Let’s build a bridge!

or

Knowledge is power

> *“Pride,” said Levin, touched to the quick by his brother’s words, “I don’t understand. If at the university they told me that others know the catenary, but I don’t know, then pride is here.*
>
> Lev Tolstoy. "Anna Karenina"

Once upon a time, not so long ago, to use sines, cosines, tangents, logarithms, and so on, in calculations was problematic. It was necessary to break away from the calculation and look into reference books—into the famous tables of Bradis, for example, on which several generations of schoolchildren grew up. It was often necessary to interpolate using discrete reference data. This was done by making calculations on pieces of paper, on adding machines, using a slide rule or a simple electronic calculator. Then the so-called scientific (engineering) calculators appeared, that, in addition to addition, subtraction, multiplication, division, square root extraction and percentage calculations, “learned” to work with the above and other similar mathematical functions.

Now there are so-called computer supercalculators (mathematical programs) that can additionally build graphs, work with matrices, find limits, take derivatives and integrals (see the epigraph[1] and the problem below), solve equations—algebraic and differential, optimize and perform other operations associated not with elementary (school), but with higher (university) mathematics, with mathematical analysis, for example.

Let's solve one beautiful engineering problem in the Mathcad environment to illustrate these features of supercalculators.

On the two banks of the river at a distance L from each other, two pylons of the future bridge with heights h_1 and h_2 were erected, to which a chain of length S and linear mass m_C is attached. A load of mass m_g is suspended from the chain at a distance x_1 from the left pylon (this, for example, is an element of the future roadway of the

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[1] In Tolstoy's novel no mention is made of a catenary, but of the integral calculus, which is also directly related to the task of this article.

bridge)[2]. Figure 1 shows a calculation where the listed variables are assigned numerical values with appropriate units. This is another significant difference between supercalculators and simple calculators—supercalculators do not just work with numbers, but with physical quantities! This speeds up calculations, makes them convenient, and eliminates possible errors in the conversion of units of measurement.

h_1 (m)	h_2 (m)	L (m)	S (m)	m_g (kg)	m_c $\left(\frac{gm}{cm}\right)$	x_1 (m)
4	3	6	8	1	3	2

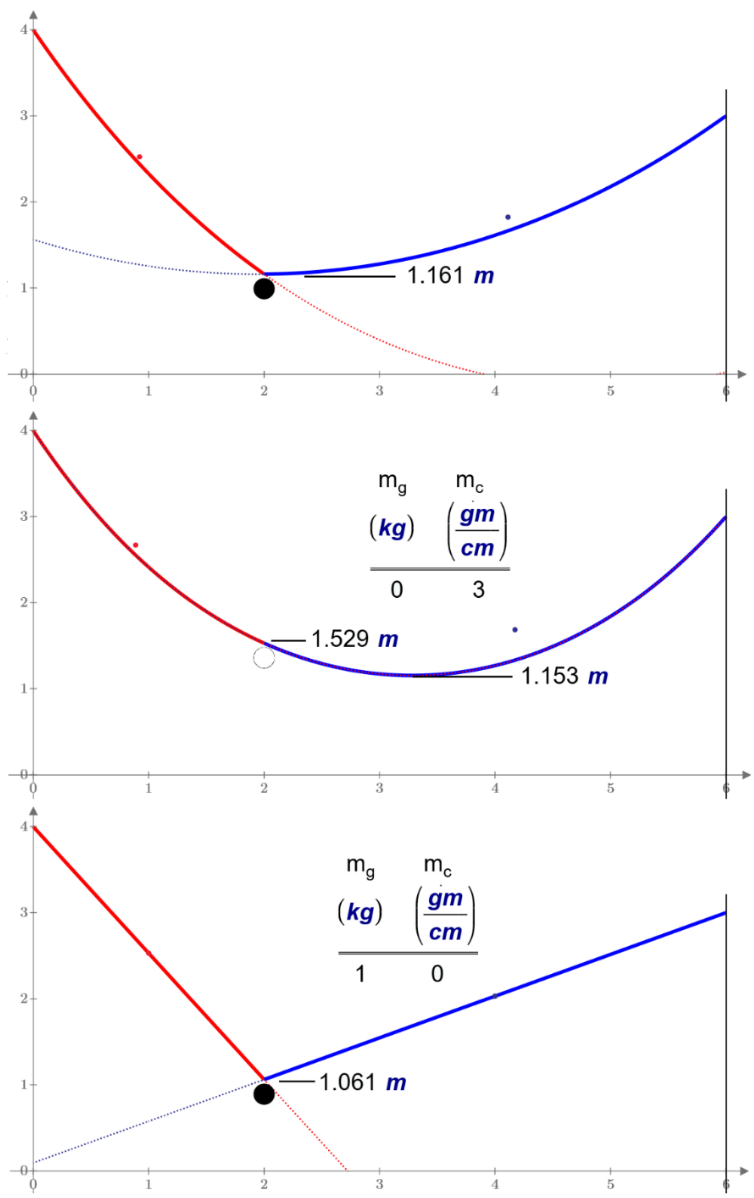


Fig. 1. Input of initial data and output of the answer in the form of a graph

[2] The task can be simplified—to build a cable car rather than a bridge.

The table with initial data is followed by a collapsed area of calculations—a plus sign in a square with a straight line on the right. Then the answer is given by three variants of the hanging chain, namely: a load is suspended from the chain so that sagging of the left (index L) and right (index R) sections of the chain are visible; the chain is free from the load and its sections merge into one sagging chain and the third option, in which the mass of the chain is negligible compared to the mass of the load and it is stretched like a string.

Let's open the collapsed calculation area and see what is written in it.

Creating and debugging *user functions* is half the solution! Figure 2 shows the functions we need:

1. Catenary with one argument x and three parameters.
2. Derivative of catenary with respect to x.
3. The length of the catenary on the segment from x_1 to x_2. The variable x_1 is already used in our calculation, but in the user function it is a local variable visible only in the function itself.
4. The ordinate of the center of gravity of the catenary on the segment from x_1 to x_2.
5. Abscissa of the center of gravity of the catenary on the segment from x_1 to x_2.
6. Potential energy PE of a chain with a load.

Dependencies 1, 3, 4 and 5 are easy to find on the Internet by searching for the corresponding keys. The function of the abscissa of the center of gravity of the chain line (item 5) is not involved in the calculation, but it is necessary to show on the graph of the sagging chain of the centers of gravity of two sections of the chain (see these centers in the graphs of Figure 1). In the third variant of the chain sagging in Figure 1, as expected, these centers were in the middle of the straight sections of the chain. One of the effective ways to test the created calculation is to set such initial data for which the answer is known in advance.

$$y(x,a,x_0,h) := a \cdot \cosh\left(\frac{x-x_0}{a}\right) - a + h \qquad (1)$$

$$y'(x,a,x_0) := \frac{d}{dx} y(x,a,x_0,h) \rightarrow \sinh\left(\frac{x-x_0}{a}\right) \qquad (2)$$

$$L_c(x_1,x_2,a,x_0) := \int_{x_1}^{x_2} \sqrt{1+y'(x,a,x_0)^2}\, dx \qquad (3)$$

$$y_{cg}(x_1,x_2,a,x_0,h) := \frac{\int_{x_1}^{x_2} y(x,a,x_0,h)\cdot\sqrt{1^2+y'(x,a,x_0)^2}\, dx}{L_c(x_1,x_2,a,x_0)} \qquad (4)$$

$$x_{cg}(x_1,x_2,a,x_0) := \frac{\int_{x_1}^{x_2} x\cdot\sqrt{1^2+y'(x,a,x_0)^2}\, dx}{L_c(x_1,x_2,a,x_0)} \qquad (5)$$

$$\begin{aligned} PE(y_1,a_L,x_{0L},h_L,a_R,x_{0R},h_R) := {} & L_c(0\,\boldsymbol{m},x_1,a_L,x_{0L})\cdot m_c\cdot \boldsymbol{g}\cdot y_{cg}(0\,\boldsymbol{m},x_1,a_L,x_{0L},h_L) \\ & + m_g\cdot \boldsymbol{g}\cdot y_1 \\ & + L_c(x_1,L,a_R,x_{0R})\cdot m_c\cdot \boldsymbol{g}\cdot y_{cg}(x_1,L,a_R,x_{0R},h_R) \quad (6) \end{aligned}$$

Fig. 2. User functions

Comments on Figure 1.

If you ask friends and acquaintances for what function the chain sags, then 99% will say that this is a parabola. This answer is prompted by the very appearance of the parabola with its apex at the origin, but also by the fact that a stone launched at an angle to the ground flies along a parabola. A hanging chain and a flying stone are affected by one main force—the force of attraction. Therefore, there is nothing surprising in the connection of the chain and the parabola—even the great Galileo thought so. True, at the end of his life he admitted that he was mistaken. After Galileo, the catenary formula was discovered almost simultaneously and independently by the three great mathematicians Bernoulli, Huygens and Leibniz [1].

The chain line in Figure 2 in paragraph 1 is given not in the canonical form a cosh(x/a), when its vertex is at the point $x=0$, $y=a$, but in the form in which this vertex will be at the point $x=x_0$, $y=h$. This is due to the fact that in our problem the origin of Cartesian coordinates is located at the base of the left pylon of the bridge—see Figure 1, and not at the point floating vertically, which is defined by the canonical view of the catenary. Therefore, the user function written in point 1 in Figure 2 has one argument x and three parameters (a, x_0 and h) instead of one (a). Rather so! The function named y has four arguments, not two. It is the person (computer user) who divides the arguments of the user function into arguments and parameters. For a computer (for Mathcad), they are all equal.

The catenary line derivative (2) is found by means of Mathcad symbolic mathematics—using the symbolic transformation operator "→". This means you don't need to do a numerical calculation of the derivative immediately, which in itself is considered a rather dubious operation from the standpoint of "pure" mathematics. In addition, this transformation—replacing the derivative itself with its expression speeds up the calculations.

It is possible to apply similar transformations to functions with integrals written in points 3-5 in Figure 2. Or you need not do this, especially since the expressions numbered 4 and 5 are not completely free of integrals.

The potential energy function of our immobile mechanical system (a chain with a load) with seven arguments has three terms, which are written in a column, and not in one long line. This can be done in the Mathcad environment. Looking ahead, let's say that the solution of our problem will be based on a special case of the d'Alembert-Lagrange principle, which says that a mechanical system takes a static position in which its potential energy will be minimal.

So, the initial data and user functions are entered—the problem can be solved!

Figure 3 shows a block of the Mathcad solver with three zones—the zone of first approximations of optimization variables, the zone of constraints, where not only equalities (as in our case), but also inequalities can be written, and the zone where one of four Mathcad built-in functions—**Find, MinErr, Maximize** and **Minimize** [2]—can be written. Our problem of a hanging chain with a load is solved using the last function. According to a special numerical algorithm, it changes the values of its last seven arguments (the first argument with the name PE is the name of the optimization objective function—see item 6 in Figure 2) so that the restrictions are met, and the objective function, according to the above principle, takes the minimum value.

Solve

Guess Values

$$y_1 := 1\ \mathbf{m}$$

$$a_L := 1\ \mathbf{m} \qquad x_{0L} := 3\ \mathbf{m} \qquad h_L := 1\ \mathbf{m}$$

$$a_R := 1\ \mathbf{m} \qquad x_{0R} := 3\ \mathbf{m} \qquad h_R := 1\ \mathbf{m}$$

Constrains

$$h_1 = y(0\ \mathbf{m}, a_L, x_{0L}, h_L) \qquad (1)$$

$$h_2 = y(L, a_R, x_{0R}, h_R) \qquad (2)$$

$$y_1 = y(x_1, a_L, x_{0L}, h_L) = y(x_1, a_R, x_{0R}, h_R) \qquad (3)$$

$$S = L_c(0\ \mathbf{m}, x_1, a_L, x_{0L}) + L_c(x_1, L, a_R, x_{0R}) \qquad (4)$$

Solver

$$\begin{bmatrix} y_1 \\ a_L \\ x_{0L} \\ h_L \\ a_R \\ x_{0R} \\ h_R \end{bmatrix} := \mathbf{Minimize}(PE, y_1, a_L, x_{0L}, h_L, a_R, x_{0R}, h_R) = \begin{bmatrix} 1.161 \\ 3.441 \\ 4.915 \\ -0.15 \\ 4.752 \\ 1.944 \\ 1.161 \end{bmatrix} \mathbf{m}$$

Fig. 3. Potential energy minimization

In the zone of restrictions, the following conditions of the problem are written in the language of mathematics:

1. The left end of the chain is fixed at height h_1.

2. The right end of the chain is fixed at height h_2.

3. Two sections of the chain converge at the point of suspension of the load x_1-y_1. Here, in fact (to save space), not two equations are written, but one. The value of x_1 is given, and the value of x_2 needs to be found.

4. Chain length S is a constant value. In principle, the chain should lengthen after it is suspended from the pylons and the load is attached, and this fact can be taken into account.

The graphs shown in Figure 1 are based on the values found by the **Minimize** function.

Based on these data, it is also easy to calculate the values of the forces that stretch the chain and build the corresponding force diagrams.

If people who know the law by which the chain sags (see above) are asked what physical meaning is inherent in the parameter, a, of the catenary line function, then again, virtually 99% will say that they do not know this or will give the wrong answer. And the correct answer is…?

Let's place a chain with a linear mass m_C equal to seventy grams per meter in a uniform gravitational field with an acceleration of gravity g equal to 9.807 meters divided by square seconds (see the first line of the calculation shown in Figure 5). And we hang the chain so that it sags, as shown in the graph in Figure 5. The ends of the chain are at the points with coordinates -1 m and 1.53 m and 1 m and 1.53 m. The length of such a chain is approximately 2.69 meters, and the weight is 188 grams.

$$g = 9.807\ \frac{m}{s^2} \qquad m_C := 70\ \frac{gm}{m} \qquad F := 0.5\ N$$

$$a := \frac{F}{g \cdot m_C} = 0.728\ m \qquad a = 0.728\ \frac{N}{\frac{m}{s^2} \cdot \frac{kg}{m}}$$

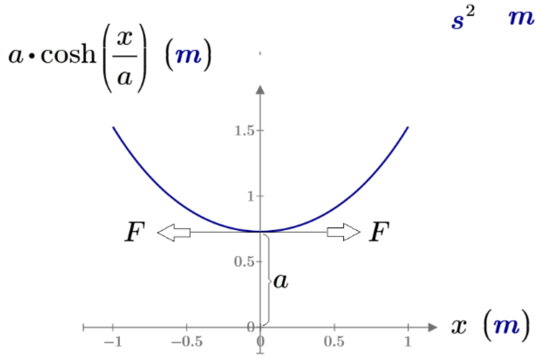


Fig. 5. Physical meaning of the parameter a of the catenary

If we measure the force F, with which our chain will stretch to the left and right at the minimum point, then it will be equal to half a newton. This force can be measured in the following way—attach a dynamometer horizontally to one of the ends of the chain and see what it shows. Force F is the horizontal projection of the force that stretches the chain anywhere. It is easy to prove that it is constant along the length of the chain. This position is based on the compilation of a differential equation, the solution of which will be a chain function. The vertical projection of the tensile force is a variable value. It varies from zero at the lowest point of the chain to a value of half the weight of the chain at its edges. These three physical quantities (F, g and m_C) will determine the value of the parameter a included in the catenary formula. But in all reference books on mathematics—paper and electronic, this constant is stubbornly considered dimensionless. But it has not only the reduced dimension of space (meters), but also the full dimension shown in Figure 5, which returns the physical meaning of the catenary formula.

When solving the differential equation, the constants F, g and m_C for simplicity were combined into one constant (parameter) a. This is the very simplicity, which, according to the proverb, turned out to be worse than theft!

Based on the foregoing, which figuratively speaking only "one percent of one percent" knows, we can create another three user functions (Figure 6) that will help us build a diagram of the forces acting on individual points of the chain with the load (Figure 7). Here it is the knowledge that is equivalent to power—see the second title of the article. More specifically, not just a force, but a diagram of forces stretching the chain.

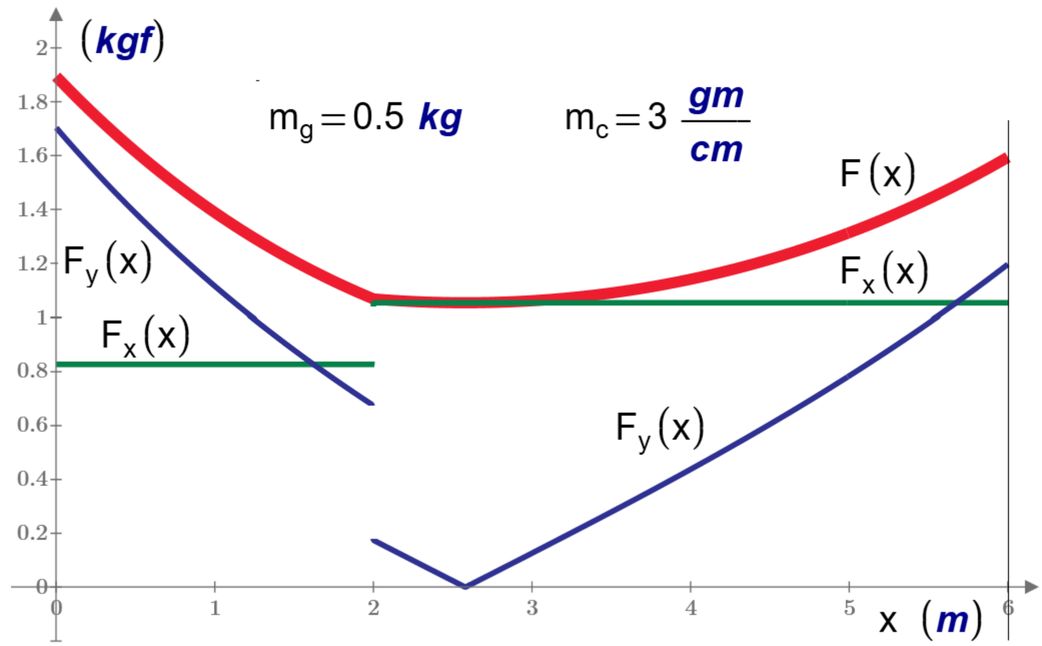
$$F_x(x) := g \cdot m_c \cdot \mathbf{if}(x < x_1, a_L, a_R)$$

$$F(x) := \frac{F_x(x)}{\cos(\operatorname{atan}(\mathbf{if}(x < x_1, y'(x, a_L, x_{0L}), y'(x, a_R, x_{0R}))))}$$

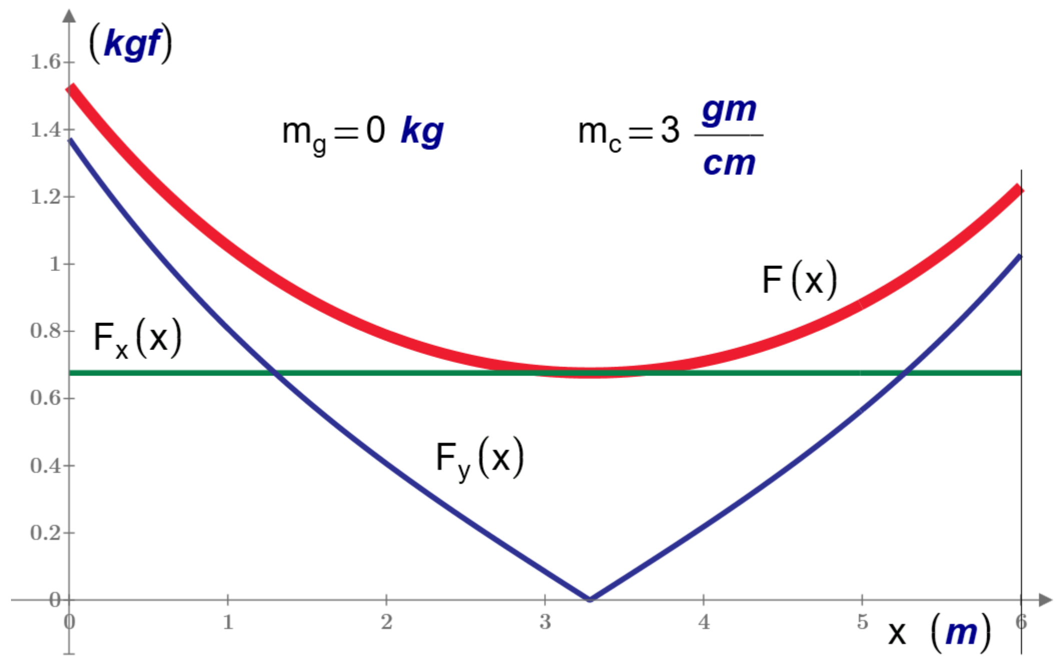
$$F_y(x) := \sqrt{F(x)^2 - F_x(x)^2}$$

Fig. 6. "Power" functions of the user

The functions in Figure 6 have one limitation—they will erroneously give zero values of forces if the chain is weightless (the third graph in Figure 1), and there is a load. But this is an extreme case that we can ignore. An error can also occur when the weight of the chain is very small compared to the weight of the load. This error is related to the accuracy of numerical calculations, which are also called approximate calculations.



a)



b)

Fig. 7. Plot of forces acting on a suspended chain with a load (a) and without load (b)

By the way, it is possible to hang not only weights, but also a kind of anti-weight on the chain—see Figures 8 and 9, which depicts such a situation. A power line (TL) is thrown across the river. A ship of oversized height floats along the river. To let it pass, the power is turned off, and the wire is temporarily lifted with a crane or a helicopter.

Another version of this task is to remove a necklace from the neck, attached a balloon to it and stretch it between the hands.

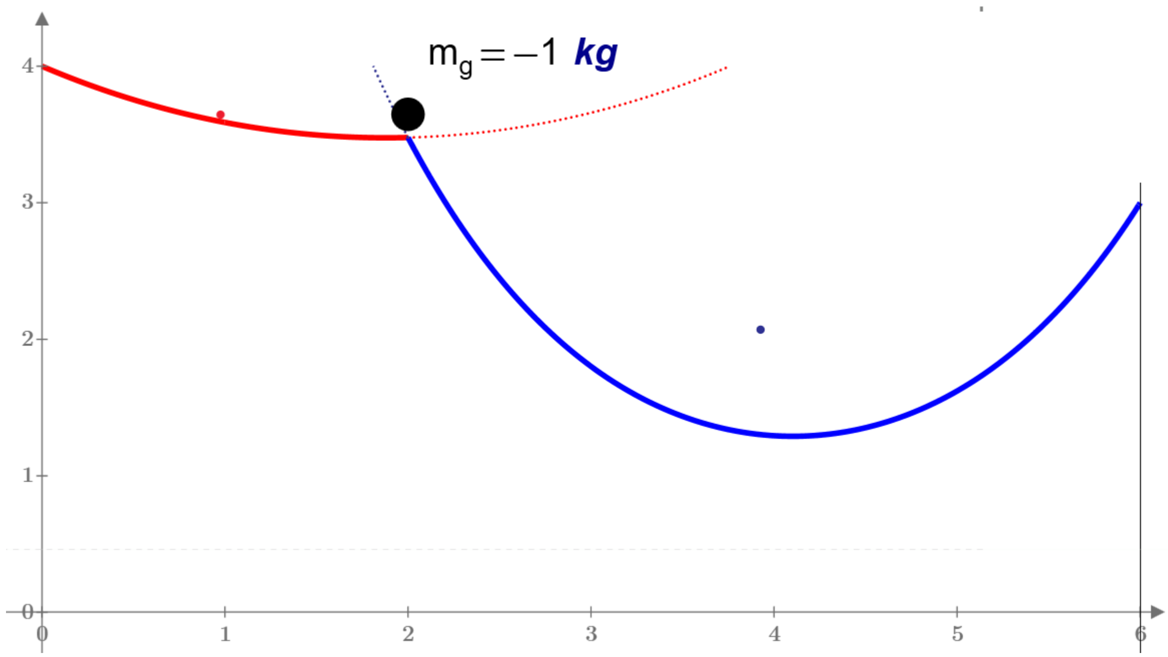


Fig. 8. Chain with anti-weight

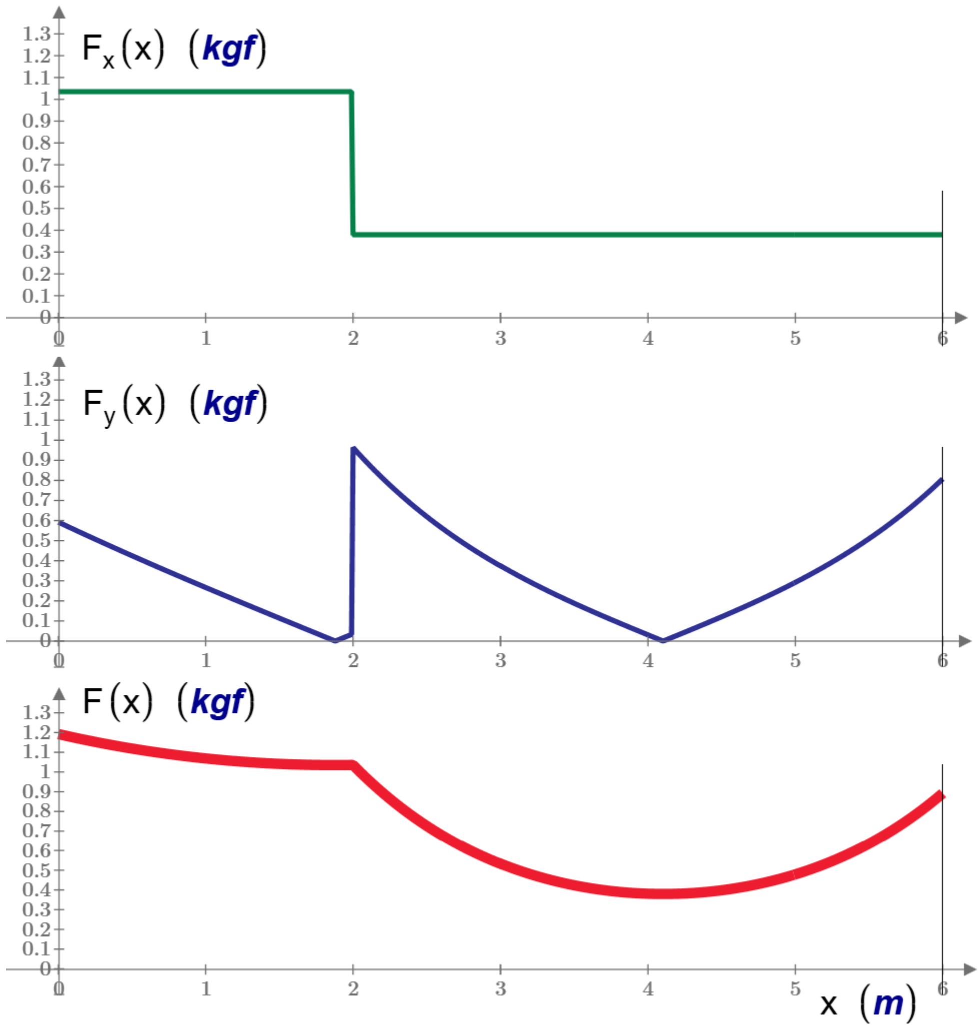


Fig. 9. Plot of forces acting on a suspended chain with an anti-weight

Speaking of knowledge equivalent to power, it should also be noted that many people forget about the physical essence of the expression parameter not only for an exotic catenary, but also for an ordinary school parabola, which, we repeat, is often confused with a catenary. The canonical equation of a parabola in a rectangular coordinate system is: $y^2=2p\,x$. The parameter p in this equation also has a clear physical meaning. It is equal to the distance from the focus of the parabola to its directrix. Many simply have not heard about the focus and directrix of the parabola! But it is a parabolic antenna that collects and focusses radio beams.

Let's return to the process of building a bridge in this vein. If more and more new loads are attached to the suspended chains or cables on the guys – the sections of the future carriageway of the bridge—then the load on the chains (cables) will be approximately the same in all sections if the sagging is along a parabola, and not along a catenary.

Finally, we can mention one more feature of the catenary, associated with an interesting constant—π. If the chain without a load is suspended with its ends at the same height, then it is easy to find the ratio S/L, at which the force F will be minimal. The approximate value of this constant is 1.26, and it is related to the root of the equation coth(x)=x [3, 4]. And one percent of people who know the physical meaning of the parameter, a, of the catenary know about this constant.

The catenary 2.5 cosh(2x/5) shown in Fig. 8 can be called *an ideal catenary*.

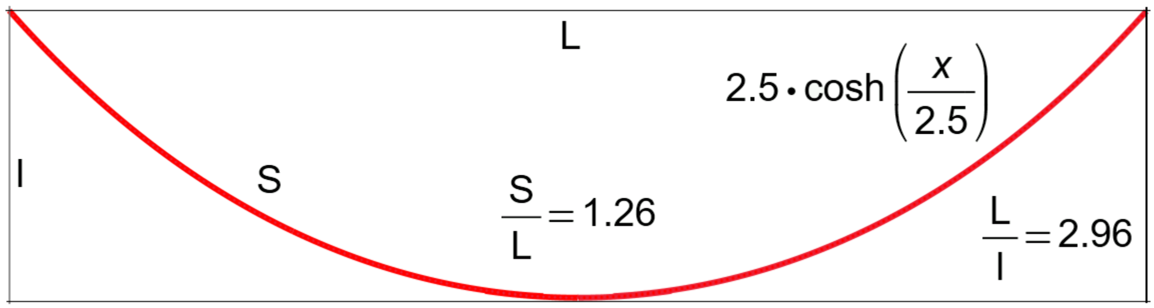


Fig. 8. An ideal catenary

If you want to enclose a monument with posts with chains suspended between its, then remember the ideal catenary Fig. 9. The fence will turn out to be ideal both in the aesthetic and in the engineering (power) sense.

Fig. 9.

Literature.

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