Solving Differential equations

⊞— dsol

dsol(DE,X,tf,N) try to solve the Differential Equation system DE for the functions in X.

$N := 100$

⊟— Example

Example Default solver is $Options(dsol, dsolver) = "rkfixed"$

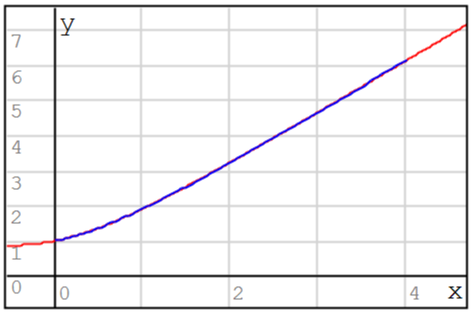
It could be changed with $Options(dsol, dsolver = "solver")$

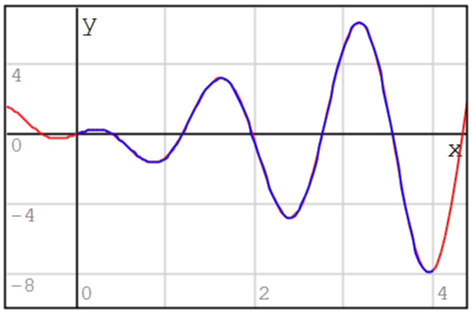
or in the solver block

$$\begin{cases} us(t) := 0.5 \cdot t + \sqrt{t^2 + 1} \\ vs(t) := 2 \cdot t \cdot \cos(4 \cdot t) \end{cases}$$

$$\left[\begin{array}{l} u(0) = 1 \qquad v(0) = 0 \qquad v'(0) = 2 \qquad dsolver = "Adams" \\ -\dfrac{v''(t)}{u(t) \cdot \sqrt{\left|v'(t)\right|}} = \\ = \dfrac{32 \cdot (2 \cdot t \cdot \cos(4 \cdot t) + \sin(4 \cdot t))}{\left(t + 2 \cdot \sqrt{1 + t^2}\right) \cdot \sqrt{\left|2 \cdot (4 \cdot t \cdot \sin(4 \cdot t) - \cos(4 \cdot t))\right|}} \\ -\dfrac{u'(t) \cdot v'(t)}{u(t)} = \\ = \dfrac{2 \cdot \left(\sqrt{1 + t^2} + 2 \cdot t\right) \cdot (4 \cdot t \cdot \sin(4 \cdot t) - \cos(4 \cdot t))}{\sqrt{1 + t^2} \cdot \left(t + 2 \cdot \sqrt{1 + t^2}\right)} \end{array}\right.$$

$$RK := dsol\left(\begin{cases} u(t) \\ v(t) \end{cases}, 4, N\right)$$



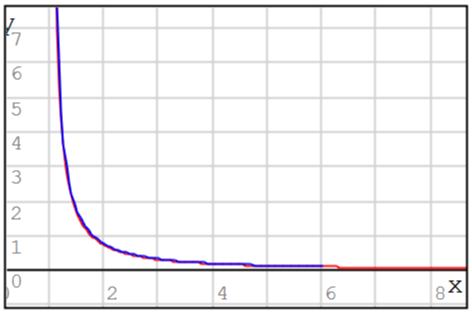


⊟— Example (Exact ODE)

Example $\varepsilon := 0.01$ $ys(x) := \dfrac{1}{x \cdot \ln(x)}$ $yo := ys(1 + \varepsilon)$

$$\left[\begin{array}{l} y(x) \cdot \ln(x) + y(x) + x \cdot \ln(x) \cdot y'(x) = 0 \\ y(1 + \varepsilon) = yo \end{array}\right.$$

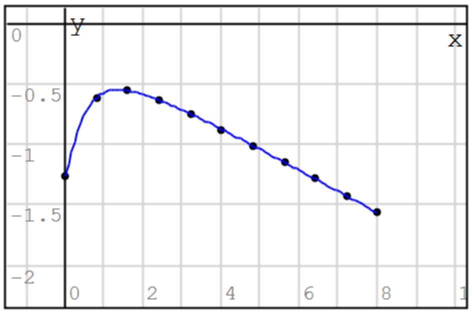
$$RK := dsol(y(x), 6, N)$$



Example $C := -2$ $ys(x) := \left|roots\left(x^2 + 5 \cdot x \cdot y + y^3 = C, y, -1\right)\right.$ $yo := ys(0)$

$$\begin{cases} 2\cdot x+5\cdot y(x)+\left(3\cdot(y(x))^{2}+5\cdot x\right)\cdot y'(x)=0 \\ y(0)=yo \end{cases}$$

$RK := dsol(y(x), 8, N)$



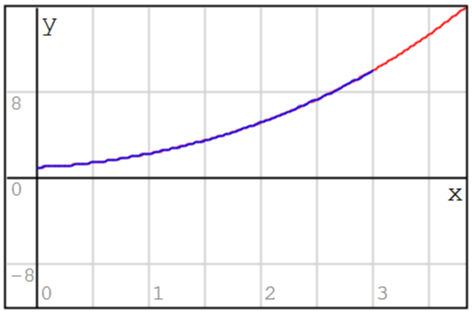
Example Guess for the higest derivatives

Example This example shows how to use the guess value for the higest derivatives

$$ys(x) := \frac{1}{9}\cdot\left(x^{3}+9+6\cdot\sqrt{3\cdot x^{3}}\right)$$

$$\begin{cases} y'(x)^{2}-x\cdot y(x)=2\cdot x \quad dsolver = \text{"Rkadapt"} \\ y(0)=1 \end{cases}$$

$RK := dsol(y(x), 3, N)$



Above solve block use

Options(dsol, "DefaultGuess") = "1"

It could be changed with

Options(dsol, DefaultGuess = −1)

$$ys(x) := \frac{1}{9}\cdot\left(x^{3}+9-6\cdot\sqrt{3\cdot x^{3}}\right)$$

or specifiying a value in the solve block

$$\begin{cases} y'(x)^{2}-x\cdot y(x)=2\cdot x \quad dsolver = \text{"Rkadapt"} \\ y(0)=1 \quad \boxed{y'(0)\approx -1} \end{cases}$$

$RK := dsol(y(x), 3, N)$

Example

Example $xo := 1$ $yo := 2$

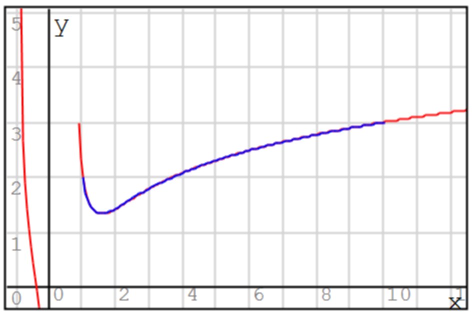
$$eq(x, C) := \ln\left(\frac{2\cdot x\cdot \mathbf{e}^{x^{2}}}{\mathbf{e}^{x^{2}}+C}\right)$$

$Co := roots(eq(xo, C)-yo, C, -2)$

$ys(x) := eq(x, Co)$

$$\begin{cases} y'(x)+\mathbf{e}^{y(x)}=2\cdot x+\frac{1}{x} \\ y(xo)=yo \end{cases}$$

$RK := dsol(y(x), 10, N)$



Example

Example $\alpha := 12$

$$\begin{cases} us(\sigma) := \ln(1+\alpha\cdot\sigma) \\ vs(\sigma) := \mathbf{e}^{-\sigma}+1 \end{cases}$$

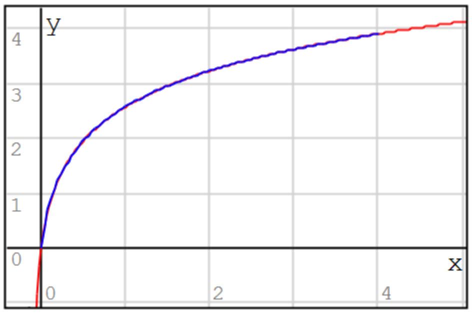
$$v''(\sigma)\cdot v(\sigma)-\sigma\cdot u(\sigma) = \frac{\mathbf{e}^{\sigma}+1-\sigma\cdot\ln(1+\alpha\cdot\sigma)\cdot\mathbf{e}^{2\cdot\sigma}}{\mathbf{e}^{2\cdot\sigma}}$$

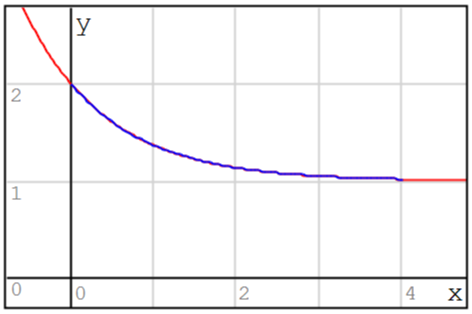
$$u'(\sigma)\cdot v''(\sigma)+\cos(2\cdot\sigma) = \frac{\alpha+\cos(2\cdot\sigma)\cdot(1+\alpha\cdot\sigma)\cdot\mathbf{e}^{\sigma}}{(1+\alpha\cdot\sigma)\cdot\mathbf{e}^{\sigma}}$$

$v(0) = 2$ $\quad v'(0) = -1$ $\quad v''(0) \approx 1$

$u(0) = 0$ $\quad u'(0) \approx \alpha$ $\quad dsolver = dn\_AdamsMoulton$

$$RK := dsol\left(\begin{cases} u(\sigma) \\ v(\sigma) \end{cases}, 4, N\right)$$





Example Abel

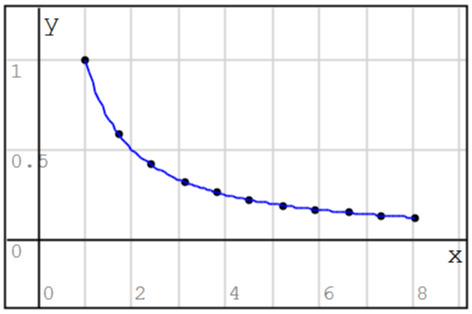
Example

$$ys(x) := \mathrm{roots}\left((1-x\cdot y)\cdot\mathbf{e}^{y\cdot\frac{y+2\cdot x^3}{2\cdot x^2}} = 0, y, 1\right)$$

$$x\cdot\left(x\cdot y(x)+x^4-1\right)\cdot y'(x) = y(x)\cdot\left(x\cdot y(x)-x^4-1\right)$$

$y(1) = 1$ $\quad y'(1) \approx 1$ $\quad dsolver = Rkadapt$

$$RK := dsol(y(x), 8, N)$$



⊟— Example

Example

$$\begin{cases} xs(t) := 2 \cdot \sin(t) \\ ys(t) := t \cdot e^{-t} \end{cases}$$

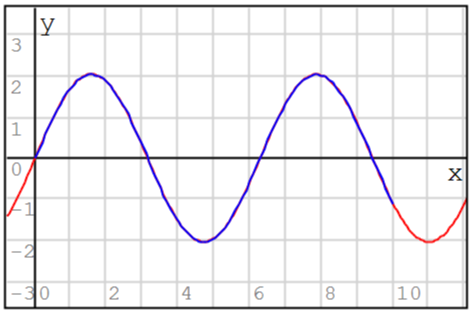
$x(0) = 0 \qquad x'(0) = 2 \qquad y(0) = 0 \qquad y'(0) = 1$

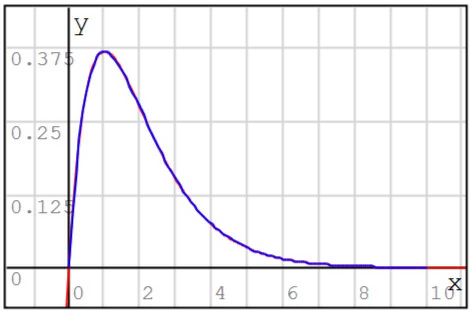
$$y''(t) \cdot t - x(t) \cdot y(t) + \frac{t \cdot (2 - t + 2 \cdot \sin(t))}{e^{t}}$$

$$x''(t) - \cos(4 \cdot t) \cdot y'(t) + \frac{2 \cdot \sin(t) \cdot e^{t} + \cos(4 \cdot t) \cdot (1 - t)}{e^{t}}$$

$x''(0) \approx 0 \qquad \boxed{y''(0) \approx -2}$

$$RK := dsol\left(\begin{cases} x(t) \\ y(t) \end{cases}, 10, N\right)$$





⊟— Example

Example

$$\begin{cases} xs(t) := t^{2} - 6 \\ ys(t) := e^{\sin(t)} \\ zs(t) := 0.5 \cdot t^{2} - 2 \end{cases}$$

$a(0) = -6 \qquad b(0) = 1 \qquad c(0) = -2$

$a'(0) = 0 \qquad b'(0) = 1 \qquad c'(0) = 0$

$$a''(t) \cdot b(t) - c(t) = \frac{4 \cdot \left(1 + e^{\sin(t)}\right) - t^{2}}{2}$$

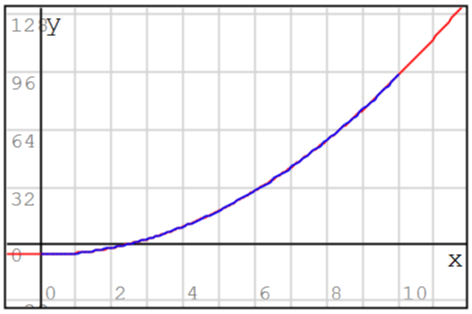
$$b''(t) - t \cdot c'(t) = e^{\sin(t)} \cdot \left((\cos(t))^{2} - \sin(t)\right) - t^{2}$$

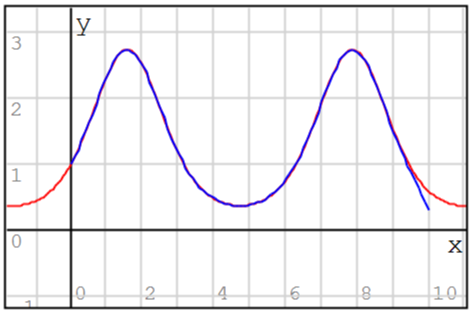
$$c''(t) \cdot a''(t) - 3 \cdot b(t) = 2 - 3 \cdot e^{\sin(t)}$$

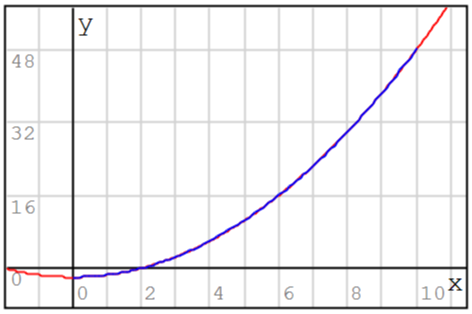
$\boxed{a''(0) \approx 2} \qquad b''(0) \approx 1 \qquad c''(0) \approx 1$

dsolver = "Rkadapt"

$$RK := dsol\left(\begin{cases} a(t) \\ b(t) \\ c(t) \end{cases}, 10, N\right)$$







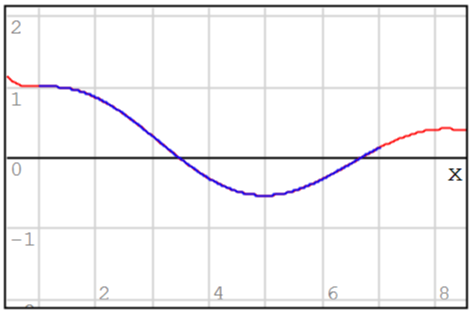
⊟— Example

Example

$$ys(x) := \frac{(Yv(1,1) - Yv(0,1)) \cdot Jv(1,x) - (Jv(1,1) - Jv(0,1)) \cdot Yv(1,x)}{Yv(1,1) \cdot Jv(0,1) - Yv(0,1) \cdot Jv(1,1)}$$

$$\begin{bmatrix} x^2 \cdot y'''(x) + (-x^3 + 3 \cdot x) \cdot y''(x) + (-x^3 + 3 \cdot x) \cdot y(x) = 0 \\ y(1) = 1 \qquad y'(1) = 0 \qquad y''(1) = 0 \end{bmatrix}$$

$RK := dsol(y(x), 7, 5 \cdot N)$



⊟— Example UpdateGuess

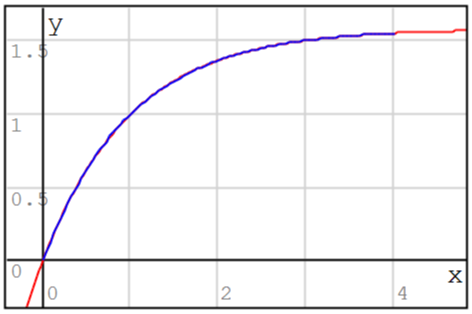
The option UpdateGuess = true is used for update the value of the y' guess at each iteration. Default value is true (1). First example requires it set to false, and second to true.

Example $Options(dsol, UpdateGuess = 0) = "0"$

$ys(t) := 0.5 \cdot \pi \cdot (1 - e^{-t})$

$$\begin{bmatrix} \sin(y'(t)) = \cos(y(t)) \\ y(0) = 0 \qquad y'(0) \approx 0.5 \end{bmatrix}$$

$RK := dsol(y(t), 4, N)$



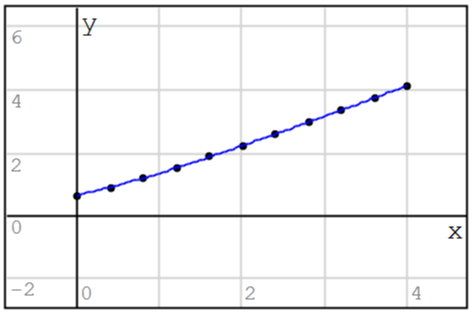
Example $Options(dsol, UpdateGuess = 1) = "1"$ $\qquad C := 0$

$$ys(t) := \left| roots\left(\sqrt{y^2 + 1} + \frac{1}{2} \cdot \ln\left(\frac{\sqrt{y^2 + 1} - 1}{\sqrt{y^2 + 1} + 1}\right) = t + C, y, 1\right)\right|$$

$yo := ys(0) = 0.6627$

$$\begin{bmatrix} \operatorname{asin}(y'(t)) = \operatorname{atan}(y(t)) \\ y(0) = yo \qquad y'(0) \approx 0 \end{bmatrix}$$

$RK := dsol(y(t), 4, N)$



⊟— bvp

⊟— bvp

Boundary Values Problem

$lbvp_2$ $\qquad lbvp_2(f(x), ab, M, N)$ solves the linear ODE in $ab = [a \ b]$

$$y'' + p \cdot y' + q \cdot y = r$$

subject to the boundary conditions

$$\begin{cases} \alpha_1 \cdot y(a) + \beta_1 \cdot y'(a) = c_1 \\ \alpha_2 \cdot y(b) + \beta_2 \cdot y'(b) = c_2 \end{cases}$$

f is such that $f(x) = [p(x)\ q(x)\ r(x)]$ and $M = \begin{bmatrix} \alpha 1 & \beta 1 & c1 \\ \alpha 2 & \beta 2 & c2 \end{bmatrix}$

bvp₂ — $bvp_2(\varphi(x, y, y'), x, Yo, M, N, \varepsilon)$ solves the non linear ODE in $x = [a\ b]$

$$y'' = \varphi(x, y, y')$$

subject to the the same boundary conditions, with Yo as guess for the solution with dimension N+1. If Yo≡0, bvp.2 try with a line between f(a) and f(b) as guess.
ε is used as the tolerance for a Newton solver.

⊞— Short hands for plots

⊟— lbvp example

Example $[a\ b] := [0\ 5]$ $h := x \cdot \cos(x)$ $h' := \frac{d}{dx} h$

Numeric solution

$$y''(x) + 2 \cdot h \cdot y'(x) + (h^2 + h') \cdot y(x) = 0$$

$$y(a) + 3 \cdot y'(a) = 0.1$$

$$y'(b) = -6$$

$dsolver = lbvp_2$

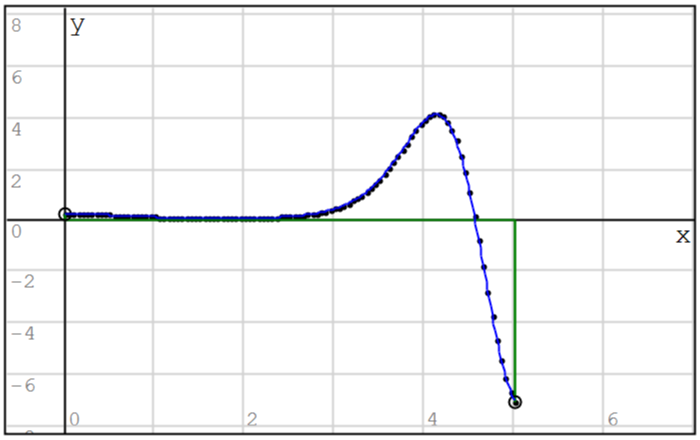
$sol := dsol(y(x), N)$

Analytic solution

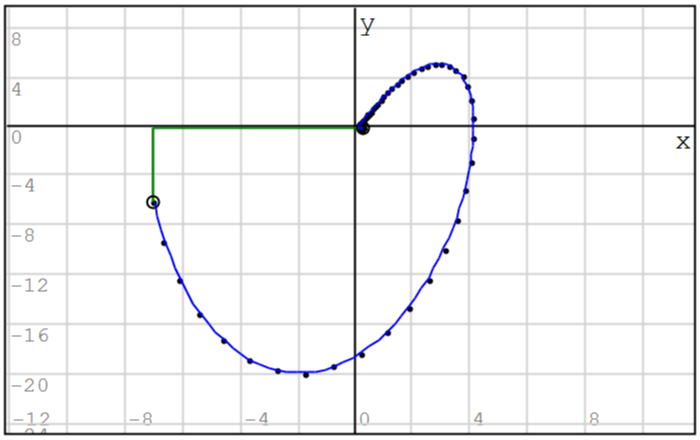
$$\psi(x) := (A + B \cdot x) \cdot e^{-(x \cdot \sin(x) + \cos(x))}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} := \text{roots}\left(\begin{bmatrix} \psi(a) + 3 \cdot \psi'(a) = 0.1 \\ \psi'(b) = -6 \end{bmatrix}, \begin{bmatrix} A \\ B \end{bmatrix}\right)$$

$Err = 0.16$



Plot ("XY", sol)



Plot ("YY'", sol)

⊟— lbvp example

Example

Numeric solution

$$y''(x) + 3 \cdot y'(x) - 4 \cdot y(x) = 0$$

$dsolver = lbvp_2$

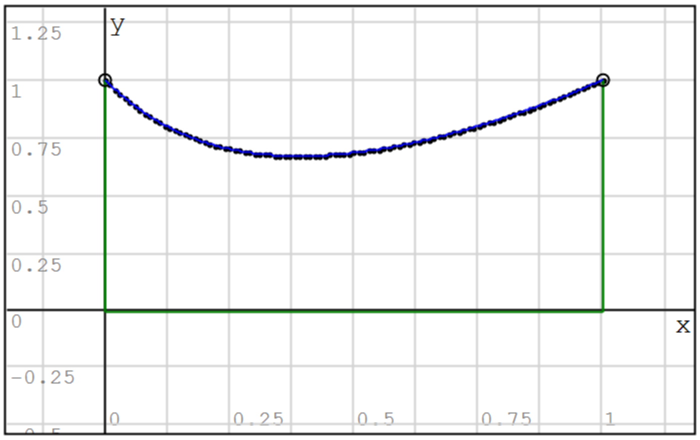
$y(0) = 1$ $y(1) = 1$

$sol := dsol(y(x), N)$

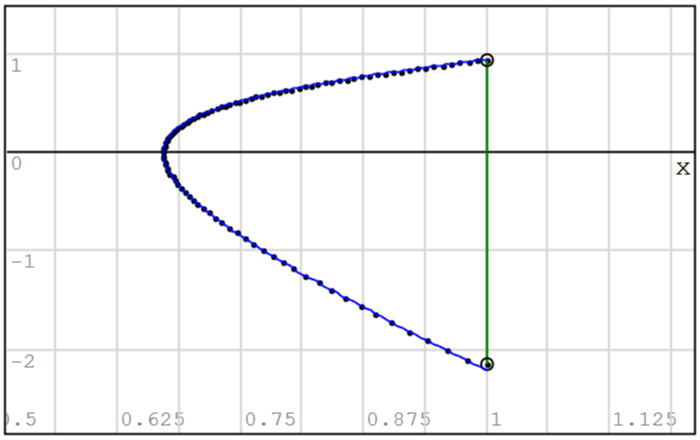
Analytic solution

$$\psi(x) := \frac{\left(e^{(4 - 4 \cdot x)} + e^{x} + e^{(x+1)} + e^{(x+2)} + e^{(x+3)}\right)}{\left(1 + e + e^2 + e^3 + e^4\right)}$$

$Err = 5.43 \cdot 10^{-5}$



Plot ("XY", sol)



Plot ("YY'", sol)

⊟— lbvp example

Example

Numeric solution

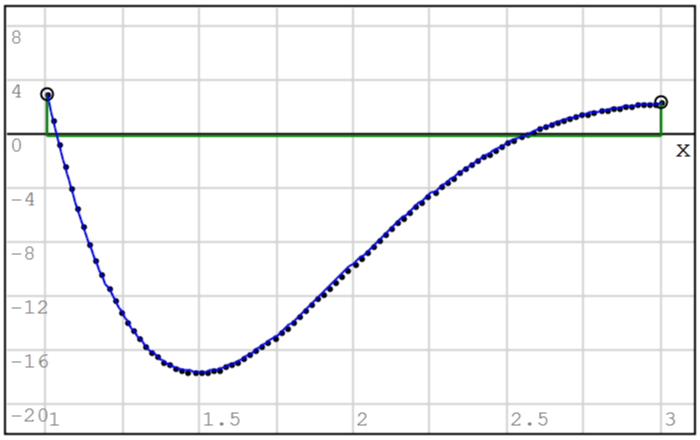
$$\left[\begin{array}{ll} x''(t)+3\cdot x'(t)+6\cdot x(t)=5 & x(1)=3 \\ dsolver=lbvp_2 & x(3)+2\cdot x'(3)=5 \end{array}\right.$$

$sol := dsol(x(t), N)$

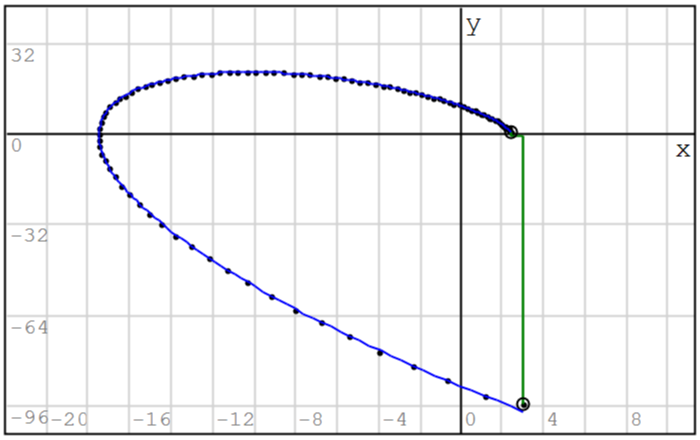
Analytic solution

$$\psi(t):=0.0696737\cdot e^{-1.5\cdot t}\cdot\left(11.9605\cdot e^{1.5\cdot t}+1243.5\cdot \sin(1.93649\cdot t)+2857.67\cdot \cos(1.93649\cdot t)\right)$$

$Err=0.21$



Plot ("XY", sol)



Plot ("YY'", sol)

⊟— lbvp example

Example

$p:=10^{-5}$ $\quad [a\ b]:=[-0.1\ 0.1]$

Analytic solution

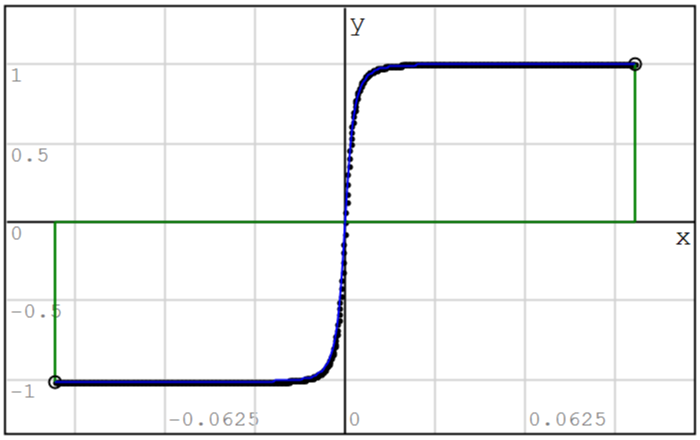
$$\psi(x):=\frac{x}{\sqrt{p+x^2}}$$

Numeric solution

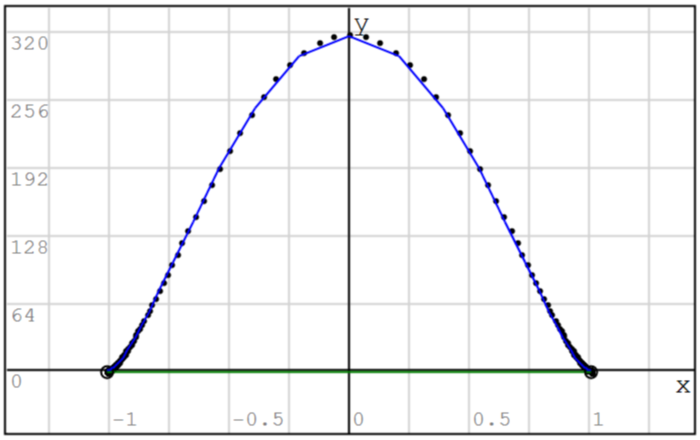
$$\left[\begin{array}{lll} y''(x)+\dfrac{3\cdot p}{\left(p+x^2\right)^2}\cdot y(x)=0 & y(a)=\psi(a) & dsolver=lbvp_2 \\ & y(b)=\psi(b) & \end{array}\right.$$

$Err=0.1$

$sol := dsol(y(x), 1000)$



Plot("XY", *sol*)



Plot("YY'", *sol*)

⊟— lbvp example

Example $[\,a\ \ b\,] := \text{eval}\left(\left[\ \frac{1}{3\cdot\pi}\ \ \frac{3}{\pi}\ \right]\right)$

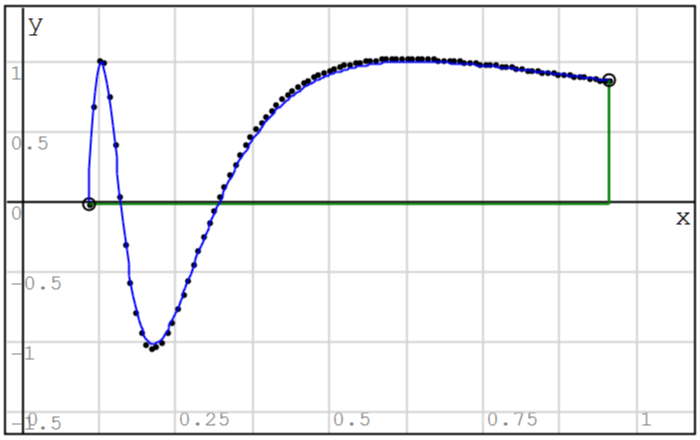
Numeric solution

$$y''(x) + \frac{2}{x}\cdot y'(x) + \frac{y(x)}{x^4} = 0 \qquad y(a) = 0 \qquad y(b) = \text{eval}\left(\frac{\sqrt{3}}{2}\right)$$

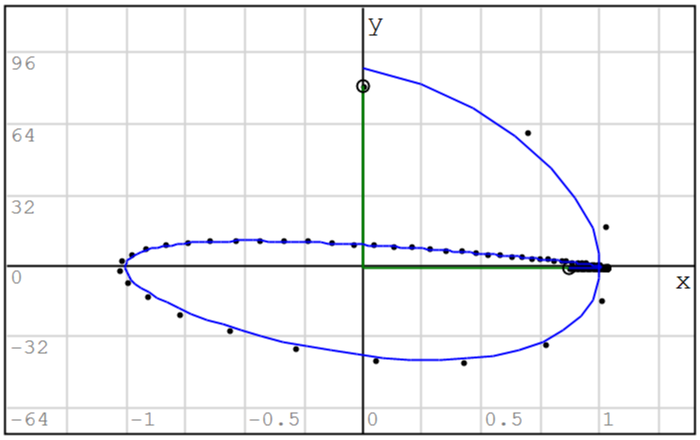
$dsolver = lbvp_2$

$sol := dsol\,(y\,(x),\ N)$

Analytic solution $\psi(x) := \sin\left(\frac{1}{x}\right)$ $Err = 0.32$



Plot("XY", *sol*)



Plot("YY'", *sol*)

⊟— bvp with two solutions

Example $[\,a\ \ b\,] := [\,0\ \ 4\,]$ $\quad y_b := -2$

Numeric solution

$$y''(x) + |y(x)| = 0 \qquad y(a) = 0 \qquad y(b) = y_b$$

$dsolver = bvp_2$

$[\,X\ \ Y\ \ Y'\,] := Cols\,(sol)$

$\text{Clear}\left(y'_a\right) = 1$ $\quad y'_a := 0$ S

$sol := dsol\,(y\,(x),\ N)$

This problem have two solutions. I compare the bvp solution with the shooting method, because can't found a symbolic expression.

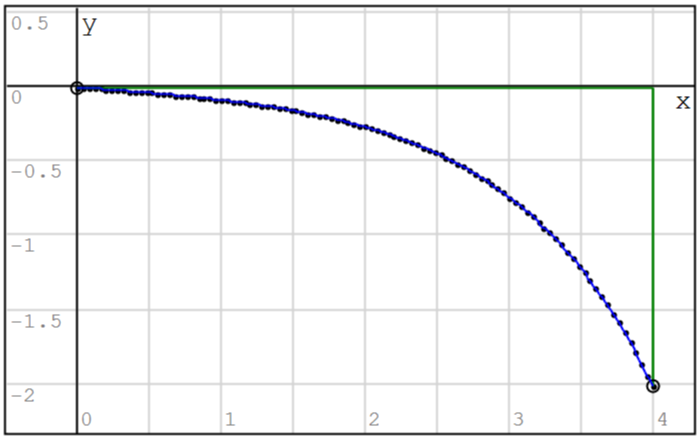
Shoot solution

$$\psi''(x)+|\psi(x)|=0 \qquad \psi(a)=0$$
$$\psi'(a)=y'_a$$

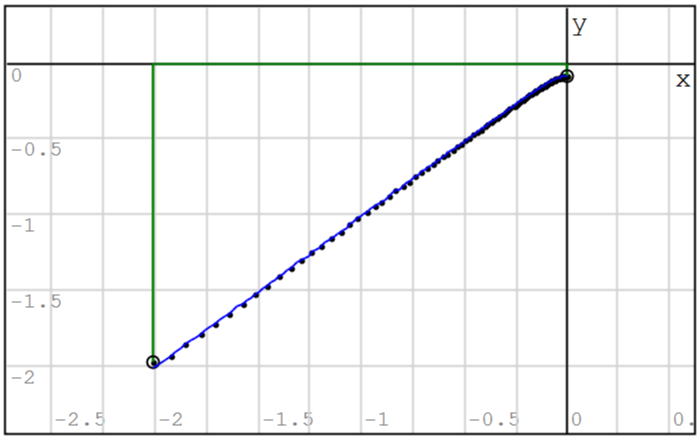
$$shoot(y'_a):=\text{Rkadapt}(\psi(x), b, N)$$

$$Eq(y'_a):=shoot(y'_a)_{N+1\,2}-y_b \qquad y'_a:=\text{al\_nleqsolve}(Y'_1, Eq)_1=-0.0733$$

$$[\Xi\ \Psi\ \Psi']:=Cols(shoot(y'_a)) \qquad \psi'(x):=\text{cinterp}(\Xi, \Psi', x) \qquad Err=0$$



Plot("XY", sol)



Plot("YY'", sol)

For the other solution: call bvp with a new guess

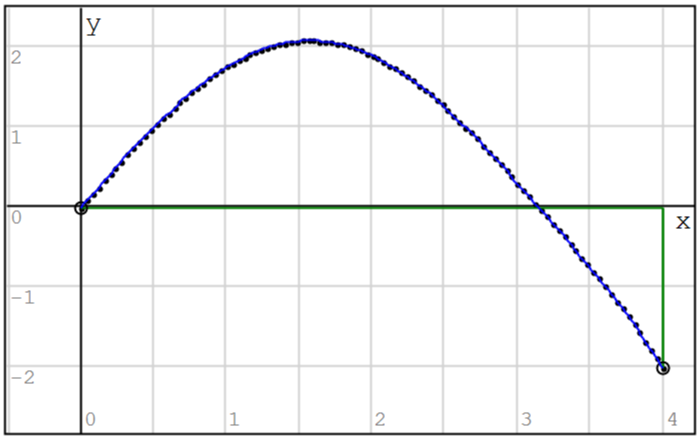
$$y''(x)+|y(x)|=0 \qquad y(a)=0 \qquad \boxed{dsolver=bvp_2} \qquad [X\ Y\ Y']:=Cols(sol)$$
$$\boxed{y'(0)\approx 1} \qquad y(b)=y_b$$

$$sol:=dsol(y(x), N)$$

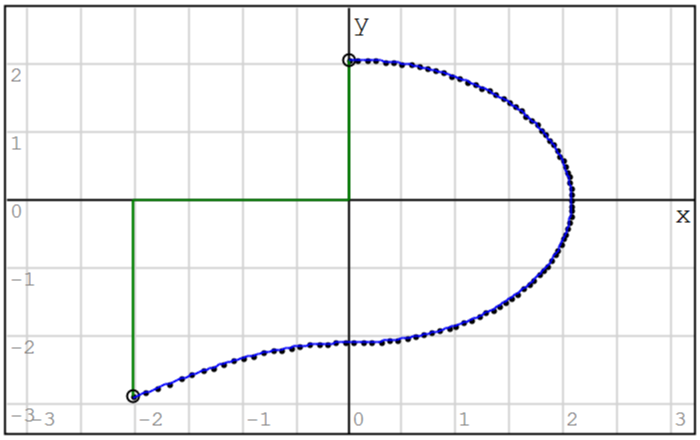
Solve the shoot with a new guess from bvp

$$y'_a:=\text{al\_nleqsolve}(Y'_1, Eq)_1=2.0666$$

$$[\Xi\ \Psi\ \Psi']:=Cols(shoot(y'_a)) \qquad \psi'(x):=\text{cinterp}(\Xi, \Psi', x) \qquad Err=0$$



Plot("XY", sol)



Plot("YY'", sol)

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