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9-6 MODULUS OF SUBGRADE REACTION

The modulus of subgrade reaction is a conceptual relationship between soil pressure and deflection that is widely used in the structural analysis of foundation members. It is used for continuous footings, mats, and various types of pilings to be taken up in later chapters. This ratio was defined on Fig. 2-43*c*, and the basic equation when using plate-load test data is

$$k_s = \frac{q}{\delta}$$

with terms identified on both Fig. 2-43*c* and Fig. 9-9*b*. Plots of q versus δ from load tests give curves of the type qualitatively shown in Fig. 9-9*b*. If this type of curve is used to obtain k_s in the preceding equation, it is evident that the value depends on whether it is a tangent or secant modulus and on the location of the coordinates of q and δ.

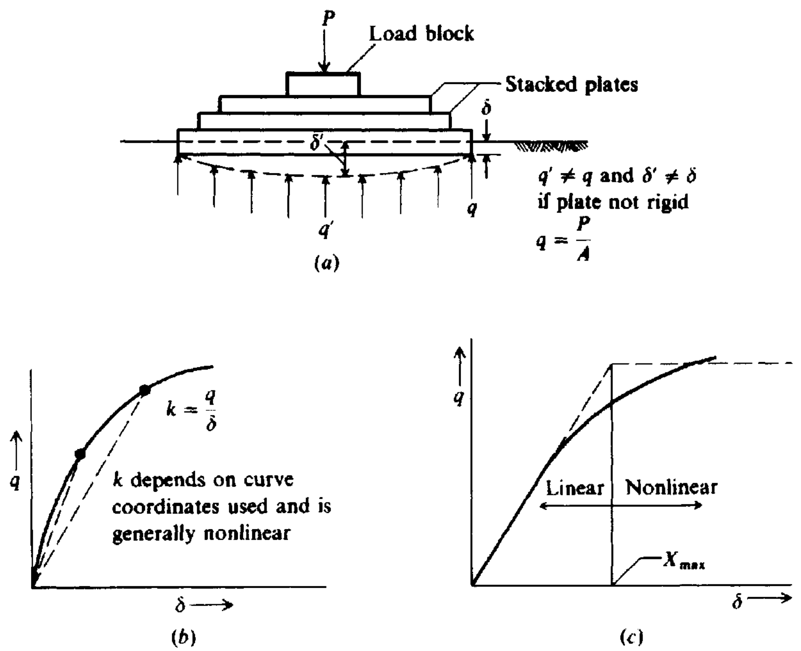
It is difficult to make a plate-load test except for very small plates because of the reaction load required. Even with small plates of, say, 450-, 600-, and 750-mm diameter it is difficult to obtain δ since the plate tends to be less than rigid so that a constant deflection across the plate (and definition of k_s) is difficult to obtain. Stacking the smaller plates concentric with the larger ones tends to increase the rigidity, but in any case the plot is of load divided by plate contact area (nominal P/A) and the *average measured deflection.*

Figure 9-9*c* is a representation of k_s used by the author where k_s is taken as a constant up to a deflection $X_{\max}$. Beyond $X_{\max}$ the soil pressure is a constant value defined by

$$q_{\text{con}} = k_s(X_{\max})$$

Obviously one could divide the q-δ curve into several regions so that k_s takes on values of the slope in the several regions; however, this approach tends to incorporate too much

Figure 9-9 Determination of modulus of subgrade reaction k_s.



refinement into the problem since most analyses proceed on the basis of estimated values or at best an approximate load test.

A number of persons do not like to use the concept of a modulus of subgrade reaction; rather, they prefer to use E_s (and μ) in some kind of finite-element analysis. The author's experience using both the finite element (of the elastic continuum) and the concept of the modulus of subgrade reaction is that, until the state of the art improves so that accurate values of E_s can be obtained, the modulus of subgrade reaction method is preferable owing to its greater ease of use and to the substantial savings in computer computation time. In the following paragraphs we will see a direct relationship between E_s and k_s.

A major problem is to estimate the numerical value of k_s. One of the early contributions was that of Terzaghi (1955), who proposed that k_s for full-sized footings could be obtained from plate-load tests using the following equations:

For footings on clay[3]

$$k_s = k_1 \frac{B_1}{B} \tag{9-3}$$

For footings on sand (and including size effects)

$$k_s = k_1 \left(\frac{B + B_1}{2B} \right)^2 \tag{9-4}$$

In these two equations use B_1 = side dimension of the square base used in the load test to produce k_1. In most cases B_1 = 0.3 m (or 1 ft), but whatever B_1 dimension was used should be input. Also this equation deteriorates when $B/B_1 \approx> 3$.

For a rectangular footing on stiff clay or medium dense sand with dimensions of $B \times L$ with $m = L/B$,

$$k_s = k_1 \frac{m + 0.5}{1.5m} \tag{9-5}$$

where k_s = desired value of modulus of subgrade reaction for the full-size (or prototype) foundation

k_1 = value obtained from a plate-load test using a 0.3 × 0.3 m (1 × 1 ft) or other size load plate

Equations (9-3), (9-4), and (9-5) are presented primarily for historical purposes and are not recommended by the author for general use.

Vesić (1961*a*, 1961*b*) proposed that the modulus of subgrade reaction could be computed using the stress-strain modulus E_s as

$$k_s' = 0.65 \sqrt[12]{\frac{E_s B^4}{E_f I_f}} \frac{E_s}{1 - \mu^2} \qquad \text{(units of } E_s\text{)} \tag{9-6}$$

[3]The B_1 is not usually seen in this equation, since at the time it was proposed by Terzaghi (1955) only Fps units were used, and with B_1 = 1 ft it did not need to be shown. The equation is dimensionally incorrect, however, without including B_1. Equation (9-3) is not correct in any case, as k_s using a 3.0 m footing would not be $\frac{1}{10}$ the value obtained from a B_1 = 0.3 m plate.

where E_s, E_f = modulus of soil and footing, respectively, in consistent units
B, I_f = footing width and its moment of inertia based on cross section (not plan) in consistent units

One can obtain k_s from k_s' as

$$k_s = \frac{k_s'}{B}$$

Since the twelfth root of any value × 0.65 will be close to 1, for all practical purposes the Vesić equation reduces to

$$k_s = \frac{E_s}{B(1-\mu^2)} \tag{9-6a}$$

One may rearrange Eq. (5-16a) and, using $E_s' = (1-\mu^2)/E_s$ as in Eqs. (5-18) and (5-19) and $m = 1$, obtain

$$\Delta H = \Delta q B E_s' I_s I_F$$

and, since k_s is defined as $\Delta q/\Delta H$, obtain

$$k_s = \frac{\Delta q}{\Delta H} = \frac{1}{BE_s' I_s I_F} \tag{9-7}$$

but carefully note the definition of E_s'. Now one can correctly incorporate the size effects that are a major concern—particularly for the mat foundations of the next chapter. As for Eqs. (5-18) and (5-19), we can write a k_s ratio from Eq. (9-7) as follows:

$$\frac{k_{s1}}{k_{s2}} = \frac{B_2 E_{s2}' I_{s2} I_{F2}}{B_1 E_{s1}' I_{s1} I_{F1}} \tag{9-8}$$

Equation (9-8) should be used instead of Eqs. (9-3) through (9-5), and Eq. (9-7) is at least as theoretically founded as Eq. (9-6). Carefully note in using these equations that their basis is in the settlement equation [Eq. (5-16a)] of Chap. 5, and use B, I_s, and I_F as defined there.

Equations (9-7) and (9-8) show a direct relationship between k_s and E_s. Since one does not often have values of E_s, other approximations are useful and often quite satisfactory if the computed deflection (directly dependent on k_s) can be tolerated for any reasonable value. It has been found that bending moments and the computed soil pressure are not very sensitive to what is used for k_s because the structural member stiffness is usually 10 or more times as great as the soil stiffness as defined by k_s. Recognizing this, the author has suggested the following for approximating k_s from the allowable bearing capacity q_a furnished by the geotechnical consultant:

$$\text{SI: } k_s = 40(\text{SF})q_a \quad \text{kN/m}^3$$
$$\text{Fps: } k_s = 12(\text{SF})q_a \quad \text{k/ft}^3 \tag{9-9}$$

where q_a is furnished in ksf or kPa. This equation is based on $q_a = q_{\text{ult}}/\text{SF}$ and the ultimate soil pressure is at a settlement $\Delta H = 0.0254$ m or 1 in. (1/12 ft) and k_s is $q_{\text{ult}}/\Delta H$. For $\Delta H = 6, 12, 20$ mm, etc., the factor 40 (or 12) can be adjusted to 160 (or 48), 83 (or 24), 50 (or 16), etc.; 40 is reasonably conservative but smaller assumed displacements can always be used.

The most general form for either a horizontal or lateral modulus of subgrade reaction is

$$k_s = A_s + B_s Z^n \tag{9-10}$$

where A_s = constant for either horizontal or vertical members
B_s = coefficient for depth variation
Z = depth of interest below ground
n = exponent to give k_s the best fit (if load test or other data are available)

Either A_s or B_s in this equation may be zero; at the ground surface A_s is zero for a lateral k_s but at any small depth $A_s > 0$. For footings and mats (plates in general), $A_s > 0$ and $B_s \cong 0$.

Equation (9-10) can be used with the proper interpretation of the bearing-capacity equations of Table 4-1 (with the d_i factors dropped) to give

$$q_{\text{ult}} = cN_c s_c + \gamma Z N_q s_q + 0.5\gamma B N_\gamma s_\gamma) \tag{9-10a}$$

Observing that

$$A_s = C(cN_c s_c + 0.5\gamma B N_\gamma s_\gamma) \qquad \text{and} \qquad B_s Z^1 = C(\gamma N_q s_q) Z^1$$

we obtain a ready means to estimate k_s. In these equations the Terzaghi or Hansen bearing-capacity factors can be used. The C factor is 40 for SI units and 12 for Fps, using the same reasoning that q_{ult} occurs at a 0.0254-m and 1-in. settlement but with no SF, since this equation directly gives q_{ult}. Where there is concern that k_s does not increase without bound with depth Z, we may adjust the $B_s Z$ term by one of two simple methods:

$$\text{Method 1: } B_s \tan^{-1}\frac{Z}{D}$$

$$\text{Method 2: } \frac{B_s}{D^n} Z^n = B_s' Z^n$$

where D = maximum depth of interest, say, the length of a pile
Z = current depth of interest
n = your best estimate of the exponent

Table 9-1 may be used to estimate a value of k_s to determine the correct order of magnitude of the subgrade modulus obtained using one of the approximations given here. Obviously if a computed value is two or three times larger than the table range indicates, the computations should be rechecked for a possible gross error. Note, however, if you use a reduced value of displacement (say, 6 mm or 12 mm) instead of 0.0254 m you may well exceed the table range. Other than this, if no computational error (or a poor assumption) is found then use judgment in what value to use. The table values are intended as guides. The reader should not use, say, an average of the range given as a "good" estimate.

The value of $X_{\max}$ used in Fig. 9-9c (and used in your diskette program FADBEMLP as XMAX) may be directly estimated at some small value of, say, 6 to 25 mm, or from inspection of a load-settlement curve if a load test was done. It might also be estimated from a triaxial test using the strain at "ultimate" or at the maximum pressure from the stress-strain plot. Using the selected strain $\epsilon_{\max}$ compute

$$X_{\max} = \epsilon_{\max}(1.5 \text{ to } 2B)$$

TABLE 9-1
Range of modulus of subgrade reaction k_s

Use values as guide and for comparison when using approximate equations

Soil	k_s, kN/m^3
Loose sand	4800–16 000
Medium dense sand	9600–80 000
Dense sand	64 000–128 000
Clayey medium dense sand	32 000–80 000
Silty medium dense sand	24 000–48 000
Clayey soil:	
$q_a \leq 200$ kPa	12 000–24 000
$200 < q_a \leq 800$ kPa	24 000–48 000
$q_a > 800$ kPa	> 48 000

The 1.5 to $2B$ dimension is an approximation of the depth of significant stress-strain influence (Boussinesq theory) for the structural member. The structural member may be either a *footing* or a *pile*.

Example 9-5. Estimate the modulus of subgrade reaction k_s for the following design parameters:

$$B = 1.22 \text{ m} \qquad L = 1.83 \text{ m} \qquad D = 0.610 \text{ m}$$
$$q_a = 200 \text{ kPa (clayey sand approximately 10 m deep)}$$
$$E_s = 11.72 \text{ MPa (average in depth } 5B \text{ below base)}$$

Solution. Estimate Poisson's ratio $\mu = 0.30$ so that

$$E_s' = \frac{1-\mu^2}{E_s} = \frac{1-0.3^2}{11.72} = 0.077\,65 \text{ m}^2/\text{MN}$$

For **center:**

$$H/B' = 5B/(B/2) = 10 \text{ (taking } H = 5B \text{ as recommended in Chap. 5)}$$
$$L/B = 1.83/1.22 = 1.5$$

From these we may write

$$I_s = 0.584 + \frac{1-2(0.3)}{1-0.3}0.023 = \mathbf{0.597}$$

using Eq. (5-16) and Table 5-2 (or your program FFACTOR) for factors 0.584 and 0.023.

At $D/B = 0.61/1.22 = 0.5$, we obtain $I_F = 0.80$ from Fig. 5-7 (or when using FFACTOR for the I_s factors). Substitution into Eq. (9-7) with $B' = 1.22/2 = 0.61$, and $m = 4$ yields

$$k_s = \frac{1}{0.61(0.077\,65)(4 \times 0.597)(0.8)} = \mathbf{11.05} \text{ MN/m}^3$$

You should note that k_s does not depend on the contact pressure of the base q_o.

For **corner:**

$$H/B' = 5B/B = 5(1.22)/1.22 = 5$$

[from Table 5-2 with $L/B = 1.5$ obtained for Eq. (5-16)]

$$I_s = 0.496 + \frac{0.4}{0.7}(0.045) = 0.522 \qquad I_F = 0.8 \text{ (as before)}$$

Again substituting into Eq. (9-7) but with $B' = B = 1.22$ m and one corner contribution, we have

$$k_s = \frac{1}{1.22 \times 0.077\,65 \times 0.522 \times 0.8} = \mathbf{25.28} \text{ MN/m}^3$$

For an average value we will use weighting, consisting of four center contributions + one corner value, giving

$$k_{s,\text{avg}} + \frac{4(11.5) + 25.28}{5} = \mathbf{13.896} \text{ MN/m}^3$$

We can also estimate k_s based on SF = 2 for sand to obtain

$$k_s = 40(\text{SF})(q_a) = 40(2)(0.200) = \mathbf{16} \text{ MN/m}^3$$

For practical usage and since these values of 13.896 and 16.0 are estimates (but reasonably close) we would use

$$k_s = \mathbf{15.0} \text{ MN/m}^3 \qquad (\mathbf{15\,000} \text{ kN/m}^3)$$

Comments. It is evident from this example that the "center" k_s is softer (or less stiff) than a corner (or edge). The center being less stiff is consistent with the *dishing* of uniformly loaded plates. One can also *zone* the area beneath a footing by computing a series of k_s values at, say, center, $\frac{1}{4}$, $\frac{1}{8}$, and edge points using for the $\frac{1}{4}$ and $\frac{1}{8}$ point the contributions from four rectangles and for the edge the contributions of two rectangles of the same size.

Note the use of $H = 5B = 5 \times 1.22 = 6.1$ m for both center and corner.

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9-7 CLASSICAL SOLUTION OF BEAM ON ELASTIC FOUNDATION

When flexural rigidity of the footing is taken into account, a solution can be used that is based on some form of a beam on an elastic foundation. This may be the classical Winkler solution of about 1867, in which the foundation is considered as a bed of springs ("Winkler foundation"), or the finite-element procedure of the next section.

The classical solutions, being of closed form, are not so general in application as the finite-element method. The basic differential equation is (see Fig. 9-10)

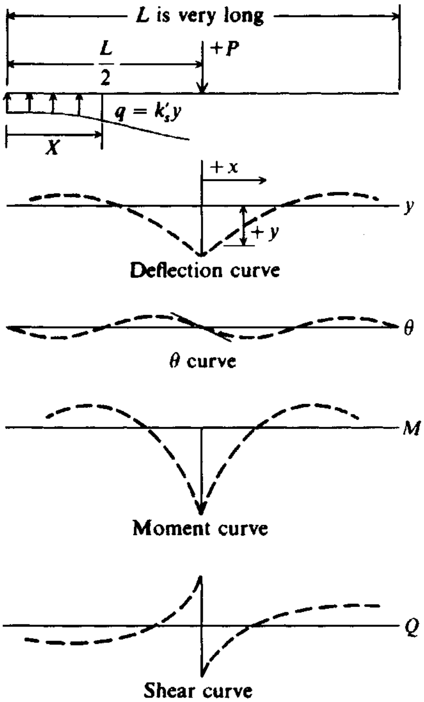
$$EI\frac{d^4y}{dx^4} = q = -k_s'y \tag{9-11}$$

where $k_s' = k_sB$. In solving the equations, a variable is introduced:

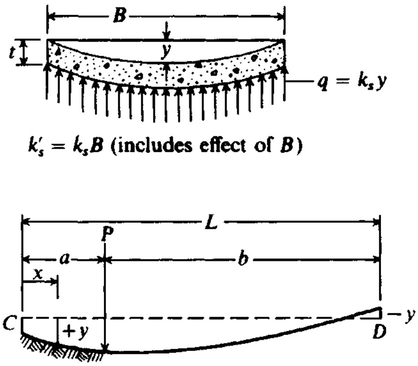
$$\lambda = \sqrt[4]{\frac{k_s'}{4EI}} \qquad \text{or} \qquad \lambda L = \sqrt[4]{\frac{k_s'L^4}{4EI}}$$

Table 9-2 gives the closed-form solution of the basic differential equations for several loadings shown in Fig. 9-10 utilizing the Winkler concept. It is convenient to express the trigonometric portion of the solutions separately as in the bottom of Table 9-2.

Hetenyi (1946) developed equations for a load at any point along a beam (see Fig. 9-10*b*) measured from the left end as follows:



(*a*) Infinite length beam on an elastic foundation with mid or center loading.



(*b*) Finite length beam on elastic foundation.

Figure 9-10 Beam on elastic foundation.

$$y = \frac{P\lambda}{k_s'(\sinh^2 \lambda L - \sin^2 \lambda L)}\{2\cosh \lambda x \cos \lambda x(\sinh \lambda L \cos \lambda a \cosh \lambda b - \sin \lambda L \cosh \lambda a \cos \lambda b) + (\cosh \lambda x \sin \lambda x + \sinh \lambda x \cos \lambda x)[\sinh \lambda L(\sin \lambda a \cosh \lambda b - \cos \lambda a \sinh \lambda b) + \sin \lambda L(\sinh \lambda a \cos \lambda b - \cosh \lambda a \sin \lambda b)]\} \tag{9-12}$$

$$M = \frac{P}{2\lambda(\sinh^2 \lambda L - \sin^2 \lambda L)}\{2\sinh \lambda x \sin \lambda x(\sinh \lambda L \cos \lambda a \cosh \lambda b - \sin \lambda L \cosh \lambda a \cos \lambda b) + (\cosh \lambda x \sin \lambda x - \sinh \lambda x \cos \lambda x) \times [\sinh \lambda L(\sin \lambda a \cosh \lambda b - \cos \lambda a \sinh \lambda b) + \sin \lambda L(\sinh \lambda a \cos \lambda b - \cosh \lambda a \sin \lambda b)]\} \tag{9-13}$$

$$Q = \frac{P}{\sinh^2 \lambda L - \sin^2 \lambda L}\{(\cosh \lambda x \sin \lambda x + \sinh \lambda x \cos \lambda x) \times (\sinh \lambda L \cos \lambda a \cosh \lambda b - \sin \lambda L \cosh \lambda a \cos \lambda b) + \sinh \lambda x \sin \lambda x[\sinh \lambda L(\sin \lambda a \cosh \lambda b - \cos \lambda a \sinh \lambda b) + \sin \lambda L(\sinh \lambda a \cos \lambda b - \cosh \lambda a \sin \lambda b)]\} \tag{9-14}$$

The equation for the slope θ of the beam at any point is not presented since it is of little value in the design of a footing. The value of x to use in the equations is from the end of the

TABLE 9-2
Closed-form solutions of infinite beam on elastic foundation (Fig. 9-10*a*)

Concentrated load at end	Moment at end
$y = \dfrac{2V_1\lambda}{k_s'} D_{\lambda x}$	$y = \dfrac{-2M_1\lambda^2}{k_s'} C_{\lambda x}$
$\theta = \dfrac{-2V_1\lambda^2}{k_s'} A_{\lambda x}$	$\theta = \dfrac{4M_1\lambda^3}{k_s'} D_{\lambda x}$
$M = \dfrac{-V_1}{\lambda} B_{\lambda x}$	$M = M_1 A_{\lambda x}$
$Q = -V_1 C_{\lambda x}$	$Q = -2M_1\lambda B_{\lambda x}$
Concentrated load at center ($+\downarrow$)	**Moment at center** ($+ \curvearrowright$)
$y = \dfrac{P\lambda}{2k_s'} A_{\lambda x}$	$y = \dfrac{M_0\lambda^2}{k_s'} B_{\lambda x}$ deflection
$\theta = \dfrac{-P\lambda^2}{k_s'} B_{\lambda x}$	$\theta = \dfrac{M_0\lambda^3}{k_s'} C_{\lambda x}$ slope
$M = \dfrac{P}{4\lambda} C_{\lambda x}$	$M = \dfrac{M_0}{2} D_{\lambda x}$ moment
$Q = \dfrac{-P}{2} D_{\lambda x}$	$Q = \dfrac{-M_0\lambda}{2} A_{\lambda x}$ shear

The A, B, C, and D coefficients (use only $+x$) are as follows:

$$A_{\lambda x} = e^{-\lambda x}(\cos \lambda x + \sin \lambda x)$$
$$B_{\lambda x} = e^{-\lambda x} \sin \lambda x$$
$$C_{\lambda x} = e^{-\lambda x}(\cos \lambda x - \sin \lambda x)$$
$$D_{\lambda x} = e^{-\lambda x} \cos \lambda x$$

beam to the point for which the deflection, moment, or shear is desired. If x is less than the distance a of Fig. 9-10*b*, use the equations as given and measure x from C. If x is larger than a, replace a with b in the equations and measure x from D. These equations may be rewritten as

$$y = \frac{P\lambda}{k_s'} A' \qquad M = \frac{P}{2\lambda} B' \qquad Q = PC'$$

where the coefficients A', B', and C' are the values for the hyperbolic and trigonometric remainder of Eqs. (9-12) to (9-14).

It has been proposed that one could use λL previously defined to determine if a foundation should be analyzed on the basis of the conventional rigid procedure or as a beam on an elastic foundation (see combined footing Example 9-1):

Rigid members: $\lambda L < \dfrac{\pi}{4}$ (bending not influenced much by k_s)

Flexible members: $\lambda L > \pi$ (bending heavily localized)

The author has found these criteria of limited application because of the influence of the number of loads and their locations along the member.

The classical solution presented here has several distinct disadvantages over the finite-element solution presented in the next section, such as

1. Assumes weightless beam (but weight will be a factor when footing tends to separate from the soil)
2. Difficult to remove soil effect when footing tends to separate from soil
3. Difficult to account for boundary conditions of known rotation or deflection at selected points
4. Difficult to apply multiple types of loads to a footing
5. Difficult to change footing properties of I, D, and B along member
6. Difficult to allow for change in subgrade reaction along footing

Although the disadvantages are substantial, some engineers prefer the classical beam-on-elastic-foundation approach over discrete element analyses. Rarely, the classical approach may be a better model than a discrete element analysis, so it is worthwhile to have access to this method of solution.

9-8 FINITE-ELEMENT SOLUTION OF BEAM ON ELASTIC FOUNDATION

The finite-element method (FEM) is the most efficient means for solving a beam-on-elastic-foundation type of problem based on Eq. (9-10) but requires a digital (or personal) computer. It is easy to account for boundary conditions (such as a point where there is no rotation or translation), beam weight, and nonlinear soil effects (either soil-beam separation or a displacement $> X_{max}$).

The FEM is more versatile than the finite-difference method (FDM), because one can write an equation model for one element and use it for each element in the beam model. With the finite-difference method all of the elements must be the same length and cross section. Different equations are required for end elements than for interior ones, and modeling boundary conditions is difficult, as is modeling nonlinear soil effects. The FDM had an initial advantage of not requiring much computer memory, because there is only one unknown at a node—the displacement. With the discovery of band matrix solution methods this advantage was completely nullified.

Only the basic elements of the FEM will be given here, and the reader is referred to Wang (1970) or Bowles (1974a) if more background is required. The computer program B-5 (FADBEMLP) on the enclosed diskette has the necessary routines already coded for the user. This program was used to obtain text output.

General Equations in Solution

For the following development refer to Fig. 9-11. At any node i (junction of two or more members at a point) on the structure we may write

$$P_i = A_i F_i$$

which states that the external node force P is equated to the contributing internal member forces F using bridging constants A. It is understood that P and F are used for either forces or moments and that this equation is shorthand notation for several values of A_iF_i summed to equal the ith nodal force.

For the full set of nodes on any structure and using matrix notation, where **P**, **F** are column vectors and **A** is a rectangular matrix, this becomes

$$\mathbf{P} = \mathbf{AF} \tag{a}$$

An equation relating internal-member deformation **e** at any node to the external nodal displacements is

$$\mathbf{e} = \mathbf{BX}$$

where both **e** and **X** may be rotations (radians) or translations. From the reciprocal theorem in structural mechanics it can be shown that the **B** matrix is exactly the transpose of the **A** matrix, which is a convenience indeed; thus,

$$\mathbf{e} = \mathbf{A}^{\mathrm{T}}\mathbf{X} \tag{b}$$

The internal-member forces **F** are related to the internal-member displacements **e** and contributing member stiffnesses **S** as

$$\mathbf{F} = \mathbf{Se} \tag{c}$$

These three equations are the fundamental equations in the finite-element method of analysis:

Substituting (b) into (c),

$$\mathbf{F} = \mathbf{Se} = \mathbf{SA}^{\mathrm{T}}\mathbf{X} \tag{d}$$

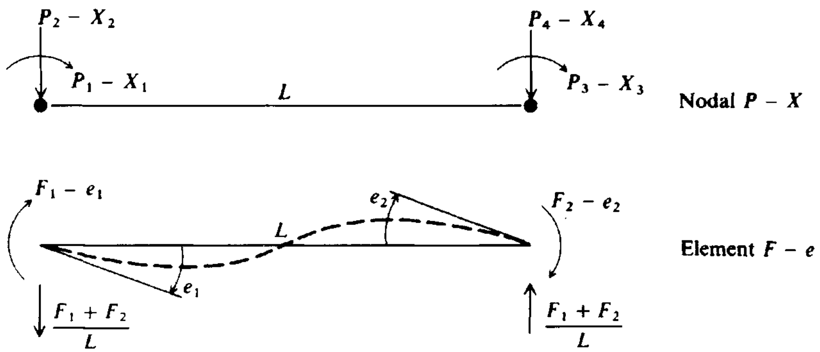
Substituting (d) into (a),

$$\mathbf{P} = \mathbf{AF} = \mathbf{ASA}^{\mathrm{T}}\mathbf{X} \tag{e}$$

Note the order of terms used in developing Eqs. (d) and (e). Now the only unknowns in this system of equations are the **X**'s; so the $\mathbf{ASA}^{\mathrm{T}}$ is inverted to obtain

$$\mathbf{X} = (\mathbf{ASA}^{\mathrm{T}})^{-1}\mathbf{P} \tag{f}$$

Figure 9-11 External (nodal) and internal (member) finite-element forces.



and with the **X**'s we can back-substitute into Eq. (*d*) to obtain the internal-member forces that are necessary for design. This method gives two important pieces of information: (1) design data and (2) deformation data.

The $\mathbf{ASA}^T$ matrix above is often called a global matrix, since it represents the system of equations for each **P** or **X** nodal entry. It is convenient to build it from one finite element of the structure at a time and use superposition to build the global $\mathbf{ASA}^T$ from the element $\mathbf{EASA}^T$. This is easily accomplished, since every entry in both the global and element $\mathbf{ASA}^T$ with a unique set of subscripts is placed into that subscript location in the $\mathbf{ASA}^T$, i.e., for $i = 2, j = 5$ all (2, 5) subscripts in $\mathbf{EASA}^T$ are added into the (2, 5) coordinate location of the global $\mathbf{ASA}^T$.

Developing the Element A Matrix

Consider the single simple beam element shown in Fig. 9-12*b* coded with four values of P–X (note that two of these P–X values will be common to the next member) and the forces on the element (Fig. 9-12*c*). The forces on the element include two internal bending moments and the shear effect of the bending moments. The sign convention used is consistent with your computer program B-5.

Summing moments on node 1 of Fig. 9-12*d*, we obtain

$$P_1 = F_1 + 0F_2$$

Similarly, summing forces and noting that the soil spring forces are global and will be included separately, we have

$$P_2 = \frac{F_1}{L} + \frac{F_2}{L}$$

$$P_3 = 0F_1 + F_2$$

$$P_4 = -\frac{F_1}{L} - \frac{F_2}{L}$$

Placed into conventional form, the element A matrix for element 1 is

EA =

P \ F	1	2
1	1	0
2	$1/L$	$1/L$
3	0	1
4	$-1/L$	$-1/L$

The **EA** matrix for member 2 would contain P_3 to P_6; it is not necessary to resubscript the F values.

Developing the S Matrix

Referring to Fig. 9-13 and using conjugate-beam (moment-area) principles, we see that the end slopes e_1 and e_2 are

$$\frac{F_1 L}{3EI} - \frac{F_2 L}{6EI} = e_1 \tag{g}$$

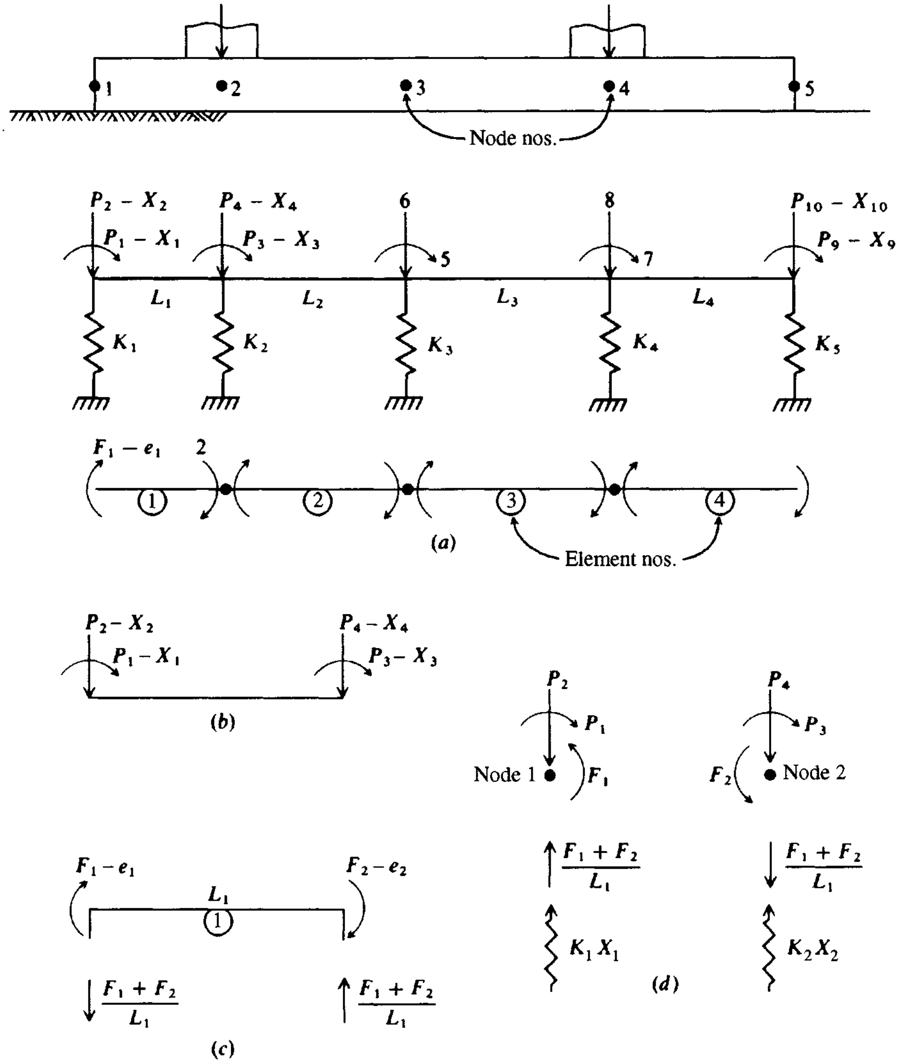


Figure 9-12 (*a*) Structure and structure broken into finite elements with global P–X; (*b*) P–X of first element; (*c*) element forces of any (including first) element; (*d*) summing nodal forces.

$$-\frac{F_1 L}{6EI} + \frac{F_2 L}{3EI} = e_2 \tag{h}$$

Solving Eqs. (*g*) and (*h*) for **F**, we obtain

$$F_1 = \frac{4EI}{L}e_1 + \frac{2EI}{L}e_2$$

$$F_2 = \frac{2EI}{L}e_1 + \frac{4EI}{L}e_2$$

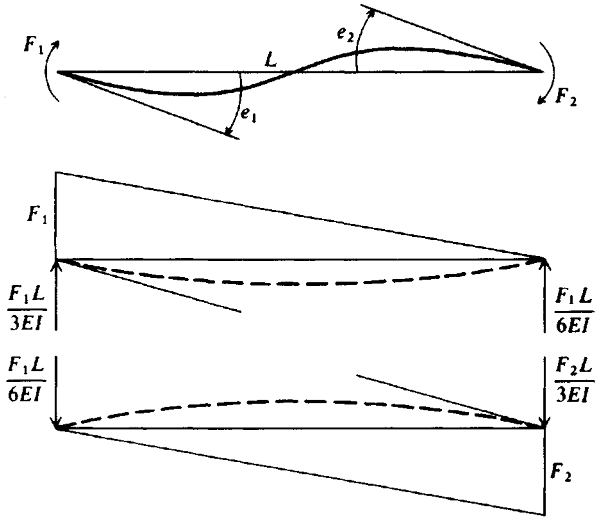


Figure 9-13 Conjugate-beam relationships between end moments and beam rotations.

The element S matrix then becomes

$$\mathrm{ES} = \begin{array}{c|c|c} F \backslash e & 1 & 2 \\ \hline 1 & \dfrac{4EI}{L} & \dfrac{2EI}{L} \\ \hline 2 & \dfrac{2EI}{L} & \dfrac{4EI}{L} \end{array}$$

Developing the Element $\mathrm{ESA^T}$ and $\mathrm{EASA^T}$ Matrices

The $\mathrm{ESA^T}$ matrix[4] is formed by multiplying the ES and the transpose of the EA matrix (in the computer program this is done in place by proper use of subscripting) as shown on the next page and noting that $\mathbf{A}^\mathrm{T}$ goes with **e** and **X**. The $\mathrm{EASA^T}$ is obtained in a similar manner[5] as shown opposite.

The node soil "spring" will have units of FL^{-1} obtained from the modulus of subgrade reaction and based on contributory node area. When k_s = constant, they can be computed as

$$K_1 = \frac{L_1}{2}Bk_s \qquad \text{and} \qquad K_2 = \frac{L_1 + L_2}{2}Bk_s$$

[4] The element arrays are prefixed with E to differentiate them from global arrays.

[5] There are several published methods to obtain the element stiffness matrix $\mathrm{EASA^T}$ (sometimes called K), including defining the 16 matrix entries directly. The method given here is easy to understand and program, but more importantly it produces the $\mathrm{ESA^T}$, which can be saved to compute element moments later.

Bowles (1974*a*) shows that *best results are obtained by doubling the end springs.* This was done to make a best fit of the measured data of Vesić and Johnson (1963) with computations. This is incorporated into the computer program on the diskette for beams.

$$\mathbf{EA}^{\mathrm{T}} = \begin{array}{c|c|c|c|c} e \backslash X & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1/L & 0 & -1/L \\ \hline 2 & 0 & 1/L & 1 & -1/L \end{array}$$

$$\mathbf{ES} = \begin{array}{c|c|c} F \backslash e & 1 & 2 \\ \hline 1 & \dfrac{4EI}{L} & \dfrac{2EI}{L} \\ \hline 2 & \dfrac{2EI}{L} & \dfrac{4EI}{L} \end{array} \qquad \mathbf{ESA}^{\mathrm{T}} = \begin{array}{|c|c|c|c|} \hline \dfrac{4EI}{L} & \dfrac{6EI}{L^2} & \dfrac{2EI}{L} & \dfrac{-6EI}{L^2} \\ \hline \dfrac{2EI}{L} & \dfrac{6EI}{L^2} & \dfrac{4EI}{L} & \dfrac{-6EI}{L^2} \\ \hline \end{array}$$

$$\mathbf{EA} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 1/L & 1/L \\ \hline 0 & 1 \\ \hline -1/L & -1/L \\ \hline \end{array} \qquad \mathbf{EASA}^{\mathrm{T}} = \begin{array}{|c|c|c|c|} \hline \dfrac{4EI}{L} & \dfrac{6EI}{L^2} & \dfrac{2EI}{L} & \dfrac{-6EI}{L^2} \\ \hline \dfrac{6EI}{L^2} & \dfrac{12EI}{L^3} + K_1 & \dfrac{6EI}{L^2} & \dfrac{-12EI}{L^3} \\ \hline \dfrac{2EI}{L} & \dfrac{6EI}{L^2} & \dfrac{4EI}{L} & \dfrac{-6EI}{L^2} \\ \hline \dfrac{-6EI}{L^2} & \dfrac{-12EI}{L^3} & \dfrac{-6EI}{L^2} & \dfrac{+12EI}{L^3} + K_2 \\ \hline \end{array}$$

There is some logic in end spring doubling (see also comments at end of Example 9-6), in that if higher edge pressures are obtained for footings, then this translates into "stiffer" end soil springs. For these matrices use $K_1 = L_2 B k_s$ and similarly for K_5 of Fig. 9-12.

From Fig. 9-12 we can see that summing vertical forces on a node (and using node 1 for specific illustration) gives

$$P_2 - \frac{F_1 + F_2}{L} - K_1 X_2 = 0$$

Since $(F_1 + F_2)/L$ is already included in the global $\mathbf{ASA}^{\mathrm{T}}$ we can rewrite the foregoing as

$$P_2 = (\mathbf{ASA}^{\mathrm{T}})_{2,2} X_2 + K_1 X_2 = [(\mathbf{ASA}^{\mathrm{T}})_{2,2} + K_1] X_2$$

or the node spring is directly additive to the appropriate diagonal [subscripted with (i, i)] term. This method is the most efficient way of including the soil springs since they can be built during element input into a "spring" array. Later the global $\mathbf{ASA}^{\mathrm{T}}$ is built (and saved for nonlinear cases) and the springs then added to the appropriate diagonal term (or column 1 of the banded matrix that is usually used).

A check on the correct formation of the $\mathbf{EASA}^{\mathrm{T}}$ and the global $\mathbf{ASA}^{\mathrm{T}}$ is that they are always symmetrical and there cannot be a zero on the diagonal. Note that the soil spring is an additive term to only the appropriate diagonal term in the global $\mathbf{ASA}^{\mathrm{T}}$ matrix. This allows easy removal of a spring for tension effect while still being able to obtain a solution, since there is still the shear effect at the point (not having a zero on the diagonal). This procedure has an additional advantage in that the $\mathbf{ASA}^{\mathrm{T}}$ does not have to be rebuilt for nonlinear soil effects if a copy is saved to call on subsequent cycles for nodal spring adjustments.

Developing the P Matrix

The **P** matrix (a column vector for each load case) is constructed by zeroing the array and then entering those node loads that are nonzero. The usual design problem may involve several different loading cases (or conditions), so the array is of the form $P_{i,j}$ where i identifies the load entry with respect to the node and P–X coding and j the load case. For example, refer to Fig. 9-12 where we have column loads at nodes 2 and 4 and two load cases ($J = 2$) as follows:

	Load case	
Column	**1**	**2**
1 (node 2)	140 kips↓	200 kips↓
1	100 ft · k ↷	110 ft · k ↶
2 (node 4)	200 kips↓	300 kips↓

Our nonzero **P** matrix entries would be (from the P–X coding diagram)

$P_{3,1} = 100$	$P_{3,2} = -110$	(moment entries)
$P_{4,1} = 140$	$P_{4,2} = 200$	(axial loads)
$P_{8,1} = 200$	$P_{8,2} = 300$	(also axial loads)

The loads acting in the same direction as the P–X coding have a (+) sign and opposed a (−) sign as for the second load case moment at column 1.

From the foregoing we see that it is necessary to know the P–X coding used in forming the **EA** matrix, or output may be in substantial error.

For columns that are intermediate between nodes, we may do one of two things:

1. Simply prorate loads to adjacent nodes using a simple beam model.
2. Prorate loads to adjacent nodes as if the element has fixed ends so the values include fixed-end moments and shears (vertical forces). This procedure is strictly correct, but the massive amount of computations is seldom worth the small improvement in computational precision.

Boundary Conditions

The particular advantage of the finite-element method is in allowing boundary conditions of known displacements or rotations. When the displacements are zero, the most expeditious method to account for them is to use P–X coding such that if NP = number of P–X codings of all the free nodes (thus, NP = 10 in Fig. 9-12) and we want to fix node 5 against both rotation and translation, we would identify

$$\mathrm{NP} = 8$$

and use P_9–X_9 for both rotation and translation P–X values at node 5 and instruct the computer that we have NP = 8. The program would then build a 9×9 array but only use the active 8×8 part. When inspecting the output we would, of course, have to know that node 5 has been specified to have zero displacements.

When displacements are of a known value (and including 0.0), a different procedure is required. Here the computer program must be set to allow known displacements. In this case have the program do the following (the computer program uses ASAT for $\mathbf{ASA}^{\mathrm{T}}$):

1. Put a 1 on the diagonal at the point of P–X coding (j, j).
2. Zero all the horizontal $\mathbf{ASA}^{\mathrm{T}}_{j,k}$ entries from $k = 1$ to n except $k = j$.
3. Insert the known displacement δ in the $\mathbf{P}$ matrix (so $\mathbf{P}_j = \delta$).
4. Augment all the other $\mathbf{P}$ matrix entries as

$$\mathbf{P}(I) = \mathbf{P}(I) - \mathbf{ASA}^{\mathrm{T}}_{i,j} \times \delta \qquad \text{for } i = 1 \text{ to NP except } i = j$$

Then set $\mathbf{ASA}^{\mathrm{T}}_{i,j} = 0 \qquad \text{for } i = 1, NP \qquad \text{except } i = j$

When this is done properly, we have an $\mathbf{ASA}^{\mathrm{T}}$ that has a horizontal and vertical row of zeros that intersect at (j, j), where there is a 1.0. The $\mathbf{P}$ matrix has been augmented everywhere except at $\mathbf{P}_j$, where there is the entry δ.

Alternatively, we can use the following (not particularly recommended) approach:

1. Multiply $\mathbf{ASA}^{\mathrm{T}}_{j,j}$ by a very large number N (say $N > 10^{10}$).
2. Replace $\mathbf{P}_j$ by $\mathbf{P}'_j = \mathbf{ASA}^{\mathrm{T}}_{j,j} \times N \times \delta$.

It is common in foundation design to have displacements that are known to be zero (beam on rock, beam embedded in an anchor of some type, etc.). Seldom do we have known displacements where $\delta \neq 0$ other than in "what if" studies.

Node Springs

All the author's finite-element programs using beam elements require concentrating the effect of k_s to the nodes as springs. The concentration method usually used is that suggested by Newmark (1943) for a general parabolic variation of k_s versus length. This method is exact for a parabolic curve and very nearly so for either a linear or cubic curve for k_s if the node spacings are not very large. The error is readily checked because the sum of the node springs (not considering any doubling or reduction of end springs) should equal the volume under the k_s curve. The equations given by Newmark (1943) include a derivation in the Appendix to his paper. For constant k_s the illustrations for K_1, K_2 previously given can be used, which are essentially average end area computations.

We can readily check the programming for the beam equations by referring to Example 9-6, which lists k_s and all node springs. The sum of the listed node springs is

$$\frac{11\,616}{2} + 11\,616 + 14\,520 + \cdots + 27\,588 + \frac{29\,040}{2} = 370\,550.4$$

The two end values were doubled in the program, since this was a beam. The volume of the k_s curve is

$$V = B \times L \times k_s$$

and, taking L = sum of element lengths = 6.38 m and B = 2.64 m, we obtain

$$V = 2.64 \times 6.38 \times 22\,000 = 370\,550$$

for very nearly an exact check. The reason for this close agreement with using a constant k_s is that the element lengths are rather short.

Spring Coupling

From a Boussinesq analysis it is evident that the base contact pressure contributes to settlements at other points, i.e., causing the center of a flexible uniformly loaded base to settle more than at the edges. Using a constant k_s on a rectangular uniformly loaded base will produce a constant settlement (every node will have the same ΔH within computer round-off) if we compute node springs based on contributing node area. This approach is obviously incorrect, and many persons do not like to use k_s because of this problem. In other words the settlement is "coupled" but the soil springs from k_s have not been coupled.

It is still desirable, however, to use k_s (some persons call this a *Winkler* foundation) in a spring concept because only the diagonal translation terms are affected. When we have true coupling, fractions of the springs K_i are in the off-diagonal terms, making it difficult to perform any kind of nonlinear analysis (soil-base separation or excessive displacements). We can approximately include coupling effects in several ways:

1. Double the end springs, which effectively increases k_s in the end zones. This approach is not applicable to the sides of very long narrow members.
2. Zone k_s with larger values at the ends that transition to a minimum at the center. This concept was illustrated in Example 9-5 where the center k_s was considerably smaller than the corner value.

For beam-on-elastic-foundation problems, where concentrated loads and moments are more common than a uniform load, doubling the end springs is probably sufficient coupling.

Finite Element Computer Program for Beam-on-Elastic Foundation

A computer program would develop the EA and ES for each finite element in turn from input data describing the member so that I, L, and computations (or read in) for K_1 and K_2 can be made. The program performs matrix operations to form the $\mathbf{ESA}^T$ and $\mathbf{EASA}^T$ and with proper instructions identifies the P–X coding so the $\mathbf{EASA}^T$ entries are correctly inserted into the global $\mathbf{ASA}^T$.

When this has been done for all the finite elements (number of members NM, a global $\mathbf{ASA}^{\mathrm{T}}$ of size NP $\times$ NP will have been developed as follows:

$$\mathbf{P}_{\mathrm{NP}} = \mathbf{A}_{\mathrm{NP}\times\cancel{\mathrm{NF}}}\mathbf{S}_{\cancel{\mathrm{NF}}\times\cancel{\mathrm{NF}}}\mathbf{A}^{\mathrm{T}}_{\cancel{\mathrm{NF}}\times\mathrm{NP}}\mathbf{X}_{\mathrm{NP}}$$

and canceling interior terms as shown gives

$$\mathbf{P}_{\mathrm{NP}} = \mathbf{ASA}^{\mathrm{T}}_{\mathrm{NP}\times\mathrm{NP}}\mathbf{X}_{\mathrm{NP}}$$

which indicates that the system of equations is just sufficient (that is, a square coefficient matrix, the only type that can be inverted). It also gives a quick estimate of computer needs, as the matrix is always the size of (NP $\times$ NP) where NP is the number of *P–X* codings. With proper coding (as in Fig. 9-12) the global $\mathbf{ASA}^{\mathrm{T}}$ is banded with all zeros except for a diagonal strip of nonzero entries that is eight values wide. Of these eight nonzero entries, four are identical (the band is symmetrical). There are matrix reduction routines to solve these types of half-band width problems. As a consequence the actual matrix required (with a band reduction method) is only **NP $\times$ 4** entries instead of NP $\times$ NP.

The $\mathbf{ASA}^{\mathrm{T}}$ is inverted (computer program FADBEMLP on the diskette reduces a band matrix) and multiplied by the **P** matrix containing the known externally applied loads. This step gives the nodal displacements of rotation and translation. The computer program then rebuilds the **EA** and **ES** to obtain the $\mathbf{ESA}^{\mathrm{T}}$ and, using Eq. (*d*), computes the element end moments. Node reactions R_i and soil pressures q_i are computed using

$$R_i = K_i X_i \qquad q_i = k_s X_i$$

It may be convenient to store the $\mathbf{ESA}^{\mathrm{T}}$ on a disk file when the $\mathbf{ASA}^{\mathrm{T}}$ is being built and recall it to compute the element end moments of the **F** matrix.

If the footing tends to separate from the soil or the deflections are larger than $X_{\max}$ it is desirable to have some means to include the footing weight, zero the soil springs where nodes separate, and apply a constant force to nodes where soil deflections exceed $X_{\max}$ of

$$P_i = -K_i(X_{\max})$$

Note the sign is negative to indicate the soil reaction opposes the direction of translation. Actual sign of the computed P matrix entry is based on the sign convention used in developing the general case as in Fig. 9-12.

A computer program of this type (FADBEMLP on your diskette) can be used to provide the output of Example 9-6 and can also be used to solve a number of structural problems by using 0.0 for k_s.

Example 9-6. Given the general footing and load data shown in Fig. E9-6*a*, assume the loads are factored and might be obtained from some kind of horizontal tank loading where the loads are from the tank supports and are the full width (2.64 m) of the footing. Take $k_s = LF \times k_s = 1.571 \times 14\,000 = 22\,000$ kN/m^3; also $f_c' = 21$ MPa $\rightarrow E_c = 21\,500$ MPa.

Comments based on Figs. E9-6b, c, and d.

1. The $\sum F_v \approx 0$ (spring forces $= 3374.7$ vs. 3375 kN input) and is within computer round-off using single precision with 6^+ digits.
2. For the far end of element 9 and near end of element 10,

$$\text{Moment difference} = 549.3 - 468.4 = 80.9 \text{ (81.0 input)}$$

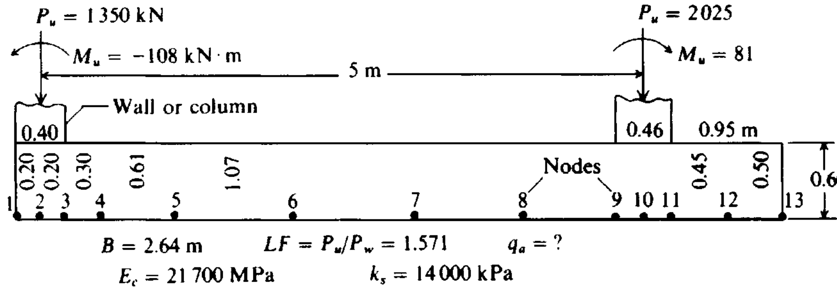


Figure E9-6*a*

3. The moments for the near end of element 1 and far end of element 12 should both be 0.0 (0.014 and −0.004).
4. If the largest soil pressure of 260.1/LF $\leq q_a$, the bearing pressure would be O.K. We must use an LF here since factored loads were input.
5. The largest node displacements are

$$\text{Translation} = 11.8 \text{ mm} \qquad \text{(at node 1)}$$
$$\text{Rotation} = -0.002\,53 \text{ rad} \qquad \text{(at nodes 1 \& 2)}$$

6. The output table of displacements from the disk plot file is used to plot the shear V and moment M diagrams shown in Fig. E9-6*c*. You should study these carefully and see how the output is interpreted—particularly at nodes with input moments. Compare the plots to the output checks shown in Fig. E9-6*d*. Refer also to the shear and moment plots of Fig. E13-1*g*.

Comments.

1. The author recently noted that Westergaard (1948) indicated that edge springs probably should be doubled. This suggestion probably did not receive the attention it should have because his observation was the last page of a "Discussion."
2. The question arises of whether one should double the edge springs or double k_s at the ends. Having checked both procedures, the author recommends doubling the edge springs for a beam-on-elastic-foundation problem. For mats one probably should double the edge k_s as that seems to give slightly better values over doubling edge springs. Doubling edge k_s for mats gives large computed edge node soil pressures that include both bearing and edge shear and may (incorrectly) give $q_i > q_a$.
3. There have been some efforts to use only one or two elements by integrating the modulus of subgrade reaction across the beam length. The author does not recommend this for three reasons:
   a. It is difficult to allow for nonlinear effects or for soil-footing separation.
   b. When using a nonlinear analysis with X_{max} the setting of a soil spring to zero introduces a discontinuity into the model. The discontinuity is minimized by using a number of closely spaced elements, the better to transition from the displacements $X > X_{max}$ and displacements $X \leq X_{max}$.
   c. It is difficult to produce a shear and moment diagram unless several elements are used. With the availability of computers, there is no justification to use a clever one-element model and have an enormous amount of hand computations to obtain the shear and moment diagrams.

   The author suggests that one should use a minimum of 10 elements for a beam—more for long beams or if it appears that any nonlinear zones are present.

```
DATA SET FOR EXAMPLE 9-6 SI-UNITS

    +++++++++++++++++ THIS OUTPUT FOR DATA FILE:  EXAM96.DTA


 SOLUTION FOR BEAM ON ELASTIC FOUNDATION--ITYPE = 0  +++++++


 NO OF NP =  26   NO OF ELEMENTS, NM = 12   NO OF NON-ZERO P, NNZP =  4
   NO OF LOAD CASES, NLC =  1                     NO OF CYCLES NCYC =  1
 NODE SOIL STARTS JTSOIL =  1
      NONLINEAR (IF > 0) =  1           NO OF BOUNDARY CONDIT NZX =  0
           MODULUS KCODE =  1                      LIST BAND IF > 0 =  0
                                                      IMET (SI > 0) =  1

 MOD OF ELASTICITY E =     21500. MPA

MEMNO   NP1    NP2    NP3    NP4   LENGTH      WIDTH    INERTIA, M**4

    1     1      2      3      4      .200     2.640    .47520E-01
    2     3      4      5      6      .200     2.640    .47520E-01
    3     5      6      7      8      .300     2.640    .47520E-01
    4     7      8      9     10      .610     2.640    .47520E-01
    5     9     10     11     12     1.070     2.640    .47520E-01
    6    11     12     13     14     1.070     2.640    .47520E-01
    7    13     14     15     16      .910     2.640    .47520E-01
    8    15     16     17     18      .610     2.640    .47520E-01
    9    17     18     19     20      .230     2.640    .47520E-01
   10    19     20     21     22      .230     2.640    .47520E-01
   11    21     22     23     24      .450     2.640    .47520E-01
   12    23     24     25     26      .500     2.640    .47520E-01

 THE INITIAL INPUT P-MATRIX ENTRIES
 NP     LC       P(NP,LC)
  3      1       -108.000
  4      1       1350.000
 19      1         81.000
 20      1       2025.000

 THE ORIGINAL P-MATRIX WHEN NONLIN > 0 ++++++
      1           .00           .00
      2       -108.00       1350.00
      3           .00           .00
      4           .00           .00
      5           .00           .00
      6           .00           .00
      7           .00           .00
      8           .00           .00
      9           .00           .00
     10         81.00       2025.00
     11           .00           .00
     12           .00           .00
     13           .00           .00

 THE NODE SOIL MODULUS, SPRINGS AND MAX DEFL:
 NODE     SOIL MODULUS     SPRING,KN/M   MAX DEFL, M
    1          22000.0      11616.0          .0500
    2          22000.0      11616.0          .0500
    3          22000.0      14520.0          .0500
    4          22000.0      26426.4          .0500
    5          22000.0      48787.2          .0500
```

Figure E9-6*b*

6	22000.0	62145.6	.0500
7	22000.0	57499.2	.0500
8	22000.0	44140.8	.0500
9	22000.0	24393.6	.0500
10	22000.0	13358.4	.0500
11	22000.0	19747.2	.0500
12	22000.0	27588.0	.0500
13	22000.0	29040.0	.0500

BASE SUM OF NODE SPRINGS = 370550.4 KN/M NO ADJUSTMENTS
* = NODE SPRINGS HAND COMPUTED AND INPUT

MEMBER MOMENTS, NODE REACTIONS, DEFLECTIONS, SOIL PRESSURE, AND LAST USED P-MATRIX FOR LC = 1

MEMNO	MOMENTS--NEAR END 1ST, KN-M		NODE	SPG FORCE, KN	ROT, RADS	DEFL, M	SOIL Q, KPA	P-, KN-M	P-, KN
1	.014	-27.486	1	137.35	-.00253	.01182	260.12	.00	.00
2	-80.742	297.008	2	131.47	-.00253	.01132	248.99	-108.00	1350.00
3	-297.074	574.550	3	157.02	-.00250	.01081	237.91	.00	.00
4	-574.568	976.292	4	266.45	-.00237	.01008	221.82	.00	.00
5	-976.300	1223.258	5	427.76	-.00190	.00877	192.89	.00	.00
6	-1223.256	983.240	6	455.11	-.00075	.00732	161.11	.00	.00
7	-983.243	404.543	7	411.62	.00040	.00716	157.49	.00	.00
8	-404.557	-194.635	8	346.31	.00102	.00785	172.60	.00	.00
9	194.540	-468.397	9	207.48	.00108	.00851	187.13	.00	.00
10	549.286	-384.339	10	116.85	.00101	.00875	192.45	81.00	2025.00
11	384.351	-141.243	11	177.07	.00090	.00897	197.27	.00	.00
12	141.230	-.004	12	257.77	.00079	.00934	205.56	.00	.00
			13	282.44	.00075	.00973	213.97	.00	.00

SUM SPRING FORCES = 3374.71 VS SUM APPLIED FORCES = 3375.00 KN

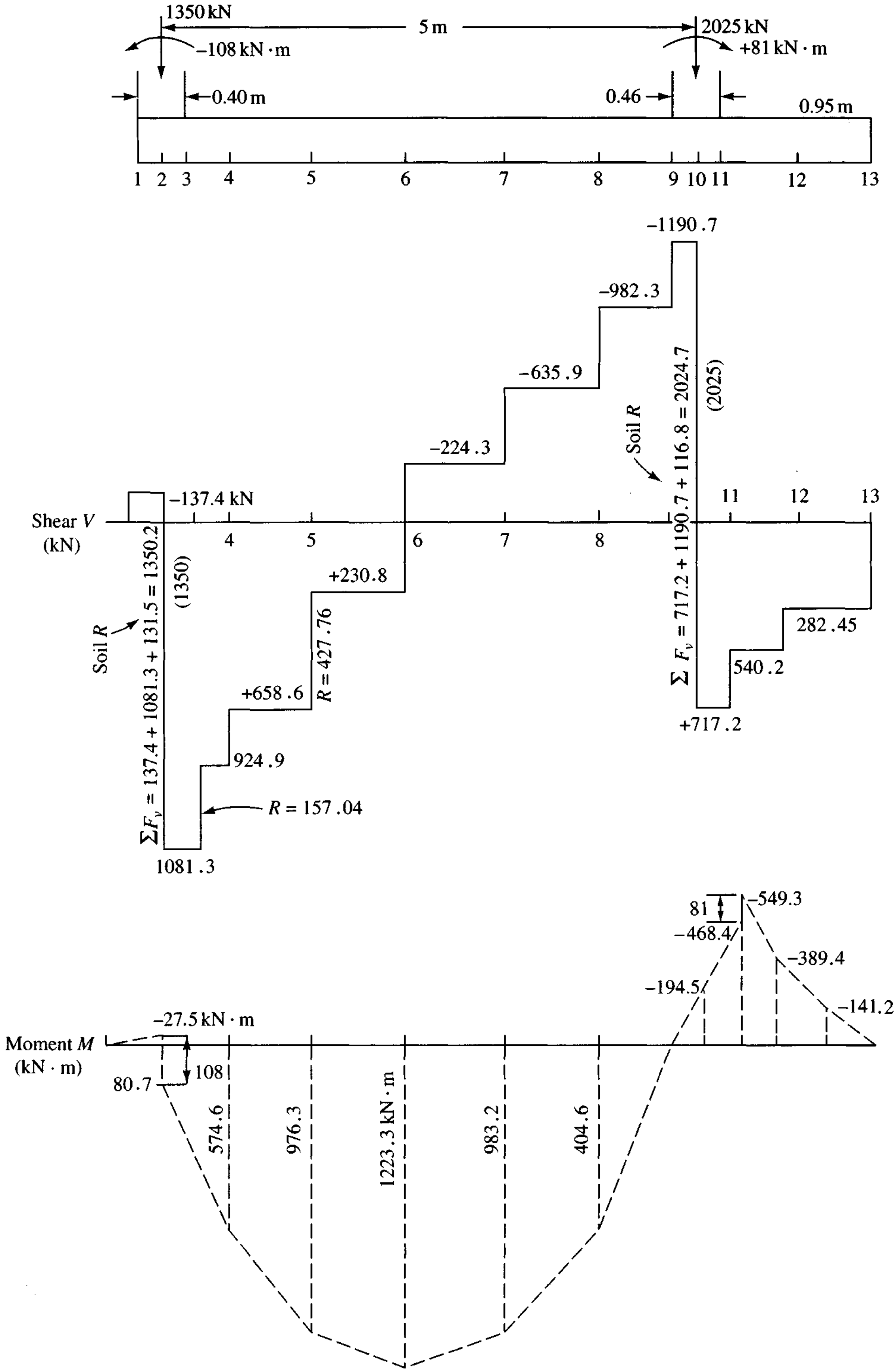
(*) = SOIL DISPLACEMENT > XMAX SO SPRING FORCE AND Q = XMAX*VALUE ++++++++++++
NOTE THAT P-MATRIX ABOVE INCLUDES ANY EFFECTS FROM X > XMAX ON LAST CYCLE ++++++++++

FOLLOWING IS DATA SAVED TO DATA FILE: BEAM1.PLT

REFER TO "READ" STATEMENT 2040 FOR FORMAT TO USE FOR PLOT PROGRAM ACCESS

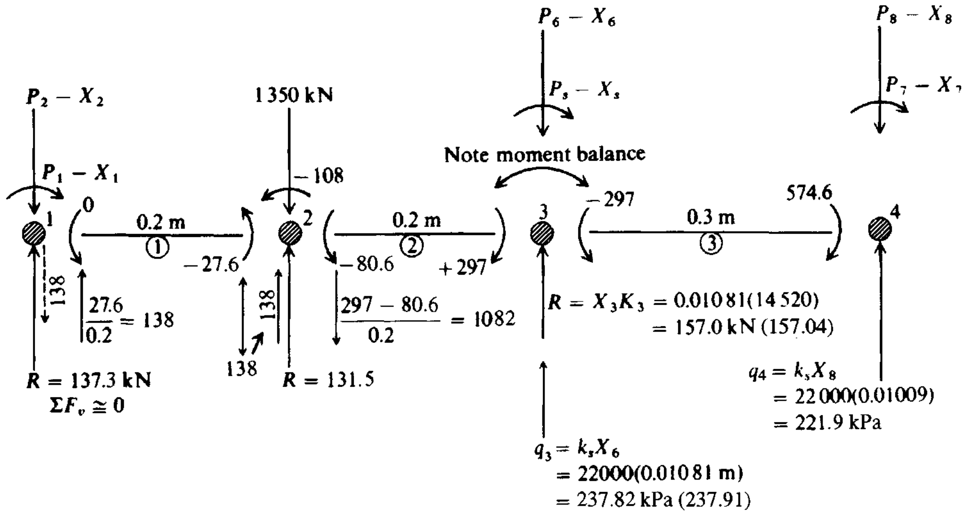
					SHEAR V(I,1),V(I,2)		MOMENT MOM(I,1),MOM(I,2)	
NODE	LENGTH	KS	COMP X,MM	XMAX	LT OR T	RT OR B	LT OR T	RT OR B
1	.000	22000.0	11.824	50.000	.00	-137.36	.0	.0
2	.200	22000.0	11.318	50.000	-137.36	1081.33	-27.5	80.7
3	.400	22000.0	10.814	50.000	1081.33	924.92	297.0	297.1
4	.700	22000.0	10.083	50.000	924.92	658.56	574.6	574.6
5	1.310	22000.0	8.768	50.000	658.56	230.80	976.3	976.3
6	2.380	22000.0	7.323	50.000	230.80	-224.31	1223.3	1223.3
7	3.450	22000.0	7.159	50.000	-224.31	-635.93	983.2	983.2
8	4.360	22000.0	7.846	50.000	-635.93	-982.28	404.5	404.6
9	4.970	22000.0	8.506	50.000	-982.28	-1190.68	-194.6	-194.5
10	5.200	22000.0	8.748	50.000	-1190.68	717.16	-468.4	-549.3
11	5.430	22000.0	8.967	50.000	717.16	540.24	-384.3	-384.4
12	5.880	22000.0	9.344	50.000	540.24	282.45	-141.2	-141.2
13	6.380	22000.0	9.726	50.000	282.45	.00	.0	.0

Figure E9-6*b* (*continued*)



1350 kN
5 m
2025 kN
−108 kN · m
+81 kN · m
0.40 m
0.46
0.95 m
1 2 3 4 5 6 7 8 9 10 11 12 13
−1190.7
−982.3
−635.9
−224.3
−137.4 kN
Shear V
(kN)
Soil R
ΣFv = 137.4 + 1081.3 + 131.5 = 1350.2
(1350)
+230.8
R = 427.76
+658.6
924.9
R = 157.04
1081.3
Soil R
Σ Fv = 717.2 + 1190.7 + 116.8 = 2024.7
(2025)
282.45
540.2
+717.2
81
−549.3
−468.4
−389.4
−194.5
−141.2
−27.5 kN · m
Moment M
(kN · m)
80.7
108
574.6
976.3
1223.3 kN · m
983.2
404.6

Figure E9-6c



Check node 2:

$\Sigma F_{v2} = 138 + 131.5 + 1082 = 1351.5 \cong 1350$ O.K.

$\Sigma M_2 = 108 - 27.6 - 80.6 \cong 0$ O.K.

$I = \dfrac{Bt^3}{12} = \dfrac{2.64(0.6)^3}{12} = 0.047\,520\ \text{m}^4$ moment of inertia of any element

$$\left.\begin{aligned} K_1 &= 22\,000\left(\frac{0.2}{2}\right)(2.64)(2) = 11\,616\ \text{kN/m} \\ K_2 &= 22\,000(0.2)(2.64) = 11\,616\ \text{kN/m} \end{aligned}\right\}\ \text{Soil spring computations for first two nodes}$$

R = node spring force

Large numbers in SI produce round-off error using single precision. Also computer values use more digits.

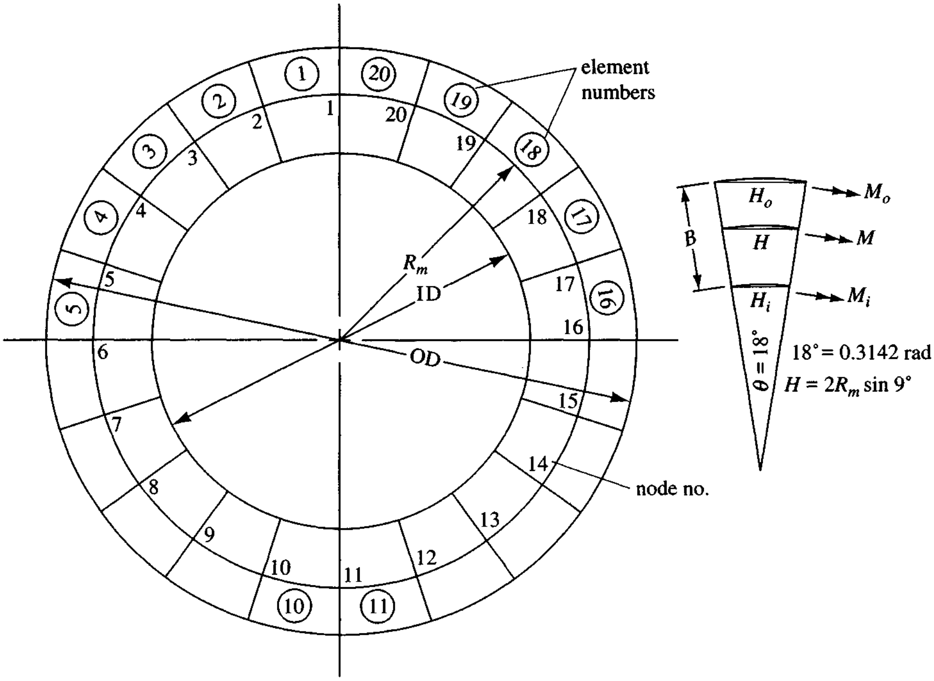
Figure E9-6*d*

////

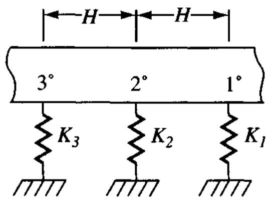
9-9 RING FOUNDATIONS

Ring foundations can be used for water tower structures, transmission towers, TV antennas, and to support various process tower superstructures. The ring foundation considered here is a relatively narrow circular beam as opposed to the circular mat considered in the next chapter.

The finite-element method (FEM) for a ring foundation is somewhat similar to the beam-on-elastic-foundation method. The node and element numbering are rather straightforward, as shown in Fig. 9-14. The computer program is considerably more lengthy since the *P–X* coding is somewhat different (see Fig. 9-15) in order to obtain a bandwidth of 9. A bandwidth of 60 is obtained if one proceeds in a continuous manner counterclockwise around the ring from node 1. The element **A** matrix is:



(*a*) Element and node numbering.

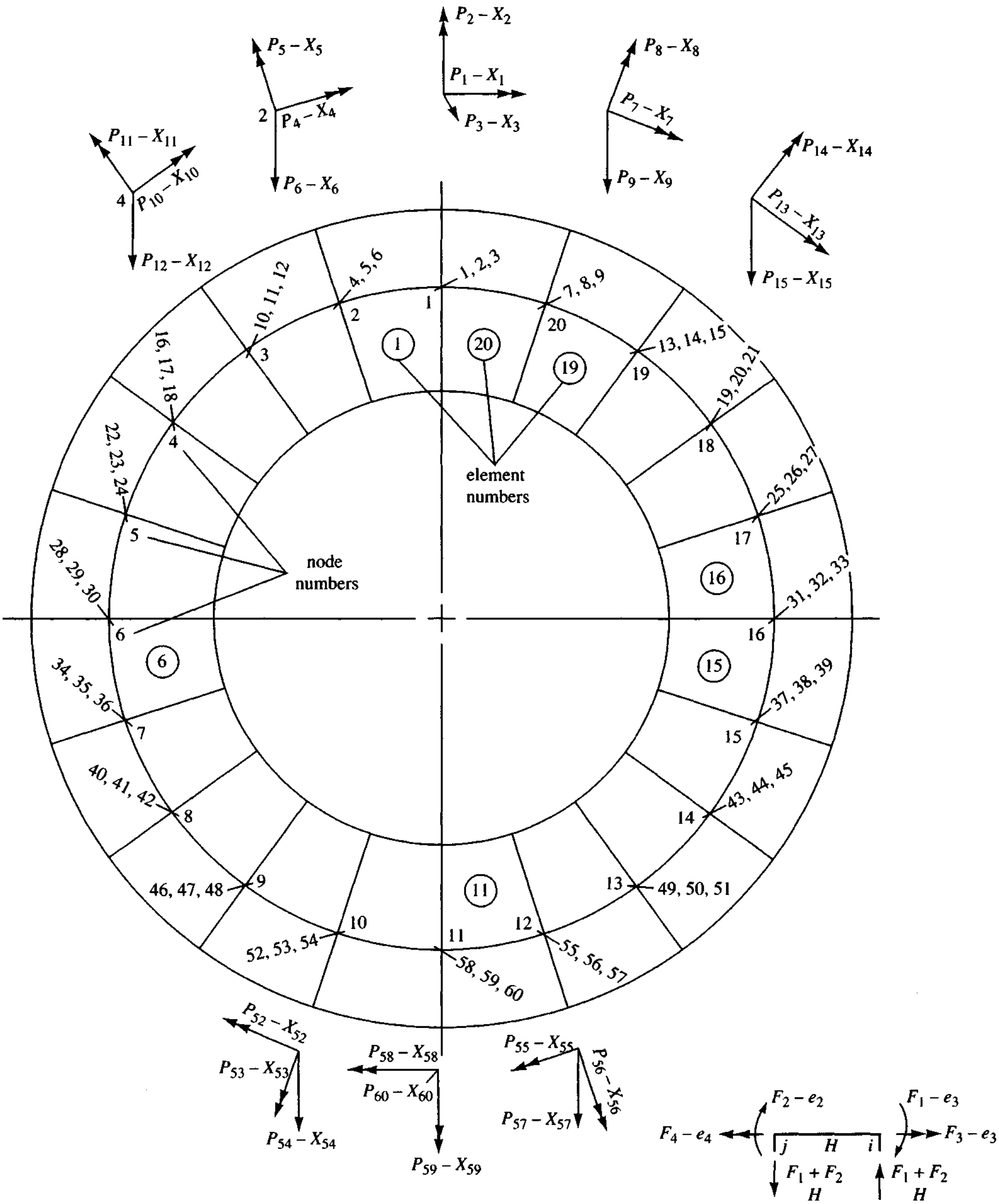


(*b*) Node springs $K = \dfrac{0.7854(\mathrm{OD}^2 - \mathrm{ID}^2)k_s}{20}$

Figure 9-14 Ring foundation configuration and definitions. Note that loads should be placed on mean radius R_m, which divides ring *area* in half, and not on average radius, which divides the ring *width* in half. Always orient your ring so node 1 is at top of page as shown here.

EA =

	1	2	3	4
1	$\sin a$	—	$\cos a$	—
2	$\cos a$	—	$-\sin a$	—
3	$-1/H$	$-1/H$	—	—
4	—	$\sin a$	—	$-\cos a$
5	—	$\cos a$	—	$\sin a$
6	$1/H$	$1/H$	—	—



(*a*) General $P - X$ coding. This coding scheme reduces the bandwidth from 60 to 6. For orientation, stand inside ring and look out along any radial line.

(*b*) General forces on element *i*. Note moment F_1 is on near end and F_2 is on far end. Also note far end of element 20 is the near end of element 1.

Figure 9-15 Ring foundation *P–X* coding and orientation and element forces.

where
$$\sin a = \sin 9^\circ$$
$$\cos a = \cos 9^\circ$$
$$H = 2R_m \sin 9^\circ$$

and, allowing for torsion, the element stiffness matrix is

ES =

$F \backslash e$	1	2	3	4
1	$\frac{4EI}{H}$	$\frac{2EI}{H}$	—	—
2	$\frac{2EI}{H}$	$\frac{4EI}{H}$	—	—
3	—	—	$\frac{GJ}{H}$	—
4	—	—	—	$\frac{GJ}{H}$

The usual matrix multiplications are carried out to produce the element $\mathbf{EASA}^{\mathrm{T}}$, which is then summed into the global $\mathbf{ASA}^{\mathrm{T}}$ matrix, which is then banded and reduced to produce the nodal displacements X_i. The displacements are then used to compute the element forces (moments and shears), soil reactions, and pressures.

To avoid twisting and for a theoretical uniform displacement across the radial line defining any node, one should place the loads on the mean radius R_m, defining the center of area and computed as

$$R_m = \sqrt{\frac{\mathrm{ID}^2 + \mathrm{OD}^2}{8}},$$

rather than on the arithmetic average radius,

$$R_a = \frac{(\mathrm{ID} + \mathrm{OD})}{4}.$$

The moments are computed at the center of area defined by the mean radius R_m (see Fig. 9-14*a*) so that the displacements can be assumed to be constant across the ring radius at the node. Since the inner and outer element lengths are different but with the same end displacements, there should be a different moment according to the central finite-difference expression given as

$$M = \frac{EI}{\Delta x^2}(y_{n+1} - 2y_n + y_{n-1})$$

Replacing Δx^2 by ΔH^2 we can readily see the moment at the inner radius defined by the ID is larger than at the mean radius, and the outer moment at the radius defined by the OD is smaller than the mean radius value. We can adjust for these values as follows:

$$M_i = \left(\frac{2R_m}{\mathrm{ID}}\right)^2 M_m \qquad \text{and} \qquad M_o = \left(\frac{2R_m}{\mathrm{OD}}\right)^2 M_m$$

where M_m = computed value on computer output sheets, and the interior moment M_i and exterior moment M_o can be computed using the preceding expressions.

The finite-element length H is taken as the chord distance and differs slightly from the arc length L_a as follows:

$$L_a = R_m \times 0.31416 \qquad H = 2 \times R_m \sin 9^\circ = R_m \times 0.31287$$

The node springs (see Fig. 9-14*b*) are computed using a constant value of modulus of subgrade reaction k_s as

$$K_i = \frac{0.7854(\mathrm{OD}^2 - \mathrm{ID}^2)k_s}{20}$$

However, one may input springs for selected nodes in the computer program.

The solution of a ring foundation will be illustrated by Example 9-7, using program B-17, described in the README.DOC file on your diskette.

Example 9-7. Find the bending moments and other data for a ring foundation given the following:

ID = 14.5 m OD = 16.0 m D_c = 0.76 m

E_c = 22 400 MPa k_s = 13 600 + 0Z^1 kN/m^3

(Assume Poisson's ratio of concrete μ = 0.15)

Three equally spaced (120°) loads of 675 kN each

Tangential moment = +200 kPa at node 1 (+) using the right-hand rule (based on the P–X coding of Fig. 9-15)

Consider the ring foundation to be *weightless* (although the computer program allows the input of the unit weight of the beam material and will then compute a weight contribution for each node).

Solution.

Step 1. We will put one load on node 1 and the other two will fall on element 7 and on element 14 (as shown on Fig. E9-7*a*). The loads on elements 7 and 14 will have to be prorated to adjacent nodes. We can use either L_a or H for the prorating. Using L_a, we write

$$R_m = \sqrt{\frac{14.5^2 + 16^2}{8}} = 7.634 \text{ m}$$

$$[\text{Average } R_a = (14.5 + 16)/4 = 7.625 \text{ m} < 7.634]$$

$$L_a = 7.634(0.31416) = 2.398 \text{ m}$$

Load location = 120° − 6 × 18° = 120° − 108° = 12° into element 7, which is exactly two-thirds of the length (either L_a or H), that is,

$$\frac{2}{3}L_a = \frac{2}{3}(2.398) = 1.599 \text{ m} \qquad \frac{L_a}{3} = \frac{2.398}{3} = 0.799 \text{ m}$$

The column loads are entered in the vertical P's so that

$$\text{Node 7: } P_{36} = \frac{0.799}{2.398}(675) \approx 224.6 \text{ kN}$$

$$\text{Node 8: } P_{42} = \frac{1.599}{2.398}(675) \approx \underline{450.4} \text{ kN}$$

$$\text{Total} = 675.0 \text{ kN}$$

$$\text{Node 14: Same as node 8} \rightarrow P_{45} = 450.4 \text{ kN}$$

$$\text{Node 15: Same as node 7} \rightarrow P_{39} = 224.6 \text{ kN}$$

$$\text{Node 1: Moment gives } P_1 = 200 \text{ kN} \cdot \text{m}$$

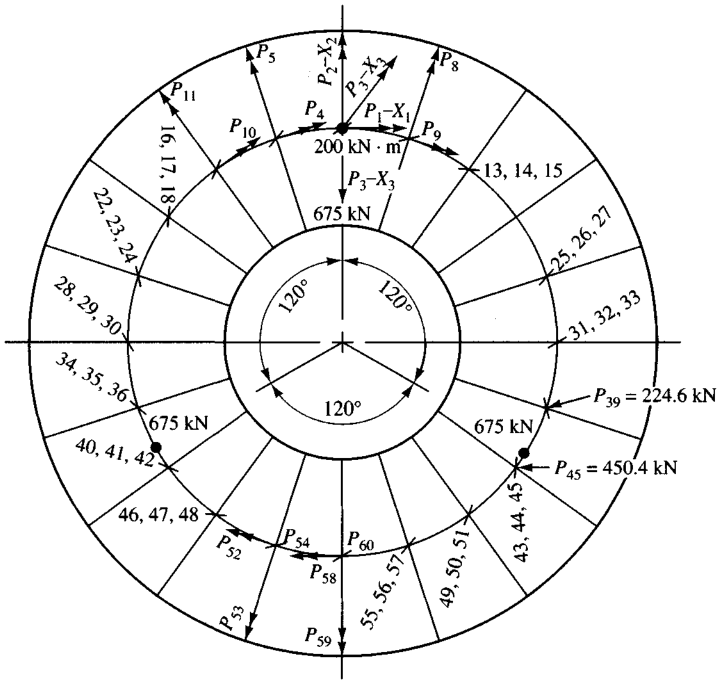


Figure E9-7*a*

These data are shown on Fig. E9-7*b* (computer output pages) where the P matrix is listed.

Step 2. Check the output. The output is partially self-checking. Note that the program converts E_s from MPa to kPa and computes the shear modulus

$$G_c' = \frac{E_c}{2(1+\mu)} = \mathbf{9\,739\,130}\text{ kPa}$$

1. First check that the sum of input vertical forces = sum of soil springs (the program sums the spring forces).
2. Since the loads are symmetrical and there is a moment only at node 1, there should be some symmetry in the soil springs (which also represents symmetry in the translation displacements).
3. All of the soil springs should be equal (unless some were input (not done here)). The springs should be

$$K_i = \frac{0.7854(16^2 - 14.5^2)(13\,600)}{20} = \mathbf{24\,433.8}\text{ kN/m}$$

(The computer value of 24 433.72 uses more digits, but in single precision.)

4. The program computes the moment of inertia using a beam with $b = (16 - 14.5)/2 = \mathbf{0.75}$ m as

$$I_i = \frac{bt^3}{12} = \frac{0.75(0.76^3)}{12} = \mathbf{0.027\,436}\text{ m}^4$$

The torsion inertia J for a rectangle is computed in the program as:

$$J = bt^3\left[\frac{1}{3} - 0.21\frac{t}{b}\left(1 - \frac{t^4}{12b^4}\right)\right],\ \text{m}^4$$

t = thickness, b = width of rectangle, and $t < b$.

```
+++++++++++ NAME OF DATA FILE USED FOR THIS EXECUTION:  EXAM97A.DTA

        EXAMPLE 9-7 RING FOUNDATION OF FAD 5/E--SI UNITS

      INPUT CONTROL PARAMETERS:
         NO OF P-MATRIX ENTRIES, NNZP =  6
                NO OF LOAD CASES, NLC =  1
          NO OF BOUND CONDITIONS, NZX =  0
      NO OF INPUT SOIL SPRINGS, ISPRG =  0
                      NONLIN (IF >0) =  1       IMET (SI>0) =  1


      THE ELEMENTS AND NPE(I):
      ELEM NO           NPE(I)
          1          1    2    3    4    5    6
          2          4    5    6   10   11   12
          3         10   11   12   16   17   18
          4         16   17   18   22   23   24
          5         22   23   24   28   29   30
          6         28   29   30   34   35   36
          7         34   35   36   40   41   42
          8         40   41   42   46   47   48
          9         46   47   48   52   53   54
         10         52   53   54   58   59   60
         11         58   59   60   55   56   57
         12         55   56   57   49   50   51
         13         49   50   51   43   44   45
         14         43   44   45   37   38   39
         15         37   38   39   31   32   33
         16         31   32   33   25   26   27
         17         25   26   27   19   20   21
         18         19   20   21   13   14   15
         19         13   14   15    7    8    9
         20          7    8    9    1    2    3

      RING FOUNDATION DATA AS FOLLOWS:
                     DIAMETER: OD =  16.000     ID =  14.500  M
                   RING DEPTH, DC =    .760  M
                   UNIT WT OF FTG =    .000 KN/M*3
                 SOIL MODULUS, SK = 13600.00 KN/M*3
       MAX LINEAR SOIL DEFL, XMAX =  .02000  M
                 MOD OF ELAS CONC = 22400000. KPA
                    POISSON RATIO =  .150
                SHEAR MODULUS, GC =  9739130. KPA

      SELECTED COMPUTED VALUES:
           MOM OF INERTIA: XI = .27436E-01     XJ = .45670E-01   M**4
             NODE SOIL SPRING = 24433.72 KN/M
              MEAN RADIUS, RM =  7.634  M
               ELEMENT: WIDTH =  .750
              W/LENGTHS  ARC = 2.398   CHORD =  2.389  M
             TOTAL RING AREA =   35.932  M**2
          MOMENT RATIOS:  RO =  .9106  RI = 1.1088


      FOR CYCLE =  1
         IF NCYC = 1 OUTPUT ORIGINAL P-MATRIX AND SPRING ARRAY
         IN NCYC > 1 OUTPUT MODIFIED P-MATRIX AND SPRING ARRAY
```

Figure E9-7*b* *(continued on next page)*

5. The nonlinear routines are not activated since X_{max} (XMAX) was set at 0.02 m (20 mm) and the largest displacement, at node 1 (as expected with a full 675 kN located at the point), is 0.007 93 m (7.93 mm).
6. With a symmetrical load and no radial moments the radial rotation at nodes 1 and 11 are both 0.000 00 as expected.
7. Note that even though the node coding is somewhat mixed, the node and element order is recovered for the output. This result makes it easy to check input node springs. Both input node springs and displacements greater than XMAX are identified on the output sheets.

THE P-MATRIX FOR NLC = 1

NODE		TANGEN MOM		RADIAL MOM		VERT P,KN	SPRING,KN/M
1	1	200.000	2	.000	3	675.000	24433.72
2	4	.000	5	.000	6	.000	24433.72
3	10	.000	11	.000	12	.000	24433.72
4	16	.000	17	.000	18	.000	24433.72
5	22	.000	23	.000	24	.000	24433.72
6	28	.000	29	.000	30	.000	24433.72
7	34	.000	35	.000	36	224.600	24433.72
8	40	.000	41	.000	42	450.400	24433.72
9	46	.000	47	.000	48	.000	24433.72
10	52	.000	53	.000	54	.000	24433.72
11	58	.000	59	.000	60	.000	24433.72
12	55	.000	56	.000	57	.000	24433.72
13	49	.000	50	.000	51	.000	24433.72
14	43	.000	44	.000	45	450.400	24433.72
15	37	.000	38	.000	39	224.600	24433.72
16	31	.000	32	.000	33	.000	24433.72
17	25	.000	26	.000	27	.000	24433.72
18	19	.000	20	.000	21	.000	24433.72
19	13	.000	14	.000	15	.000	24433.72
20	7	.000	8	.000	9	.000	24433.72

DISPLACEMENT MATRIX FOR CYCLE = 1 AND NLC = 1

NODE	X1	TANGENT	X2	RADIAL	X3	VERTICAL
1	1	.00002	2	.00000	3	.00793
2	4	-.00004	5	.00131	6	.00595
3	10	.00014	11	.00108	12	.00292
4	16	.00019	17	.00028	18	.00127
5	22	.00017	23	-.00059	24	.00164
6	28	.00008	29	-.00120	30	.00385
7	34	-.00020	35	-.00091	36	.00666
8	40	-.00036	41	.00045	42	.00738
9	46	.00002	47	.00128	48	.00491
10	52	.00016	53	.00084	54	.00226
11	58	.00019	59	.00000	60	.00124
12	55	.00016	56	-.00084	57	.00226
13	49	.00002	50	-.00128	51	.00491
14	43	-.00036	44	-.00045	45	.00738
15	37	-.00020	38	.00091	39	.00666
16	31	.00008	32	.00120	33	.00385
17	25	.00017	26	.00059	27	.00164
18	19	.00019	20	-.00028	21	.00127
19	13	.00014	14	-.00108	15	.00292
20	7	-.00004	8	-.00131	9	.00595

ELEMENT MOMENTS (KN-M) AND OTHER COMPUTED DATA FOR LC = 1

ELEM #	F(1)	F(3)*	F(2)	F(4)*	SHEAR, KN
1	-619.876	3.068	45.214	46.367	-240.595
2	-57.329	30.126	-170.079	5.180	-95.209
3	160.153	57.484	-217.019	-25.966	-23.808
4	214.420	42.368	-197.204	-48.704	7.208
5	202.603	14.619	-89.486	-50.005	47.359
6	100.559	-19.905	237.422	10.376	141.503
7	-229.008	-63.500	419.104	79.895	79.588
8	-423.281	-53.526	-31.873	33.073	-190.561
9	20.093	41.304	-188.497	-4.847	-70.506
10	180.769	53.639	-217.039	-34.376	-15.185
11	217.039	34.376	-180.768	-53.639	15.186
12	188.497	4.847	-20.092	-41.304	70.506
13	31.873	-33.073	423.282	53.526	190.561
14	-419.104	-79.895	229.009	63.500	-79.588
15	-237.422	-10.376	-100.559	19.905	-141.503
16	89.487	50.005	-202.603	-14.619	-47.359
17	197.205	48.704	-214.421	-42.368	-7.208
18	217.019	25.966	-160.153	-57.484	23.808
19	170.079	-5.180	57.330	-30.126	95.210
20	-45.215	-46.367	619.876	-3.068	240.595

* = TORSION MOMENT (MAY NOT BE ZERO WHEN INPUT MOMENTS)
+++ MOMENTS ABOVE AT MEAN RADIUS RM USE RO*F(I) AND
RI*F(I) FOR OUTSIDE AND INSIDE VALUES

Figure E9-7*b* *(continued)*

```
NODE SOIL DATA AND DISPLACEMENTS FOR NLC =  1
                                                     ROTAT., RADS
NODE       SOIL Q,KPA  DISPV, M    SPRING R,KN     RADIAL    TANGENT.
 1   3       107.88    .007932       193.81       .000000    .000017
 2   6        80.92    .005950       145.39       .001313   -.000044
 3  12        39.74    .002922        71.40       .001076    .000142
 4  18        17.26    .001269        31.02       .000282    .000186
 5  24        22.35    .001643        40.15      -.000585    .000172
 6  30        52.40    .003853        94.14      -.001200    .000082
 7  36        90.55    .006658       162.68      -.000912   -.000201
 8  42       100.33    .007377       180.25       .000453   -.000363
 9  48        66.82    .004913       120.05       .001277    .000022
10  54        30.79    .002264        55.32       .000837    .000159
11  60        16.90    .001243        30.37       .000000    .000187
12  57        30.79    .002264        55.32      -.000837    .000159
13  51        66.82    .004913       120.05      -.001277    .000022
14  45       100.33    .007377       180.25      -.000453   -.000363
15  39        90.55    .006658       162.68       .000912   -.000201
16  33        52.40    .003853        94.14       .001200    .000082
17  27        22.35    .001643        40.15       .000585    .000172
18  21        17.26    .001269        31.02      -.000282    .000186
19  15        39.74    .002922        71.40      -.001076    .000142
20   9        80.92    .005950       145.39      -.001313   -.000044

* = NON-LINEAR SOIL SPRING FORCE FOR XMAX*SPRNG1(I)

THE SUM OF INPUT VERTICAL LOADS (INCL FTG WT) =   2025.00
                COMPUTED SOIL SPRING REACTIONS =   2025.00 KN
IF INPUT SUM EQUALS COMPUTED SUM YOU HAVE A STATICS CHECK
   CHECK NODE SOIL PRESSURE Q <= QALLOW
```

8. One should never accept FEM output as correct without at least some internal checks. Here Fig. E9-7*c* (next page) illustrates checking nodes 1 and 11 for statics ($\sum M = 0$ and $\sum F_v = 0$). To orient the moments and end shears, you should be inside the ring and look outward at the element or node of interest. Element end shears are computed (but watch the signs—both the moments and shear direction have one) as $(F_1 + F_2)/H$. For any element the shear at each end is the same but reversed in direction (refer also to Figs. 9-11, 9-12, and 9-15*b*). For element 1 we have the numerical value of the shear using $H = 2(7.634)\sin 9° = 2.388$ m as

$$V = \frac{-619.876 + 45.214}{2.388} = -240.6 \text{ kN } (-240.595 \text{ computer})$$

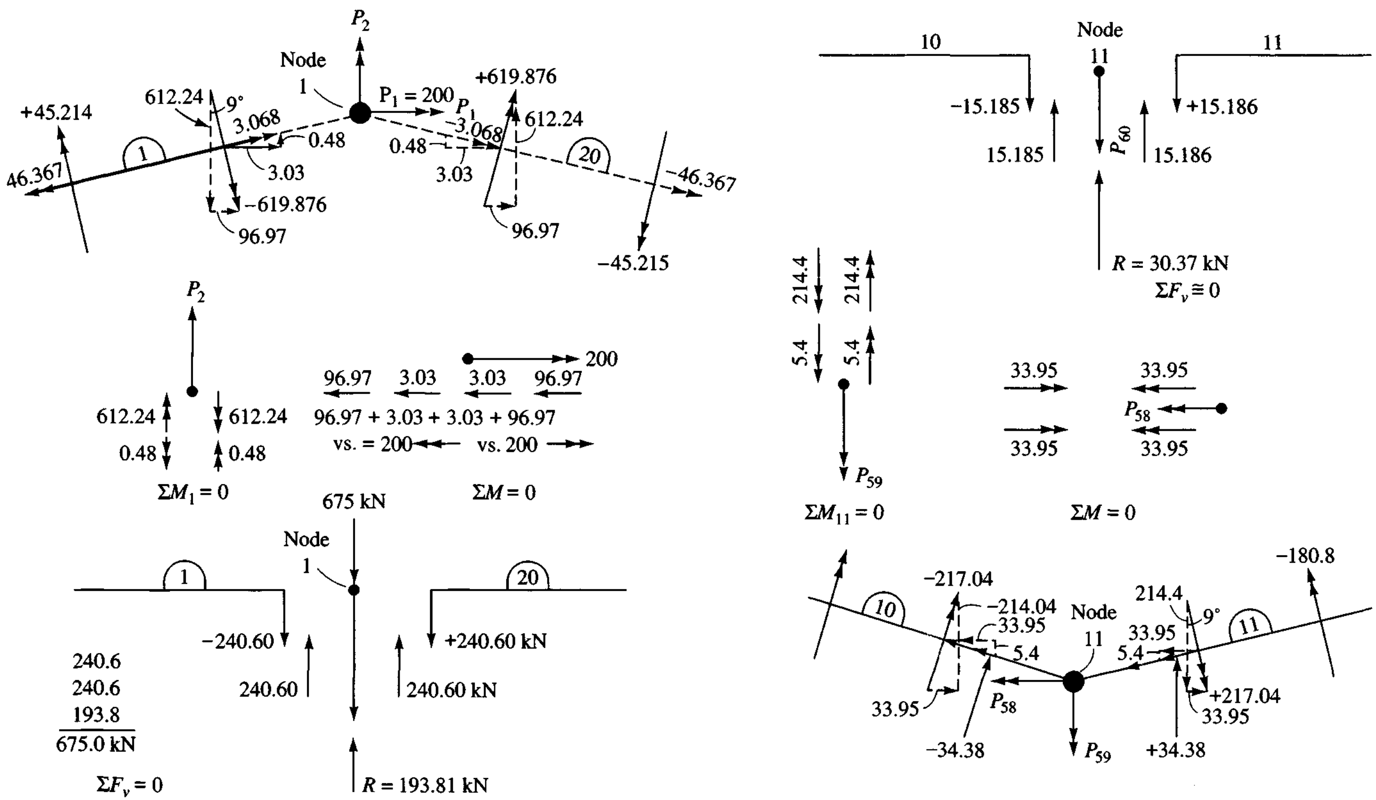
Although the foregoing came out (−), you must look at the element to assign the correct direction (up or down).

Comment. For design one might use the computed displacements from an analysis such as this, depending on how much confidence the user has in the value of k_s. Many designers use some alternative method for computing settlements that often includes both "immediate" and "consolidation" settlement components. A method for "immediate" settlements was illustrated in Example 5-13 using a case history.

////

9-10 GENERAL COMMENTS ON THE FINITE-ELEMENT PROCEDURE

Strictly, the finite-element model used in this chapter should be termed a *beam-element* model. It is a *beam-column* model when axial forces are included as a part of the element force model. The finite-element method is practical only when written into a computer program, because there are usually too many equations for hand solving. The following comments are



P2
Node
1
P1 = 200
P1
+45.214
612.24
9°
3.068
0.48
3.03
46.367
-619.876
96.97
+619.876
612.24
-3.068
0.48
3.03
96.97
20
1
-46.367
-45.215
612.24
612.24
0.48
0.48
ΣM1 = 0
96.97
3.03
3.03
96.97
200
96.97 + 3.03 + 3.03 + 96.97
vs. = 200
vs. 200
ΣM = 0
675 kN
Node
1
-240.60
+240.60 kN
240.60
240.60 kN
240.6
240.6
193.8
675.0 kN
ΣFv = 0
R = 193.81 kN
10
Node
11
11
-15.185
+15.186
15.185
P60
15.186
R = 30.37 kN
ΣFv ≅ 0
214.4
214.4
5.4
5.4
P59
ΣM11 = 0
33.95
33.95
P58
33.95
33.95
ΣM = 0
-180.8
-217.04
-214.04
33.95
5.4
Node
11
214.4
9°
33.95
5.4
+217.04
33.95
33.95
P58
-34.38
P59
+34.38

Figure E9-7c

observations made from solving a large number of different problems using the finite-element method.

1. One must always check finite-element program output. A finite-element computer program should be somewhat self-checking. This is accomplished by echoing back the input and comparing sums of input versus output forces.
   a. Carefully check the input data for correct dimensions, elastic properties, and units.
   b. Check the $\sum M = 0$ at nodes and the sum of soil reactions equal to applied loads ($\sum F_v = 0$). Note how applied moments were treated in Examples 9-6 and 9-7. Also in these examples observe that select nodes were given statics checks.
   c. When the program seems to have been working and a new problem gives obviously incorrect output, compare the *P–X* coding to be sure you are inputting the loads with the correct signs.
2. One should use at a minimum 8 to 10 finite elements, but it is not usually necessary to use more than 20. The number of finite elements used (NM) depends on the length of the member. Also more elements (and closely spaced) are needed if you consider soil nonlinearity or have shear and moment diagrams to plot.
3. One should not use a very short element next to a long element. Use more finite elements and effect a transition between short and long members. Try to keep the ratio

$$\frac{L_{\text{long}}}{L_{\text{short}}} \leq 2 \text{ and not more than } 3.$$

4. The value of k_s directly affects the deflection but has very little effect on the computed bending moments—at least for reasonable values of k_s such as one might obtain from using $k_s = 40(\text{SF})q_a$ (or $k_s = 12(\text{SF})q_a$). If one must obtain accurate displacements one must input a good estimation of k_s.

There are a large number of published solutions claimed by their authors to be better than the simple one proposed here for the beam-on-elastic foundation. A recent claim [Chiwanga and Valsangkar (1988)] has a reference list that may be of some value. Generally, these solutions require additional soil data (which are usually estimated) or obscure soil parameters that are not clearly defined. As a consequence, the solution that is the simplest and requires the minimum of soil properties is going to be the best one—regardless of claims to the contrary. After all, if one must guess at soil values, keep it simple.

As previously stated, a number of beam-on-elastic-foundation solutions claim to allow the user to model the foundation with one or two elements. This is too few for practical purposes in general and too few for realism when including nonlinear effects or where critical values of shear or moment are required for plotting shear and moment curves. In using one or two elements in the beam model these authors do some form of integration, so the result is often a difficult equation containing hyperbolic (and sometimes Bessel and/or Hankel) functions and with strange symbols. These models seldom have any provision for soil-footing separation or for modeling the case where the displacements $X > X_{\max}$.

A note of caution—since k_s is usually estimated—is that the use of refined methods may give undeserved confidence in the computed results.

The only ring solutions known to the author are a closed-form procedure given by Voltera (1952) and Voltera and Chung (1955) and the finite-element method given in the preceding section.

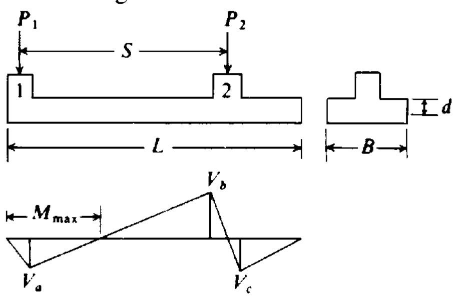
PROBLEMS

TABLE P9-1

				Loads				
Prob.	**Col No.**	**Col Size**	**Spacing S**	**DL**	**LL**	f'_c	f_y	**Allow soil q_a**
a	1	12 in.		100 kip	60			
	2	14	16 ft	160	80	3.0 ksi	50	2.0 ksf
b	1	340 mm		580 kN	310			
	2	380	4.85 m	670	425	21 MPa	400	175 kPa
c	1	340 mm		400 kN	720			
	2	380	5.50 m	780	440	21	350	145
d	1	440 mm		720 kN	890			
	2	440	6.10 m	1120	900	21	400	150

Units: Column Size = in. or mm DL, LL = kips or kN
f'_c = ksi or MPa f_y = ksi or MPa
q_a = ksf or kPa Column spacing S = ft or m

9-1. Design a continuous rectangular footing for the conditions shown in Fig. P9-1 using the assigned data given in Table P9-1 and the method of Sec. 9-2.



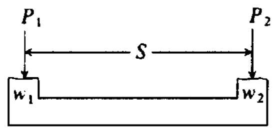


Figure P9-1

Figure P9-2

9-2. Proportion a trapezoidal-shaped footing using the assigned problem data in Table P9-2, as identified on Fig. P9-2. Draw the shear and moment diagrams.

TABLE P9-2

			Loads			
Prob.	**Col**	**Col Size**	**DL**	**LL**	**Allow soil q_a**	**Col spacing S**
a	1	22 in.	250	200 kips		
	2	18	180	150	4.0 ksf	20.0 ft
b	1	18 in.	180	170 kips		
	2	18	150	110	3.0	15.0 ft
c	1	500 mm	1400	1250 kN		
	2	480	1150	700	120 kPa	5.20 m
d	1	500 mm	2020	1100 kN		
	2	480	1125	1150	195 kPa	4.90 m

Units: Column size: in. or mm DL, LL = kips or kN q_a = ksf or kPa
Column spacing S = ft or m

9-3. Design the trapezoid footing for which the shear and moment diagrams were drawn in Prob. 9-2. Use $f_c' = 21$ MPa or 3 ksi; $f_y = 400$ MPa or 60 ksi.

9-4. What would the dimensions of the two footings of Example 9-3 (strap footing design) be if you used e = value assigned by the instructor (half the class should use 1.0 m and half use 1.4 m) instead of 1.2 m that was used in the example? Compute the volume of concrete for the two footings in Example 9-3 and for your value of e. Swap values of concrete volume with the other group, and for the three points plot e versus concrete volume and see if there might be an optimum e.

9-5. Proportion a strap footing for the following conditions:

	DL	LL
w_1 = 16in. + 6 in. edge distance	50	65 kips
w_2 = 16in.	85	60 kips

9-6. Design d and A_s for footings, and strap for Prob. 9-5. Use $f_c' = 3$ and $f_y = 60$ ksi. Make strap moment of inertia I at least two times I of footing $(BD_c^3/12)$.

9-7. Proportion a strap footing for the following conditions:

	DL	LL
w_1 = 400 mm + 150 mm edge dist	190 kN	300 kN
w_2 = 420 mm	385 kN	270 kN

9-8. Take $f_c' = 21$ and $f_y = 400$ MPa; and design d, A_s, and strap for Prob. 9-7. Make strap moment of inertia I at least two times I of footing $(BD_c^3/12)$.

9-9. Check if $D_c = 0.560$ m is an adequate total depth for the octagon footing for the process tower of Example 9-4.

9-10. Reproportion the octagon footing of Example 9-4 if $q_a = 120$ kPa and the importance factor $I = 1.0$ (instead of 1.15 of the example). Find A_s and make a neat drawing showing how you would place the reinforcing bars.

Modulus of subgrade reaction, k_s

9-11. Referring to Example 9-5, compute k_s for a midside and the $\frac{1}{4}$ and $\frac{1}{8}$ points along the midside to center of the base. Using these three points + the center point of the example, make a plot of k_s versus location and comment on its shape. How close is the edge k_s value to double that of the center?

9-12. Estimate k_s for a soil with $\phi = 34°$ and $c = 25$ kPa.

9-13.* Estimate k_s for the soil of Prob. 3-10.

9-14.* Estimate k_s for the soil of Prob. 3-11.

9-15.* Estimate k_s using the dilatometer data of Prob. 3-14.

* Since there will be a number of different values for any of Problems 9-13, 9-14, and 9-15, the individual values should be turned in or placed on the blackboard, and a statistical average of all the values used should be obtained, with each student computing the statistical class average and comparing it to his or her own value. Any student whose value is more than two standard deviations from the average should give an explanation for the divergence.

Beam-on-elastic foundation

9-16. Refer to the computer output of Fig. E9-6*c* (beam-on-elastic foundation) and perform a statics check at nodes 10 and 13.

9-17. Refer to the computer output of Fig. E9-6*c* and verify the node reaction and soil pressure at nodes 8 and 13.

9-18. Using program B-5 (FADBEMLP) on your program diskette, solve Example 9-1 as a beam-on-elastic foundation. Use nodes at column faces and other locations as necessary and estimate $k_s = 120q_{\text{ult}}$. Compare the moments output with those in the table in the example. Also compare the node soil pressures with the uniform values assumed in Example 9-1. Note that you should use ultimate loads and moments for consistency in comparing the example table and computing k_s.

9-19. Make a beam-on-elastic-foundation solution for Example 9-1 using computer program B-5 (FADBEMLP) on your program diskette. For the first trial use $k_s = 120q_{\text{ult}}$. Make two additional runs using (*a*) $k_s = 0.5$ and (*b*) two times the initially estimated value. Make one additional run where you input the undoubled values of the two end springs using a program option. Can you draw any conclusions after inspecting the moment and displacement output about the effect of doubling end springs and what is used for k_s?

9-20. Using program B-5 (FADBEMLP) on your program diskette, solve the trapezoidal footing of Example 9-2 as a beam-on-elastic foundation. You will have to use average element widths and estimate $k_s = 120q_i$. Compare the output moments with the moment table in Example 9-2. Also compare the soil node pressures with the uniform value assumed in the example.

9-21. Use program B-5 (FADBEMLP) on the enclosed diskette and analyze the strap footing you designed in Problem 9-6 or 9-8. Use at least four nodes across each footing. Based on the footing displacements, do you think your strap has a sufficient moment of inertia I?

9-22. Refer to the computer output of Fig. E9-6*c* of Example 9-6 and rerun the example using XMAX = 0.011 m. Plot the vertical displacements to a large scale such as 0.01 m = 10 mm (or 2 cm), and superimpose on this displacement plot the horizontal line of XMAX = 0.011 (11 mm).

Ring foundations

9-23. Perform a statics check of the ring foundation of Example 9-7 at node 6 or node 16 as assigned.

9-24. If you have access to the ring foundation computer program (B-17), redo Example 9-6 for k_s = 0.5, 1.5, and 2.0 times the value used of 13 600 kPa. Can you draw any conclusions about the effect of k_s?

9-25. If you have access to computer program B-17, design a ring foundation similar to Example 9-7 assuming a water tower with four equally spaced columns. Other data:

$R_m = 7.5$ m.
The tank holds 378 m^3 of water.
The empty tank, appurtenances, and legs weigh 2200 kN.
The wind moment is 2250 kN · m.
Take the maximum allowable soil pressure (SF = 2) as 200 kPa.
$f'_c = 28$ MPa.
$f_y = 400$ MPa.

Required: Find the ID, OD, and foundation depth D_c of the base. Be sure to check at least two nodes (not adjacent) and draw a neat sketch showing column locations and other critical data.