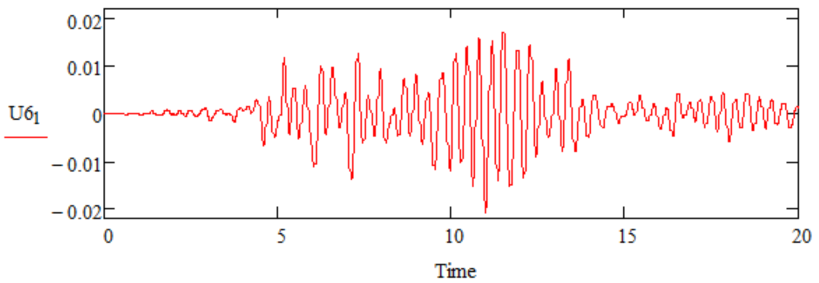
The total overturning moment in kN · m at instant **t = 3.08 s**, is obtained from:

$Moment_{308} = -140.867 kN \cdot mt$

The same procedures can be used to obtain the response at any instant. If this is performed systematically, results such as shown in Figure 17 are obtained. There the displacement response for the roof of the building is shown for the first 20 sec. of the EW component of Hector-Mine record. From this figure, it is evident that the significant portion of the response is contributed solely by the first two modes, with thee second contributing marginally.

Roof Displacements

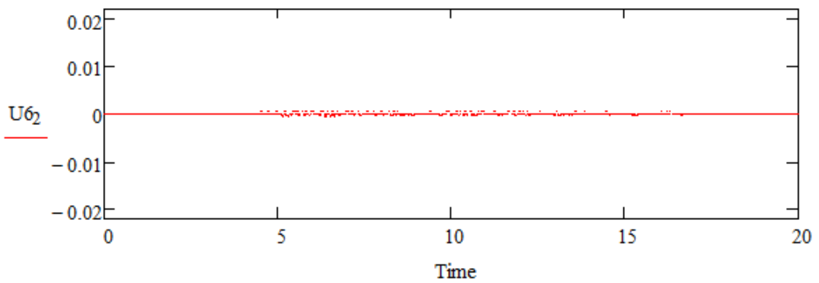
Mode 1



$max(U6_1) = 0.01703 m$

$min(U6_1) = -0.02092 m$

Mode 2



$max(U6_2) = 0.00052 m$

$min(U6_2) = -0.00068 m$

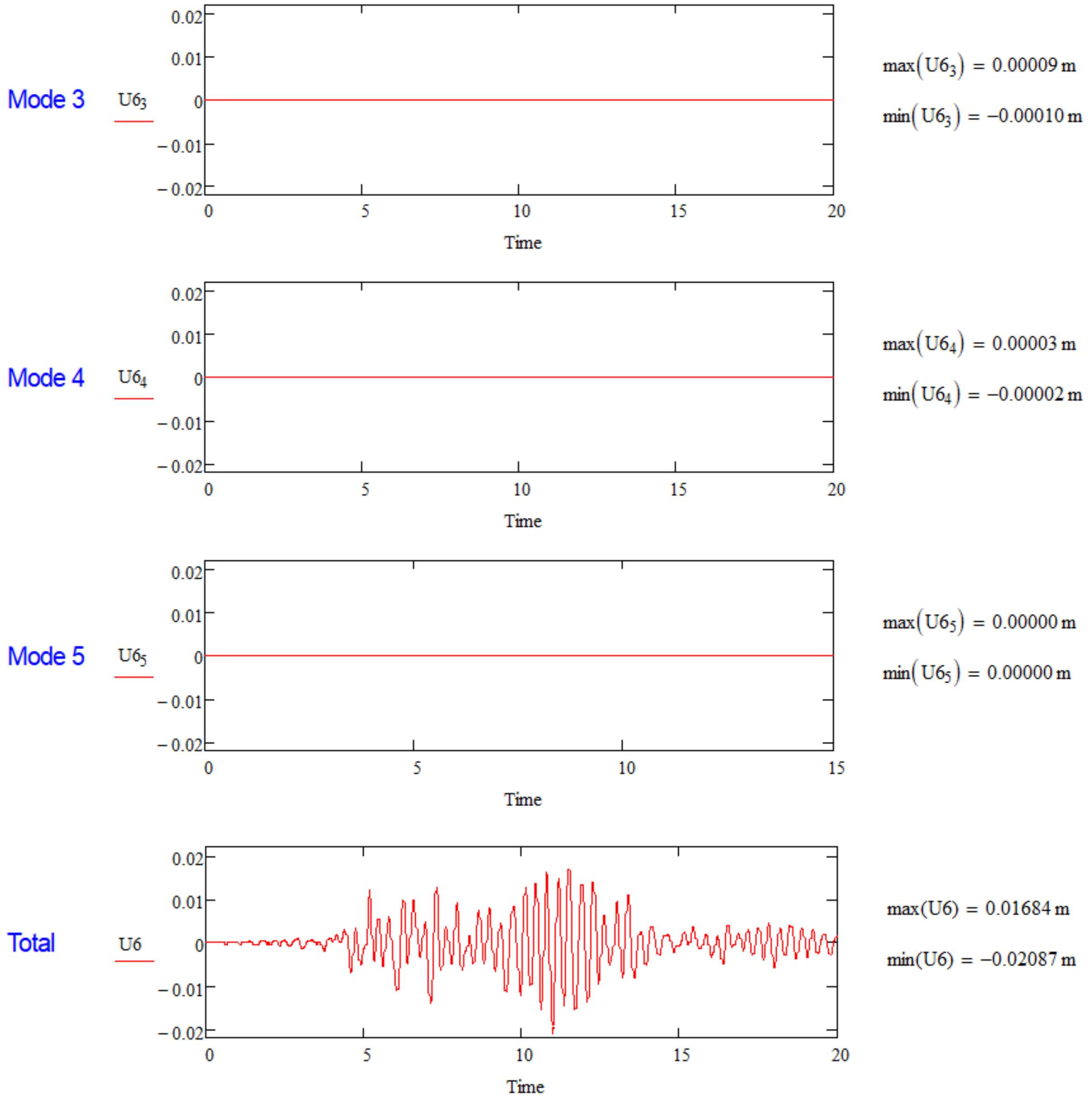


Figure 17. Roof displacements from each mode and total response.

Figure 18 shows the variation of the base shear of the building during the first 20 sec. of response to the EW component of Hector-Mine record.

Base Shear

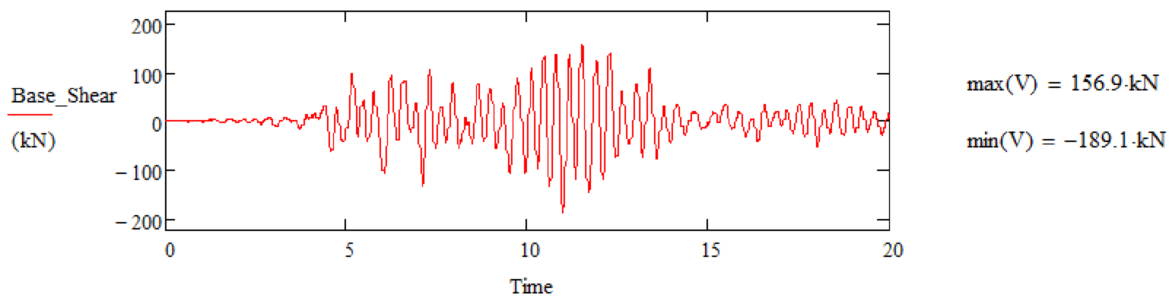
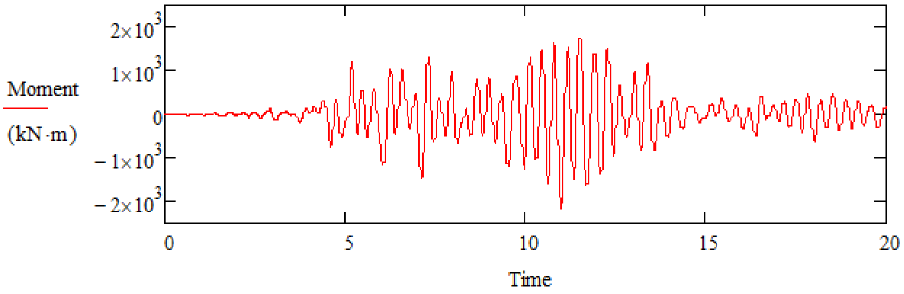


Figure 18. Base shear of the structure.

Figure 19 shows the variation of overturning moment for the first 20 sec. of response to the EW component of El Centro record.

Overturning Moment



$\max(M) = 1763.0 \cdot kN \cdot mt$

$\min(M) = -2174.2 \cdot kN \cdot mt$

Figure 19. Overturning moment of the structure

4.6. Example 6: Modal Spectral Analysis

In this section, we rework Example 5 using the displacement response spectra of the Hector-Mine record. The results are the same up to the point where the dynamic equilibrium equations were uncoupled.

$$\eta''_1 + 2\cdot\xi\cdot\omega_1\cdot\eta'_1 + \omega_1^{\,2}\cdot\eta_1 = -188.968x'(t)$$

$$\eta''_2 + 2\cdot\xi\cdot\omega_2\cdot\eta'_2 + \omega_2^{\,2}\cdot\eta_2 = -67.437x'(t)$$

$$\eta''_3 + 2\cdot\xi\cdot\omega_3\cdot\eta'_3 + \omega_3^{\,2}\cdot\eta_3 = -42.599x'(t)$$

$$\eta''_4 + 2\cdot\xi\cdot\omega_4\cdot\eta'_4 + \omega_4^{\,2}\cdot\eta_4 = -26.418x'(t)$$

$$\eta''_5 + 2\cdot\xi\cdot\omega_5\cdot\eta'_5 + \omega_5^{\,2}\cdot\eta_5 = -11.954x'(t)$$

$$\omega := \sqrt{\lambda_{value}} = \begin{pmatrix} 17.727 \\ 51.046 \\ 78.672 \\ 99.185 \\ 112.480 \end{pmatrix}$$

The response for each of the uncoupled equations is obtained using the displacement response spectra for the EW component of the Hector-Mine record. Figure 20 shows the Displacement Response Spectrum for the record of interest obtained using SeismoSignal software. Table 5 shows the period for each mode and the displacement read from the spectrum for each period.

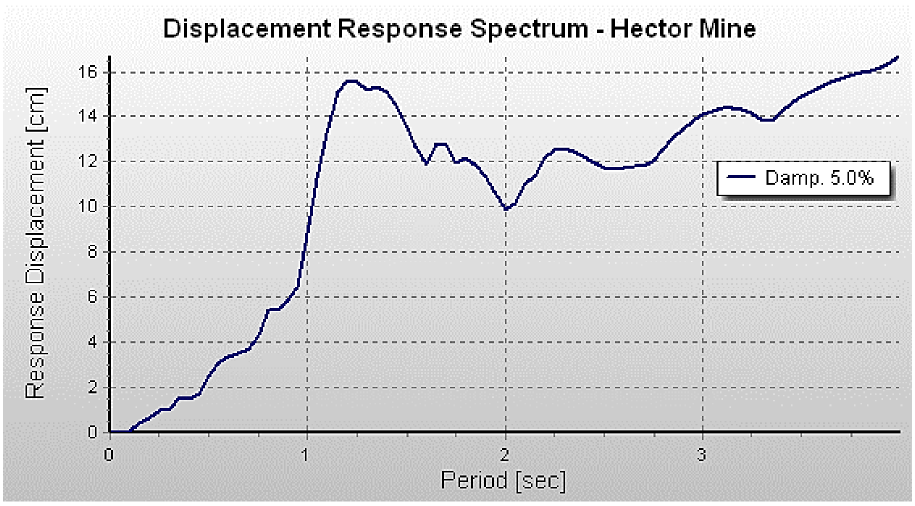


Figure 20. Displacement response spectrum for Hector-Mine NS record.

Table 5. Values read from the Displacement Spectrum.

Mode	Period T (s)	Sd (Ti, ξi) (m)
1	0.359	0.015850
2	0.114	0.001684
3	0.064	0.000646
4	0.043	0.000383
5	0.033	0.000274

With this information, it is possible to compute the maximum displacement that the uncoupled degrees of freedom can attain:

Table 6. Maximum displacement values for the uncoupled degrees of freedom.

Mode	α_i	Sd (Ti, ξi) (m)	$(\eta_i)_{max} = \alpha_i$ xSd (Ti, ξi) (m)
1	188.968	0.015850	2.9951
2	67.437	0.001684	0.1136
3	42.599	0.000646	0.0275
4	26.418	0.000383	0.0101
5	11.954	0.000274	0.0033

Maximum modal displacements (m)

The maximum displacements for each mode are obtained from:

$$[\mathbf{U_{mod}}] = [\mathbf{\Phi}][\mathbf{H_{mod}}] = \left[\left\{ \mathbf{U^{(1)}_{mod}} \right\} \middle| \left\{ \mathbf{U^{(2)}_{mod}} \right\} \middle| \cdots \middle| \left\{ \mathbf{U^{(n)}_{mod}} \right\} \right]$$

$$H_{mod_{j,j}} := \eta_{max_j} \qquad H_{mod} = \begin{pmatrix} 2.995 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.114 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.028 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.010 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.003 \end{pmatrix} \mathrm{m}$$

Then, the values for [Umod] are:

$$U_{mod} := \Phi \cdot H_{mod} = \begin{pmatrix} 0.020114 & -0.000683 & 0.000133 & -0.000035 & 0.000000 \\ 0.018275 & -0.000165 & -0.000106 & 0.000065 & -0.000016 \\ 0.014764 & 0.000478 & -0.000154 & -0.000021 & 0.000021 \\ 0.009904 & 0.000759 & 0.000076 & -0.000047 & -0.000020 \\ 0.004138 & 0.000464 & 0.000169 & 0.000061 & 0.000012 \end{pmatrix} \mathrm{m}$$

Figure 21 shows the maximum lateral displacements for each mode.

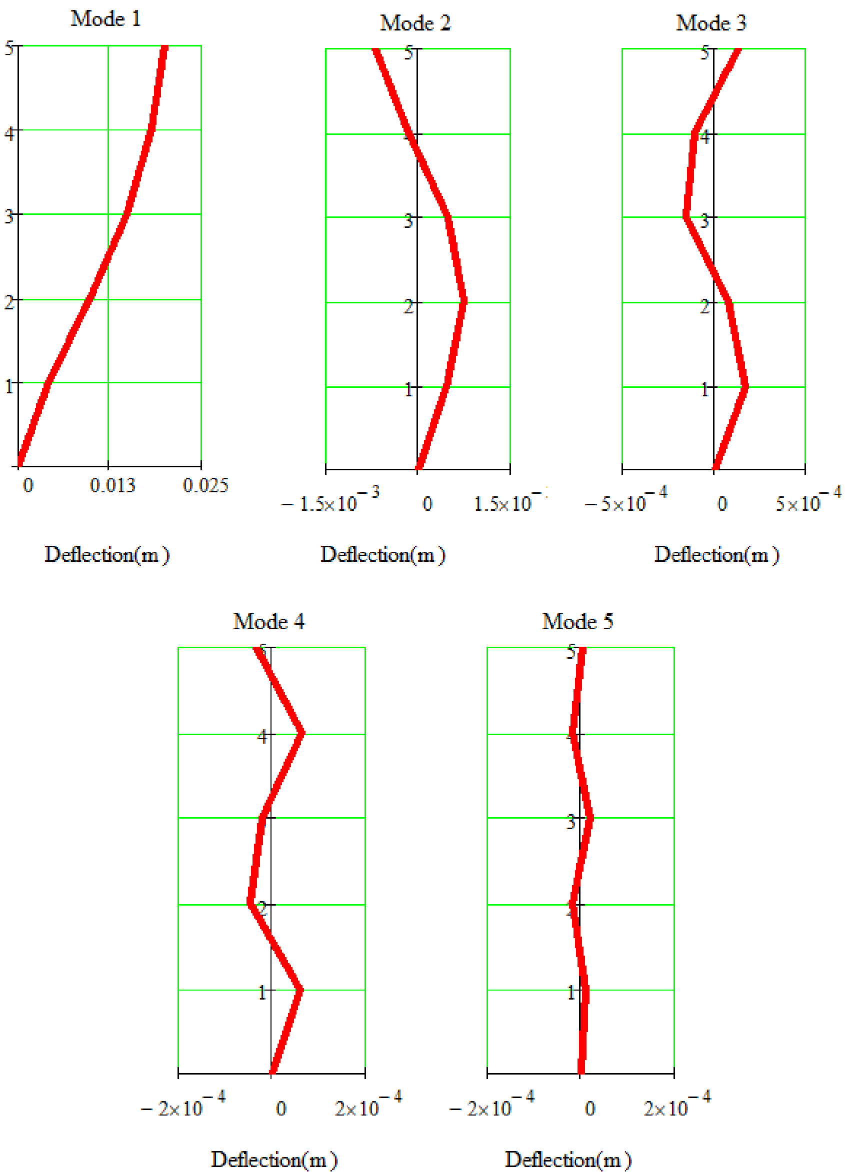


Figure 21. Maximum lateral displacements for each mode.

<u>Maximum story drift as a percentage of story height (%h)</u>

Using the displacements just computed the story drift for each story and mode could be computed as the algebraic difference of the displacement of two consecutive stories. Drift is usually expressed as percentage of the inter-story height. Table 7 and Figure 22 show the story drifts for each mode.

Table 7. Maximum displacement values for the uncoupled degrees of freedom.

Story	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
5	0.0613	-0.0173	0.0080	-0.0033	0.0007
4	0.1170	-0.0214	0.0016	0.0029	-0.0013
3	0.1620	-0.0094	-0.0076	0.0008	0.0014
2	0.1922	0.0098	-0.0031	-0.0036	-0.0010
1	0.1379	0.0155	0.0056	0.0020	0.0004

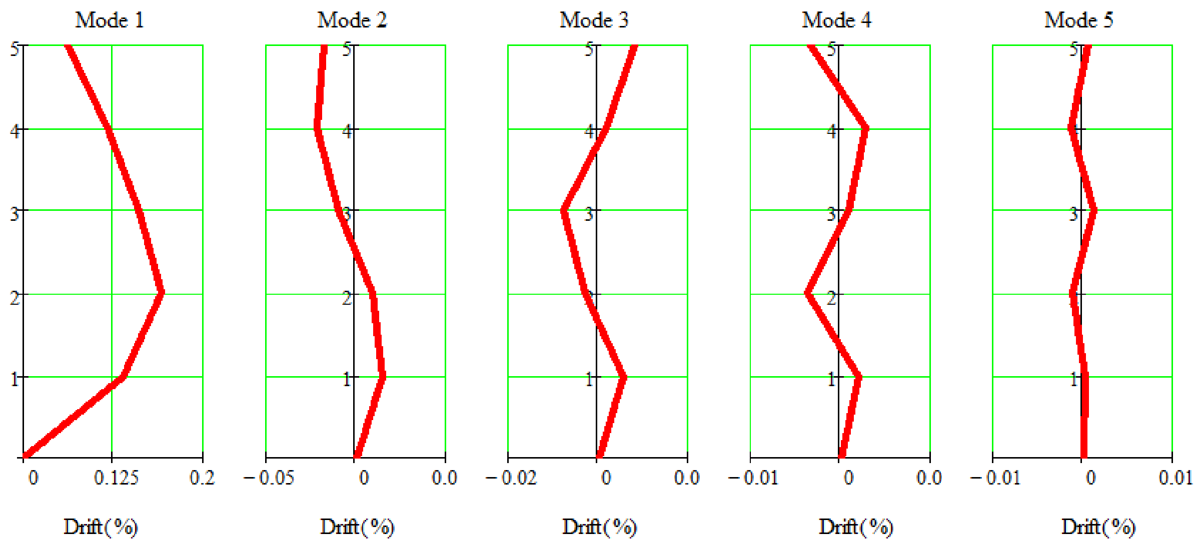


Figure 22. Maximum story drift (%h) for each mode.

<u>Maximum modal lateral forces (kN)</u>

To obtain the maximum modal lateral forces imposed on the structure by the ground motions the stiffness matrix of the structure is multiplied by the modal lateral displacements. Results are obtained in kN.

$$[\mathbf{F}_{\text{mod}}] = [\mathbf{K}_E]\left[\mathbf{U}^{(1)}_{\text{mod}} \mid \mathbf{U}^{(2)}_{\text{mod}} \mid \cdots \mid \mathbf{U}^{(6)}_{\text{mod}}\right] = \left[\mathbf{F}^{(1)}_{\text{mod}} \mid \mathbf{F}^{(2)}_{\text{mod}} \mid \cdots \mid \mathbf{F}^{(6)}_{\text{mod}}\right]$$

$$\text{Fmod} := \text{K}\cdot\text{U}_{\text{mod}} = \begin{pmatrix} 52.8 & -14.9 & 6.9 & -2.9 & 0.6 \\ 48.0 & -3.6 & -5.5 & 5.3 & -1.7 \\ 38.7 & 10.4 & -7.9 & -1.7 & 2.2 \\ 26.0 & 16.5 & 3.9 & -3.8 & -2.1 \\ 12.4 & 11.5 & 9.9 & 5.7 & 1.4 \end{pmatrix}\cdot\text{kN}$$

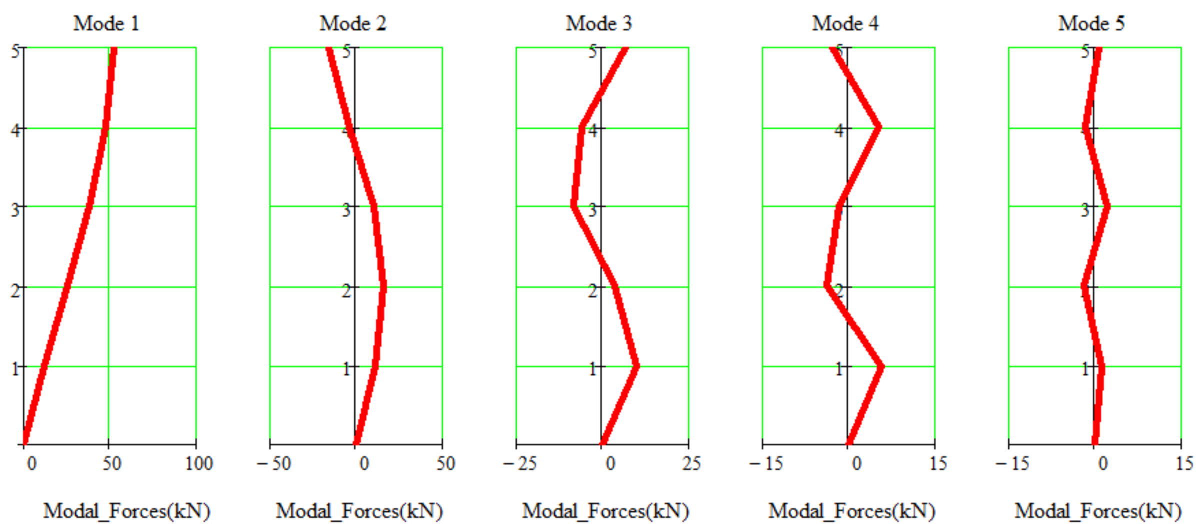


Figure 23. Maximum modal forces for each mode (kN).

Maximum modal story shear (kN)

The maximum modal story shear is obtained from: $$V_j^{(i)} = \sum_{k=j}^{n} F_k^{(i)}$$

Table 8. Maximum modal values for story shear.

Story	V^1_{mod} (kN)	V^2_{mod} (kN)	V^3_{mod} (kN)	V^4_{mod} (kN)	V^5_{mod} (kN)
5	52.789	-14.870	6.856	-2.853	0.640
4	100.752	-18.466	1.365	2.461	-1.077
3	139.502	-8.061	-6.584	0.730	1.170
2	165.496	8.456	-2.675	-3.091	-0.891
1	177.855	19.955	7.255	2.630	0.495
0	177.855	19.955	7.255	2.630	0.495

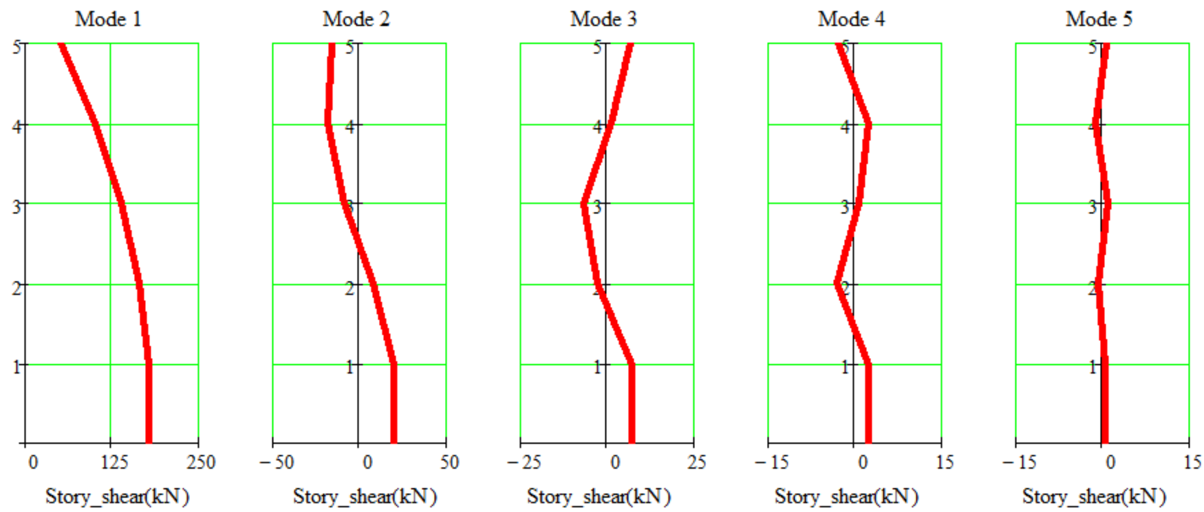


Figure 24. Maximum story shear for each mode (kN).

Base shear (kN)

The base shear in kN for each mode is obtained from: $\{V_{mod}\} = \{1\}^T [F_{mod}]$

$one_{1,j} := 1$

$Vmod := one \cdot Fmod = (177.9\ \ 20.0\ \ 7.3\ \ 2.6\ \ 0.5) \cdot kN$

It is the same value obtained for the first story when the story shears were computed.

Overturning moment (kN · m)

The overturning moment for each story is obtained from: $$M_j^{(i)} = \sum_{k=j+1}^{n} (h_k - h_j) \cdot F_j^{(i)}$$

Table 9. Maximum story modal overturning moment.

Story	M^1_{mod} (kN·m)	M^2_{mod} (kN·m)	M^3_{mod} (kN·m)	M^4_{mod} (kN·m)	M^5_{mod} (kN·m)
5	0.0	0.0	0.0	0.0	0.0
4	316.7	-89.2	41.1	-17.1	3.8
3	762.9	-155.4	28.8	6.2	-4.5
2	1297.6	-148.4	-14.8	3.2	5.7
1	1872.1	-73.5	-11.1	-17.5	-3.1
0	2442.7	20.9	40.4	7.5	2.5

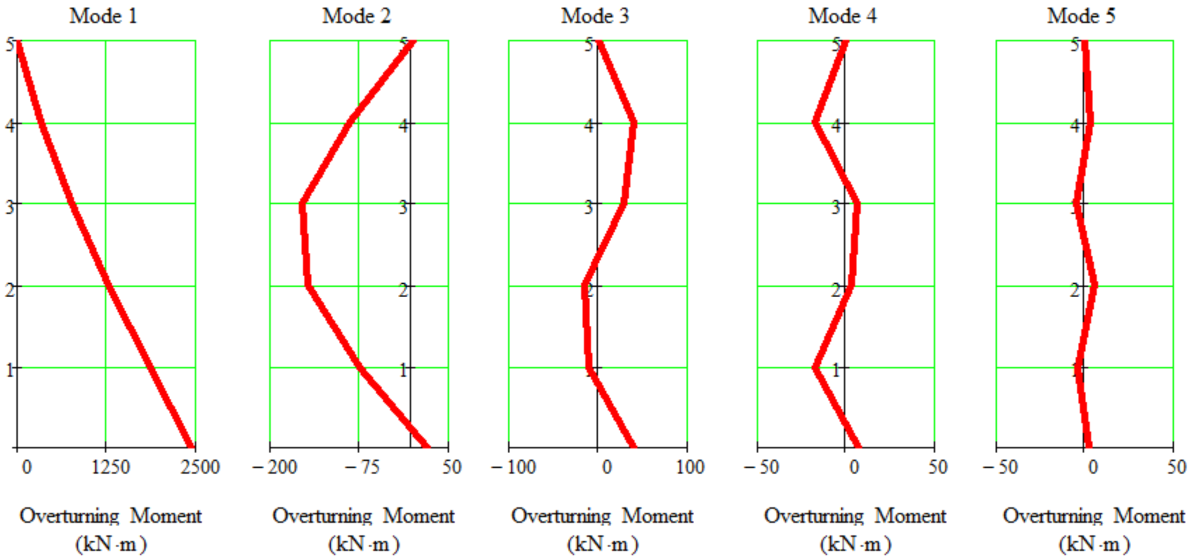


Figure 25. Overturning moment for each mode (kN·m).

The maximum overturning moment at the base, in kN · m, contributed by each mode can be obtained from:

$$h := (16\ \ 13\ \ 10\ \ 7\ \ 4)\cdot m$$

$$M\_base_{mod} := h\cdot Fmod = (2087.0\ \ -19.0\ \ 25.9\ \ 2.3\ \ 1.5)\cdot kN\cdot m$$

This is the same result obtained for the overturning moment previously.

In Example 5 the step-by-step response of the building was obtained for the same earthquake record used to compute the spectrum in this example, it is interesting to make some comparisons of the results obtained in both cases. Table 10 lists the values obtained in Example 5 and Example 6 for each of the uncoupled degrees of freedom.

Table 10. Comparison of values obtained in Examples 5 and 6.

Uncoupled degree of freedom	Example 5			Example 6
		η_i (m)	t (s)	$(\eta_i)_{max} = \alpha_i \, xSd\,(Ti, \xi i)$ (m)
$\eta 1$	max	2.53626	10.99	2.99514
	min	-3.11463	11.51	
$\eta 2$	max	0.11266	8.34	0.11356
	min	-0.08605	6.37	
$\eta 3$	max	0.01932	5.99	0.02752
	min	-0.01998	5.09	
$\eta 4$	max	0.00688	6.00	0.01012
	min	-0.00841	5.10	
$\eta 5$	max	0.00239	5.99	0.00328
	min	-0.00266	5.10	

In Table 10, we observe that the results are essentially the same, and the differences obey to precision rounding in the numerical procedures because the algorithm employed to obtain the response is different from the one used to compute the spectrum.

It is important to notice that the maximum values for each uncoupled degree of freedom in Example 5 were obtained at different time instants. Also, note that the maximum value obtained from the spectrum in some cases correspond to the maximum value and in some to the minimum obtained in the step-by-step procedure (Example 5), this is because the value carried by the spectrum is the absolute value.

The maximum lateral displacement of the roof obtained in Example 5 was 0.017 m. The algebraic sum of the values obtained for the MDOF system in Example 6 is 0.020 m, and the sum of the absolute values is 0.021 m. The algebraic sum of the modal response usually underestimates the value obtained using a time step-by-step procedure and the sum of the absolute modal values overestimate it. In this case, we observe that both values are overestimating the displacements, this is probably because the displacement response spectrum is very sensitive to small periods, like for this example.

The maximum value for the base shear of the building obtained in Example 5 using a time step-by-step procedure was 157 kN. The sum of the maximum modal base shears obtained in Example 6 was 208 kN. This value overestimates the time step value by a factor of 1.3. In the time step procedure of Example 5 the base shear is controlled by the first mode with the other modes contributing very little when the first mode peak occurs. For the overturning moment at the base in Example 5 a value of 1,763 kN · m was obtained In Example 6 the algebraic sum of the maximum modal values is 2,098 kN · m, and the sum of the absolute values is 2,136 kN · m. For the overturning moment, the contribution of the higher modes is small in both examples.

4.7. Example 7: Modal Combination (SRSS)

In this section, we apply the square root of the sum of the squares SRSS procedure to the results obtained in Example 6. The use of the SRSS technique produces the following results:

Maximum credible lateral displacements (m)

Maximum credible lateral displacements (m) $\{U^{(i)}_{mod}\} = \{\phi^{(i)}\}(\eta_i)_{max}$

$$U_{mod} = \begin{pmatrix} 0.020114 & -0.000683 & 0.000133 & -0.000035 & 0.000000 \\ 0.018275 & -0.000165 & -0.000106 & 0.000065 & -0.000016 \\ 0.014764 & 0.000478 & -0.000154 & -0.000021 & 0.000021 \\ 0.009904 & 0.000759 & 0.000076 & -0.000047 & -0.000020 \\ 0.004138 & 0.000464 & 0.000169 & 0.000061 & 0.000012 \end{pmatrix} m$$

We now apply the SRSS procedure to each of the row of previous matrix.

```
U_SRSS := | for i ∈ 1..N
          |   | U_SRSS_i ← 0
          |   | for j ∈ 1..N
          |   |   U_SRSS_i ← U_SRSS_i + (U_mod_i,j)^2
          | return U_SRSS^0.5
```

$$U_{SRSS} = \begin{pmatrix} 0.02013 \\ 0.01828 \\ 0.01477 \\ 0.00993 \\ 0.00417 \end{pmatrix} m$$

This value compares fairly well with the values obtained from the step-by-step procedure in Example 6.

$\max(U6) = 0.01684\,m$

Maximum credible story drift

The modal spectral story drifts are computed from the values shown in [U_{mod}]. The following result are obtained:

```
Drift := | for i ∈ 1..N − 1
         |   for j ∈ 1..N
         |     Drift_i,j ← U_mod_i,j − U_mod_i+1,j
         | for k ∈ 1..N
         |   Drift_N,k ← U_mod_N,k
         | return Drift
```

$$Drift = \begin{pmatrix} 0.0018 & -0.0005 & 0.0002 & -0.0001 & 0.0000 \\ 0.0035 & -0.0006 & 0.0000 & 0.0001 & -0.0000 \\ 0.0049 & -0.0003 & -0.0002 & 0.0000 & 0.0000 \\ 0.0058 & 0.0003 & -0.0001 & -0.0001 & -0.0000 \\ 0.0041 & 0.0005 & 0.0002 & 0.0001 & 0.0000 \end{pmatrix} \cdot mt$$

```
Drift_SRSS := | for i ∈ 1..N
              |   | Drift_SRSS_i ← 0
              |   | for j ∈ 1..N
              |   |   Drift_SRSS_i ← Drift_SRSS_i + (Drift_i,j)^2
              | return Drift_SRSS^0.5
```

$$\Delta_{SRSS} := \frac{Drift_{SRSS}}{H} = \begin{pmatrix} 0.06 \\ 0.12 \\ 0.16 \\ 0.19 \\ 0.14 \end{pmatrix} \cdot \%$$

Now, for the sake of discussion, lets compute erroneously the story drift from lateral displacements already combined, $\{U_{SRRS}\}$. The following are the results for story drift as a percentage of the story height (%h) thus computed:

$$\text{Wrong\_}\Delta_{SRSS} := \left|\begin{array}{l} \text{for } i \in 1..N-1 \\ \quad Drift_{SRSS_i} \leftarrow \dfrac{U_{SRSS_i} - U_{SRSS_{i+1}}}{H} \\ Drift_{SRSS_N} \leftarrow \dfrac{U_{SRSS_N}}{H} \\ \text{return } Drift_{SRSS} \end{array}\right.$$

$$\text{Wrong\_}\Delta_{SRSS} = \begin{pmatrix} 0.06 \\ 0.12 \\ 0.16 \\ 0.19 \\ 0.14 \end{pmatrix} \cdot \%$$

Maximum credible story forces (kN)

The maximum modal spectral forces were obtained for each mode in Example 6 multiplying the stiffness matrix by the modal spectral displacements of each mode, obtaining there the following forces in kN:

$$Fmod = \begin{pmatrix} 52.789 & -14.870 & 6.856 & -2.853 & 0.640 \\ 47.962 & -3.596 & -5.491 & 5.315 & -1.717 \\ 38.750 & 10.405 & -7.949 & -1.731 & 2.247 \\ 25.994 & 16.517 & 3.909 & -3.822 & -2.061 \\ 12.360 & 11.499 & 9.931 & 5.721 & 1.386 \end{pmatrix} \cdot kN$$

A sensible recommendation is to keep these modal forces separated by mode and never combine them using SRSS. This way the danger of using the combined forces in the computation of story shears and overturning moments is avoided.

Maximum credible story shear (kN)

Story shear modal spectral values: $\mathbf{V}_j^{(i)} = \sum_{k=j}^{p} \mathbf{F}_k^{(i)}$

$$VShear = \begin{pmatrix} 52.789 & -14.870 & 6.856 & -2.853 & 0.640 \\ 100.752 & -18.466 & 1.365 & 2.461 & -1.077 \\ 139.502 & -8.061 & -6.584 & 0.730 & 1.170 \\ 165.496 & 8.456 & -2.675 & -3.091 & -0.891 \\ 177.855 & 19.955 & 7.255 & 2.630 & 0.495 \\ 177.855 & 19.955 & 7.255 & 2.630 & 0.495 \end{pmatrix} \cdot kN$$

Applying the SRSS procedure we obtain:

$$VShear_{SRSS} := \left|\begin{array}{l} \text{for } i \in 1..N \\ \quad \left|\begin{array}{l} VShear_{SRSS_i} \leftarrow 0 \\ \text{for } j \in 1..N \\ \quad VShear_{SRSS_i} \leftarrow VShear_{SRSS_i} + \left(VShear_{i,j}\right)^2 \end{array}\right. \\ \text{return } VShear_{SRSS}^{\,0.5} \end{array}\right.$$

$$VShear_{SRSS} = \begin{pmatrix} 55.348 \\ 102.474 \\ 139.896 \\ 165.765 \\ 179.138 \end{pmatrix} \cdot kN$$

Maximum credible base shear

$$\{V_{mod}\} = \{1\}^T [F_{mod}]$$

$$Vmod = (177.855\ \ 19.955\ \ 7.255\ \ 2.630\ \ 0.495)\cdot kN$$

$$V_{SRSS} := \left[\sum_{i=1}^{N} \left(Vmod^{\langle i\rangle}\right)^2\right]^{0.5} = (179.138)\cdot kN$$

Maximum credible overturning moment

Modal story overturning moments: $M_j^{(i)} = \sum_{k=j+1}^{n} \left[(h_k - h_j)\cdot F_j^{(i)}\right]$

$$Mmod = \begin{pmatrix} 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 316.736 & -89.222 & 41.135 & -17.121 & 3.843 \\ 762.879 & -155.407 & 28.755 & 6.208 & -4.540 \\ 1297.634 & -148.372 & -14.844 & 3.206 & 5.712 \\ 1872.104 & -73.452 & -11.143 & -17.533 & -3.143 \\ 2442.749 & 20.912 & 40.416 & 7.519 & 2.502 \end{pmatrix}\cdot kN\cdot mt$$

Applying the SRSS procedure we obtain:

```
M_SRSS := | for i ∈ 1.. N + 1
          |   | M_SRSS_i ← 0
          |   | for j ∈ 1.. N
          |   |   M_SRSS_i ← M_SRSS_i + (Mmod_i,j)^2
          | return M_SRSS^0.5
```

$$M_{SRSS} = \begin{pmatrix} 0.0 \\ 332.1 \\ 779.1 \\ 1306.2 \\ 1873.7 \\ 2443.2 \end{pmatrix} m\cdot kN$$

Maximum credible base overturning moment

$$\{M_{mod}\} = \{h\}^T [F_{mod}]$$

$$M\_base_{mod} = (2087.0\ \ -19.0\ \ 25.9\ \ 2.3\ \ 1.5)\cdot kN\cdot mt$$

$$M\_base_{SRSS} := \left[\sum_{i=1}^{N} \left(M\_base_{mod}^{\langle i\rangle}\right)^2\right]^{0.5} = (2087.3)\ m\cdot kN$$

Static equivalent lateral forces

These forces, in kN, are computed using the story shears obtained by using the SRSS procedure:

$$F_j^E = \begin{cases} V_j^{max} & \text{for } j = p \\ V_j^{max} - V_{j+1}^{max} & \text{for } j \neq p \end{cases}$$

$$VShear_{SRSS} = \begin{pmatrix} 55.348 \\ 102.474 \\ 139.896 \\ 165.765 \\ 179.138 \end{pmatrix}\cdot kN$$

$$F_{Static} := \left| \begin{array}{l} F_{Static_1} \leftarrow VShear_{SRSS_1} \\ \text{for } i \in 2..N \\ \quad F_{Static_i} \leftarrow VShear_{SRSS_i} - VShear_{SRSS_{i-1}} \\ \text{return } F_{Static} \end{array} \right.$$

$$F_{Static} = \begin{pmatrix} 55.348 \\ 47.126 \\ 37.422 \\ 25.868 \\ 13.374 \end{pmatrix} \cdot kN$$

The overturning moment, in kN·m, computed for these equivalent lateral loads is:

$$h = (16.000 \;\; 13.000 \;\; 10.000 \;\; 7.000 \;\; 4.000)\,m$$

$$M_{Static} := h \cdot F_{Static} = 2107.0\,kN \cdot m$$

The overturning moment, in this case, is slightly larger than the one obtained using the SRSS procedure with the modal spectral overturning moments.

$$M\_base_{SRSS} = (2087.287) \cdot kN \cdot m$$

5. COMMENTS

In Example 5 the step-by-step response of the system to the Hector-Mine record was computed, in Example 6 the individual modal spectral responses were computed for the spectrum of the same record — thus permitting the computation of the absolute maximum spectral response —, and in Example 7 the SRSS procedure was applied to the results obtained in Example 6. Now some comparisons can be made between the results of the three examples.

Table 11. Comparison of the results from Examples 5, 6, and 7.

Parameter	Example 5 Step-by-step Analysis	Example 6 Modal Spectral Absolute value	Example 7 Modal Spectral SRSS
Roof lateral displacement	0.017 m	0.021 m	0.020 m
Base Shear	157 kN	208 kN	179 kN
Overturning Moment	1763 kN·m	2136 kN·m	2087 kN·m

In this case, we observe that both the modal spectral absolute and SRSS value are overestimating the displacements, base shear and overturning moment. This is probably because the displacement response spectrum, as mentioned before, is very sensitive to small periods. However, we can say that, for this case the match between the step-by step analysis values and the values obtained using the SRSS procedure is reasonable good.

If we compare our results with the examples presented in the reference report we find that the response in terms of displacements, forces and moments, are very small. This is because the structure selected for this case is very stiff and has a small mass (only self-weight has been considered). Then, the fundamental period is small $T_1 = 0.354s$, and so the response. For instance, the spectral displacement for a structure with period of 1.0 and 0.35s are 1.6cm and 8.5cm respectively (5.3 times greater). This is a reason why we got small response values.

Therefore, we recommend to replicate the example with a less stiff structure, or increase the mass to have a fundamental period greater than 1 second; so that the response contribution of each mode will be more significant, and the comparisons between the step-by-step analyses and modal spectral analysis would be more reliable.

An additional recommended exercise, would be to perform the time history analysis of the model in ETABS. So we can compare the results and accuracy of the theorical time stepping method called Newmark's Beta Method with the linear modal time history analysis performed using ETABS software.

6. REFERENCES

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