Schur Decomposition of Square Matrices

Preliminaries

$\mathrm{ORIGIN} := 0$

Return mxn matrix filled with constant value c

$$fill_c(m,n,c) := \left\| \begin{array}{l} f(i,j) \leftarrow c \\ \text{return matrix}(m,n,f) \end{array}\right.$$

Return i-th unit vector of order n

$$e_n(n,i) := \left\| \begin{array}{l} O \leftarrow \mathrm{ORIGIN} \\ ev \leftarrow fill_c(n,1,0) \\ ev_{i-1+O} \leftarrow 1 \\ ev \end{array}\right.$$

$$e_n(4,2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Return Strictly Lower triangle of matrix

$$LT(A) := \left\| \begin{array}{l} O \leftarrow \mathrm{ORIGIN} \\ nr \leftarrow \mathrm{rows}(A) \\ ir \leftarrow nr - 1 + O \\ L \leftarrow fill_c(nr, nr, 0) \\ \text{for } i \in O+1 .. ir \\ \quad \left\| \begin{array}{l} \text{for } j \in O .. ir \\ \quad \left\| \begin{array}{l} \text{if } j < i \\ \quad \left\| L_{i,j} \leftarrow A_{i,j} \right. \end{array}\right. \\ \text{continue} \end{array}\right. \\ L \end{array}\right.$$

Exact eigenvalues

Guess Values

$\mu := 1$

Constraints

$\det(A - \mu \cdot \mathrm{identity}(rows(A))) = 0$

Solver

$\mathrm{EV}(A) := \mathbf{find}(\mu)^{\mathrm{T}}$

Matrix Normality check

$$Normality_{check}(A) := \left\| \begin{array}{l} \text{if } A \cdot \overline{A}^{\mathrm{T}} = \overline{A}^{\mathrm{T}} \cdot A \\ \quad \left\| \text{return "yes"} \right. \\ \text{"no"} \end{array}\right.$$

A unitary matrix is a square matrix such that $A^{-1} = \overline{A}^{\mathrm{T}}$

$$Unitary_{check}(A) := \left\| \begin{array}{l} \text{if round}\left(\left\|A^{-1}\right\| - \left\|\overline{A}^{\mathrm{T}}\right\|, 14\right) = 0 \\ \quad \left\| \text{return "yes"} \right. \\ \text{"no"} \end{array}\right.$$

Test convergence with Norm of lower off-center diagonal of matrix

$$test_H(A,d) := \left\| \begin{array}{l} O \leftarrow \mathrm{ORIGIN} \\ nr \leftarrow \mathrm{rows}(A) - 1 + O \\ \varepsilon \leftarrow 10^{-d} \\ \varepsilon_H \leftarrow \mathrm{submatrix}(A, O+1, nr, O, nr-1) \\ \varepsilon_H \leftarrow \mathrm{norme}(\mathrm{diag}(\mathrm{diag}(\varepsilon_H))) \\ \text{if } \varepsilon_H \le \varepsilon \\ \quad \left\| \text{return } 1 \right. \\ \varepsilon_H \end{array}\right.$$

Test results of multiple test values in vector

$$test(v) := \left\| \begin{array}{l} O \leftarrow \mathrm{ORIGIN} \\ rn \leftarrow \mathrm{rows}(v) - 1 + O \\ \text{for } i \in O .. rn \\ \quad \left\| \begin{array}{l} \text{if } v_i = 0 \\ \quad \left\| \text{return } 0 \right. \end{array}\right. \\ 1 \end{array}\right.$$

Percent error in Schur calculated eigenvalues

$rs(\lambda) := \mathrm{reverse}(\mathrm{sort}(\lambda))$

$$\lambda_{error}(\lambda, T) := \left\| \begin{array}{l} \lambda_s \leftarrow rs(\lambda) \\ \Delta \leftarrow rs(\mathrm{diag}(T)) - \lambda_s \\ \text{if } \det(\mathrm{diag}(\lambda)) = 0 \\ \quad \left\| \text{return} \begin{bmatrix} \text{"Difference"} \\ \Delta \end{bmatrix} \right. \\ \begin{bmatrix} \text{"Percent error"} \\ \mathrm{diag}(\lambda_s)^{-1} \cdot \Delta \cdot 100 \end{bmatrix} \end{array}\right.$$

Return column vectorization of matrix A (stack cols of A on top of each other)

$$cvec(A) := \left\| \begin{array}{l} O \leftarrow \mathrm{ORIGIN} \\ cA \leftarrow \mathrm{cols}(A) \\ \text{if } cA = 1 \\ \quad \left\| \text{return } A \right. \\ v \leftarrow A^{\langle O \rangle} \\ \text{for } i \in O+1 .. cA - 1 + O \\ \quad \left\| v \leftarrow \mathrm{stack}(v, A^{\langle i \rangle}) \right. \\ v \end{array}\right.$$

Switch n-th and m-th element in vector z

$$switch(z,n,m) := \left\| \begin{array}{l} O \leftarrow \mathrm{ORIGIN} \\ in \leftarrow n - 1 + O \\ im \leftarrow m - 1 + O \\ sz \leftarrow z \\ sz_{in} \leftarrow z_{im} \\ sz_{im} \leftarrow z_{in} \\ sz \end{array}\right.$$

Upper Hessenberg decomposition $A = \overline{U}^{T} \cdot H \cdot U$

https://en.wikipedia.org/wiki/Hessenberg_matrix

```
Hessenberg(A) := | nr ← rows(A)
                 | if nr≠cols(A)
                 |   | return "Error: matrix is not square"
                 | U ← identity(nr)
                 | H ← A
                 | if nr < 3
                 |   | return [H; U]
                 | O ← ORIGIN
                 | n ← nr − 1 + O
                 | for k ∈ O .. n − 2
                 |   | a_k ← submatrix(H, k+1, n, k, k)
                 |   | kr ← rows(a_k)
                 |   | e_k ← e_n(kr, 1)
                 |   | if a_k_O ≠ 0
                 |   |   | w ← ‖a_k‖ · e_k + (conj(a_k_O) / |a_k_O|) · a_k
                 |   | else
                 |   |   | w ← ‖a_k‖ · e_k − a_k
                 |   | "Householder matrix"
                 |   | V_k ← identity(kr) − 2 · (w · conj(w)^T) / ‖w‖^2
                 |   | kn ← k + 1 − O
                 |   | I_k ← identity(kn)
                 |   | z_k ← fill_c(kn, n − k, 0)
                 |   | Iz_k ← augment(I_k, z_k)
                 |   | zV_k ← augment(z_k^T, V_k)
                 |   | U_k ← stack(Iz_k, zV_k)
                 |   | U ← U_k · U
                 |   | H ← U_k · H · conj(U_k)^T
                 | [H; U]
```

Matrix Symmetry check

```
Sym(A) := | if A = A^T
          |   | return "yes"
          | "no"
```

Hessenberg decomposition Example from: Businger, P.A. (1969) "Reducing a Matrix to Hessenberg Form," Bell Telephone Laboratories, Inc. Murry Hill, New Jersey.

$$B := \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$|\mathrm{tr}(B)| = 0$ $\quad |||B||| = 4.5$ $\quad Sym(B) = \text{“no”}$

$Normality_{check}(B) = \text{“no”}$

$$\lambda_B := \mathrm{eigenvals}(B) = \begin{bmatrix} 1 \\ 0.47473445 - 1.43725651i \\ 0.47473445 + 1.43725651i \\ -0.38126774 + 1.2285915i \\ -0.38126774 - 1.2285915i \\ -1.18693341 \end{bmatrix}$$

2 real distinct eigenvalues and 2 pairs of complex conjugate eigenvalues

$$v_B := \mathrm{eigenvecs}(B) = \begin{bmatrix} 0.5145 & 0.27201 + 0.43872i & 0.27201 - 0.43872i & -0.19138 - 0.36044i & -0.19138 + 0.36044i & 0.33717 \\ 0.68599 & -0.17286 + 0.29697i & -0.17286 - 0.29697i & -0.20933 + 0.38312i & -0.20933 - 0.38312i & -0.54229 \\ 0.343 & 0.04273 - 0.27303i & 0.04273 + 0.27303i & 0.57037 & 0.57037 & 0.48273 \\ 0 & 0.44498 - 0.25589i & 0.44498 + 0.25589i & 0.10628 - 0.49061i & 0.10628 + 0.49061i & -0.38086 \\ -0.343 & 0.51758 & 0.51758 & -0.15104 - 0.03302i & -0.15104 + 0.03302i & 0.34672 \\ -0.1715 & -0.07274 + 0.04928i & -0.07274 - 0.04928i & 0.19951 + 0.04281i & 0.19951 - 0.04281i & -0.3065 \end{bmatrix}$$

$$v_B \cdot \operatorname{diag}(\lambda_B) \cdot v_B^{-1} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0 \end{bmatrix} \quad v_B \cdot v_B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad Unitary_{check}(v_B) = \text{“no”}$$

$$\begin{bmatrix} H \\ U \end{bmatrix} := Hessenberg(B) \quad H = \begin{bmatrix} 0 & 1 & 0.3015 & 0 & 0.6224 & 1.8766 \\ -2 & 0.25 & -0.0754 & -0.3536 & -1.1336 & 0.5213 \\ 0 & 0.8292 & -0.25 & -0.3198 & -0.2212 & -0.4087 \\ 0 & 0 & 0.8528 & -0.5 & 0.0314 & -0.1474 \\ 0 & 0 & 0 & 0.723 & 0.3696 & -0.0278 \\ 0 & 0 & 0 & 0 & -1.7334 & 0.1304 \end{bmatrix}$$

$Normality_{check}(H) = \text{“no”}$

$Unitary_{check}(H) = \text{“no”}$

$Normality_{check}(U) = \text{“no”}$

$Unitary_{check}(U) = \text{“yes”}$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0.5 & 0.5 & 0.5 & 0 \\ 0 & 0.1508 & -0.7538 & 0.4523 & 0.4523 & 0 \\ 0 & 0 & 0 & -0.7071 & 0.7071 & 0 \\ 0 & -0.1778 & -0.0889 & -0.0445 & -0.0445 & 0.978 \\ 0 & 0.8341 & 0.417 & 0.2085 & 0.2085 & 0.2085 \end{bmatrix} \quad \lambda_H := \operatorname{eigenvals}(H) = \begin{bmatrix} 1 \\ 0.47473445 - 1.43725651i \\ 0.47473445 + 1.43725651i \\ -0.38126774 + 1.2285915i \\ -0.38126774 - 1.2285915i \\ -1.18693341 \end{bmatrix}$$

$$v_H := \operatorname{eigenvecs}(H) = \begin{bmatrix} 0.5145 & -0.0291+0.5154i & -0.0291-0.5154i & 0.1595-0.3756i & 0.1595+0.3756i & 0.3372 \\ -0.343 & 0.7194 & 0.7194 & 0.5836 & 0.5836 & 0.4954 \\ -0.3103 & 0.2314+0.3269i & 0.2314-0.3269i & -0.1642-0.487i & -0.1642+0.487i & -0.4611 \\ -0.2425 & -0.0618+0.1776i & -0.0618-0.1776i & -0.3659+0.0624i & -0.3659-0.0624i & 0.5145 \\ -0.305 & -0.0891-0.0245i & -0.0891+0.0245i & 0.1179+0.1408i & 0.1179-0.1408i & -0.2447 \\ 0.6079 & -0.0036+0.1083i & -0.0036-0.1083i & -0.1102+0.2123i & -0.1102-0.2123i & -0.322 \end{bmatrix}$$

$Unitary_{check}(v_H) = \text{“no”}$

$$H' := v_H \cdot \operatorname{diag}(\lambda_H) \cdot v_H^{-1} = \begin{bmatrix} 0 & 1 & 0.3015 & 0 & 0.6224 & 1.8766 \\ -2 & 0.25 & -0.0754 & -0.3536 & -1.1336 & 0.5213 \\ 0 & 0.8292 & -0.25 & -0.3198 & -0.2212 & -0.4087 \\ 0 & 0 & 0.8528 & -0.5 & 0.0314 & -0.1474 \\ 0 & 0 & 0 & 0.723 & 0.3696 & -0.0278 \\ 0 & 0 & 0 & 0 & -1.7334 & 0.1304 \end{bmatrix} \quad \overline{U}^{T} \cdot H' \cdot U = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0 \end{bmatrix}$$

$$v_H \cdot v_H^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \lambda_{exact} := \operatorname{EV}(B) \rightarrow \begin{bmatrix} 1.0 \\ -1.1869334139818197152 \\ -0.3812677408218209518 + 1.2285914951694575107i \\ -0.3812677408218209518 - 1.2285914951694575107i \\ 0.47473444781273080941 - 1.4372565145936822087i \\ 0.47473444781273080941 + 1.4372565145936822087i \end{bmatrix}$$

$$LT(B) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0 \end{bmatrix} \quad LT(H) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8292 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8528 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.723 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.7334 & 0 \end{bmatrix}$$

Schur Decomposition (with and without shifts) of square matrix (real or complex) using iterative QR method. **A** is square matrix, **n** is maximum number to iterations for solution and **d** is number of digits for accuracy convergence.

```
Schur_S(A, n, d) :=  nr ← rows(A)
                     if nr ≠ cols(A)
                        return "Error: matrix is not square"
                     [H; U] ← Hessenberg(A)
                     O ← ORIGIN
                     in ← nr − 1 + O
                     k ← 0
                     t ← H
                     I ← identity(nr)
                     u ← I
                     λ_i ← 0
                     λ_j ← 0
                     S ← 0·I
                     for i ∈ 0..|n|
                        [Q; R] ← QR(t − S, 0)
                        t ← R·Q + S
                        u ← u·Q
                        if n > 0
                           λ_j ← 0
                           if |t_(in, in−1)| = 1
                              λ_i ← 1
                           else if in − 1 ≥ O
                              t_s ← submatrix(t, in−1, in, in−1, in)
                              λ_s ← eigenvals(t_s)
                              if Im(λs_O) = 0 ∨ Im(λs_(O+1))
                                 "Wilkinson single shift"
                                 if |λs_O − t_(in,in)| ≤ |λs_(O+1) − t_(in,in)|
                                    λ_i ← λs_O
                                 else
                                    λ_i ← λs_(O+1)
                              else
                                 "Double shift"
                                 λ_i ← λs_O
                                 λ_j ← λs_(O+1)
                              k ← k + 10^−5
                           else
                              "Rayleigh single shift"
                              λ_i ← t_(in,in)
                           S ← λ_i·I + λ_j·I
                           if round(|t_(in,in−1)|, |d|) ≤ 0
                              in ← in − 1
                        if round(norme(LT(t)), |d| + 1) ≤ 0
                           return [i+k  conj(U)^T·u  t]^T
                        if in − 1 < O
                           in ← nr − 1 + O
                     [i+k  conj(U)^T·u  t]^T
```

$A = u \cdot t \cdot \overline{u}^{T} = Q \cdot T \cdot \overline{Q}^{T}$ where $\overline{Q}^{T}$ is the complex conjugate transpose

Schur with first transforming matrix to Hessenberg form and introducing shifting to increase convergence

If n>0 shifting is performed.
If -n is input no shifting is performed.

returned

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix}$

- i: Number of iterations and shifts performed
- Q: Unitary matrix ($Q^{-1} = \overline{Q}^{T}$)
- T: Upper triangular matrix with eigenvalues along diagonal

S is shift matrix in QR decomposition

Shifting sequences are preformed based on eigenvalues from 2x2 submatrix $\begin{bmatrix} t_{in-1,in-1} & t_{in-1,in} \\ t_{in,in-1} & t_{in,in} \end{bmatrix}$ starting at bottom right lower off-diagonal of Hessenberg matrix to upper left as $t_{in,in-1}$ converges to zero.

For Wilkinson single shift, select eigenvalue λ_i from 2x2 submatrix $\begin{bmatrix} t_{in-1,in-1} & t_{in-1,in} \\ t_{in,in-1} & t_{in,in} \end{bmatrix}$ that is closest to $t_{in,in}$

The number of iterations (i) and number of number of shifts (k) performed is returned as i.k.

If not converged when upper left shift window is reached, reset index for shifting to restart at bottom right.

Schur with precalculated eigenvectors

```
Schur_V(A, n, d) := | nr ← rows(A)
                    | if nr ≠ cols(A)
                    |   | return "Error: matrix is not square"
                    | [H; U] ← Hessenberg(A)
                    | V_R ← eigenvecs(H, "R")
                    | [Q_V; R_V] ← QR(V_R, 0)
                    | T_V ← conj(Q_V)^T · H · Q_V
                    | if round(norme(LT(T_V)), |d| − 1) ≤ 0
                    |   | return [0  conj(U)^T · Q_V  T_V]^T
                    | Schur_S(A, n, d)
```

Eigenvectors for Schur decomposition

```
E_Vec(Q, T) := | O ← ORIGIN
               | nr ← rows(T)
               | in ← nr − 1 + O
               | λ ← diag(T)
               | U ← identity(nr)
               | for i ∈ O + 1 .. in
               |   | M ← λ_i · identity(i − 1 + 1 − O)
               |   |       − submatrix(T, O, i − 1, O, i − 1)
               |   | v ← M^(−1) · submatrix(T, O, i − 1, i, i)
               |   | if i = O + 1
               |   |   | U_(O,i) ← v
               |   | else
               |   |   | for k ∈ O .. i − 1
               |   |   |   | U_(k,i) ← v_k
               | V ← Q · U
```

https://math.stackexchange.com/questions/3947108/how-to-get-eigenvectors-using-qr-algorithm#:~:text=It%20is%20not%20hard%20to%20obtain%20eigenvectors%20when,%28%E2%88%97%20%E2%88%97%20%E2%8B%AE%20%E2%88%97%201%200%20%E2%8B%AE%200%29

p-th power or primary p-th root of a square matrix with eigenvalues λ and corresponding eigenvectors V

$$M_p(\lambda, V, p) := V \cdot \mathrm{diag}(\lambda^p) \cdot V^{-1}$$

Summary results: This version of Schur decomposition (with shifts) of square matrix $A = u \cdot t \cdot \overline{u}^{\mathrm{T}} = Q \cdot T \cdot \overline{Q}^{\mathrm{T}}$ returns eigenvalues of matrix along diagonal of converged upper triangular matrix T if the eigenvalues of A are distinct real/complex eigenvalues or complex pairs. Repeated or very close eigenvalues (real/complex) may be obtained but likely at reduced accuracy and increased iterations.

If A is a normal matrix ($A \cdot \overline{A}^{\mathrm{T}} = \overline{A}^{\mathrm{T}} \cdot A$) then T is diagonal and Q are the eigenvectors.

$n := 10^6$ $\qquad$ $d := 14$

Examples

$A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\lambda := \text{eigenvals}(A) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $|\text{tr}(A)| = 0$ $|||A||| = 1$ $Sym(A) = \text{“yes”}$

$Normality_{check}(A) = \text{“yes”}$ $\text{conde}(A) = 2$

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d)$ $i = 1000000$ $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ no shifting => no solution

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d)$ $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $Q \cdot T \cdot \overline{Q}^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$i = 1.00001$

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d)$ $i = 0$ $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$A := \begin{bmatrix} 3 & 2 & 1 \\ 4 & 2 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ $\lambda := \text{eigenvals}(A) = \begin{bmatrix} 7.27491722 \\ 1 \\ -0.27491722 \end{bmatrix}$ $|\text{tr}(A)| = 8$ $|||A||| = 2$ $Sym(A) = \text{“no”}$

$Normality_{check}(A) = \text{“no”}$ $\text{conde}(A) = 58.8048$

no shifting => greater iterations

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d)$ $i = 27$ $T = \begin{bmatrix} 7.27491722 & 4.05917951 & 1.86494765 \\ 0 & 1 & 1.43004616 \\ 0 & 0 & -0.27491722 \end{bmatrix}$ $\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 7 \cdot 10^{-14} \\ -2 \cdot 10^{-14} \\ 4 \cdot 10^{-13} \end{bmatrix} \end{bmatrix}$

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d)$ $T = \begin{bmatrix} 7.27491722 & 4.05917951 & 1.86494765 \\ 0 & 1 & 1.43004616 \\ 0 & 0 & -0.27491722 \end{bmatrix}$ $\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 1 \cdot 10^{-13} \\ -7 \cdot 10^{-14} \\ 4 \cdot 10^{-13} \end{bmatrix} \end{bmatrix}$

$i = 5.00006$

$Q \cdot T \cdot \overline{Q}^T = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 2 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ check $Q = \begin{bmatrix} 0.3986 & 0.4439 & 0.8026 \\ 0.4534 & 0.6653 & -0.5931 \\ 0.7972 & -0.6003 & -0.064 \end{bmatrix}$ $\overline{Q}^T \cdot A \cdot Q = \begin{bmatrix} 7.2749 & 4.0592 & 1.8649 \\ 0 & 1 & 1.43 \\ 0 & 0 & -0.2749 \end{bmatrix}$

$V := E_{Vec}(Q, T) = \begin{bmatrix} 0.3986 & 0.186 & 0.4466 \\ 0.4534 & 0.372 & -1.178 \\ 0.7972 & -1.116 & 0.8932 \end{bmatrix}$ $V^{-1} \cdot A \cdot V = \begin{bmatrix} 7.2749 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.2749 \end{bmatrix}$

$Z := M_p\left(\text{diag}(T), V, \frac{1}{3}\right) = \begin{bmatrix} 1.1252 + 0.2018i & 0.4272 - 0.1492i & 0.1633 - 0.0161i \\ 1.0556 - 0.5324i & 0.811 + 0.3934i & 0.1129 + 0.0424i \\ 0.2505 + 0.4037i & 0.8543 - 0.2983i & 1.3265 - 0.0322i \end{bmatrix}$ $Z^3 = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 2 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ check

$M_p(\text{diag}(T), V, 10) = \begin{bmatrix} 162583013 & 109995046 & 63762184 \\ 184931448 & 125114812 & 72526845 \\ 325166024 & 219990092 & 127524369 \end{bmatrix}$ $A^{10} = \begin{bmatrix} 162583013 & 109995046 & 63762184 \\ 184931448 & 125114812 & 72526845 \\ 325166024 & 219990092 & 127524369 \end{bmatrix}$ check

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d)$ $i = 0$ $T = \begin{bmatrix} 7.27491722 & 4.05917951 & -1.86494765 \\ 0 & 1 & -1.43004616 \\ 0 & 0 & -0.27491722 \end{bmatrix}$ $\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 2 \cdot 10^{-14} \\ -2 \cdot 10^{-13} \\ 2 \cdot 10^{-13} \end{bmatrix} \end{bmatrix}$

Schur Decomposition of Square Matrices

$$A := \begin{bmatrix} -5 & 7 & 3 & 4 & -8 \\ 5 & 8 & 3 & 6 & 8 \\ 3 & -7 & 9 & -4 & 5 \\ -3 & 0 & 4 & 5 & 3 \\ 7 & 4 & 5 & 9 & 5 \end{bmatrix} \qquad \begin{bmatrix} H \\ U \end{bmatrix} := Hessenberg(A) \qquad H = \begin{bmatrix} -5 & 2.5022 & -10.0741 & 3.5819 & -4.1738 \\ -9.5917 & 8.6848 & 6.9063 & -0.1452 & -8.0904 \\ 0 & 6.8555 & 3.7544 & -0.2831 & -6.1491 \\ 0 & 0 & 11.2241 & 7.4208 & 0.5547 \\ 0 & 0 & 0 & 4.5914 & 7.14 \end{bmatrix}$$

$$\lambda := \mathrm{eigenvals}(A) = \begin{bmatrix} 13.14066209 + 4.93688069i \\ 13.14066209 - 4.93688069i \\ 4.8798093 \\ -4.58056674 + 6.94205086i \\ -4.58056674 - 6.94205086i \end{bmatrix}$$

$|\mathrm{tr}(A)| = 22 \qquad \||A\|| = 66515 \qquad Sym(A) = \text{"no"}$

$Normality_{check}(A) = \text{"no"} \qquad \mathrm{conde}(A) = 9.4955$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 1000000 \qquad rs(\mathrm{diag}(T)) = \begin{bmatrix} 15.86194524 \\ 10.41937893 \\ 4.8798093 \\ -3.10459163 \\ -6.05654185 \end{bmatrix}$$

Bad

$$\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 6 + 4i \cdot 10 \\ -3 \cdot 10 - 3i \cdot 10 \\ 4 \cdot 10^{-14} \\ -8 \cdot 10 + 3i \cdot 10 \\ -6 \cdot 10 - 6i \cdot 10 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 34.00035 \qquad rs(\mathrm{diag}(T)) = \begin{bmatrix} 13.14066209 + 4.93688069i \\ 13.14066209 - 4.93688069i \\ 4.8798093 \\ -4.58056674 + 6.94205086i \\ -4.58056674 - 6.94205086i \end{bmatrix}$$

$$\lambda_s := switch(\lambda, 4, 5) = \begin{bmatrix} 13.14066209 + 4.93688069i \\ 13.14066209 - 4.93688069i \\ 4.8798093 \\ -4.58056674 - 6.94205086i \\ -4.58056674 + 6.94205086i \end{bmatrix} \qquad rs\left(\overline{\lambda_s}\right) = \begin{bmatrix} 13.14066209 + 4.93688069i \\ 13.14066209 - 4.93688069i \\ 4.8798093 \\ -4.58056674 + 6.94205086i \\ -4.58056674 - 6.94205086i \end{bmatrix}$$

$$\lambda_{error}\left(\overline{\lambda_s}, T\right) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 9 \cdot 10^{-14} + 1i \cdot 10^{-14} \\ 4 \cdot 10^{-14} + 4i \cdot 10^{-14} \\ 1 \cdot 10^{-13} - 5i \cdot 10^{-14} \\ -7 \cdot 10^{-14} - 9i \cdot 10^{-14} \\ 1 \cdot 10^{-13} - 3i \cdot 10^{-13} \end{bmatrix} \end{bmatrix}$$

check

$$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} -5 & 7 & 3 & 4 & -8 \\ 5 & 8 & 3 & 6 & 8 \\ 3 & -7 & 9 & -4 & 5 \\ -3 & 0 & 4 & 5 & 3 \\ 7 & 4 & 5 & 9 & 5 \end{bmatrix}$$

$$T = \begin{bmatrix} -4.5806 - 6.9421i & 5.3152 + 5.8028i & -0.7908 + 0.7898i & 0.9505 + 7.3613i & 0.0765 + 0.6092i \\ 0 & -4.5806 + 6.9421i & 6.9805 - 4.5504i & 1.2195 + 0.0592i & -0.4644 - 0.5182i \\ 0 & 0 & 13.1407 - 4.9369i & 4.2543 + 5.036i & -0.2992 - 2.0082i \\ 0 & 0 & 0 & 13.1407 + 4.9369i & 1.4514 + 0.0232i \\ 0 & 0 & 0 & 0 & 4.8798 \end{bmatrix}$$

$$V := E_{Vec}(Q, T) = \begin{bmatrix} -0.418 + 0.6048i & 0.4758 + 0.6984i & 0.0347 - 0.0561i & 0.023 - 0.0759i & 0.1968 + 0.2457i \\ 0.1417 - 0.1721i & -0.1616 - 0.1989i & 0.5383 - 0.5609i & 0.0788 - 0.9314i & 0.4243 + 0.5297i \\ 0.233 - 0.0938i & -0.2671 - 0.1096i & -0.508 - 0.0603i & 0.4578 + 0.4107i & 0.1796 + 0.2243i \\ 0.0746 + 0.308i & -0.0881 + 0.3534i & -0.0542 - 0.2161i & 0.2382 - 0.1227i & -0.4411 - 0.5507i \\ -0.2542 - 0.4262i & 0.2955 - 0.4879i & 0.2092 - 0.4729i & 0.2608 - 0.5644i & -0.025 - 0.0312i \end{bmatrix}$$

$$V^{-1} \cdot A \cdot V = \begin{bmatrix} -4.5806 - 6.9421i & 0 & 0 & 0 & 0 \\ 0 & -4.5806 + 6.9421i & 0 & 0 & 0 \\ 0 & 0 & 13.1407 - 4.9369i & 0 & 0 \\ 0 & 0 & 0 & 13.1407 + 4.9369i & 0 \\ 0 & 0 & 0 & 0 & 4.8798 \end{bmatrix}$$

$$Z := M_p\left(\mathrm{diag}(T), V, \frac{1}{3}\right) = \begin{bmatrix} 1.2113 & 1.0188 & 0.2972 & 0.9713 & -1.5286 \\ 0.2456 & 1.8538 & 0.0732 & 0.2445 & 0.8199 \\ 0.0194 & -0.561 & 2.1126 & -0.3936 & 0.5693 \\ -0.4219 & 0.3311 & 0.3868 & 1.9976 & -0.2434 \\ 0.9648 & -0.1138 & 0.1631 & 0.3496 & 2.3626 \end{bmatrix} \qquad Z^3 = \begin{bmatrix} -5 & 7 & 3 & 4 & -8 \\ 5 & 8 & 3 & 6 & 8 \\ 3 & -7 & 9 & -4 & 5 \\ -3 & 0 & 4 & 5 & 3 \\ 7 & 4 & 5 & 9 & 5 \end{bmatrix}$$

check

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 0 \qquad rs(\mathrm{diag}(T)) = \begin{bmatrix} 13.14066209 + 4.93688069i \\ 13.14066209 - 4.93688069i \\ 4.8798093 \\ -4.58056674 - 6.94205086i \\ -4.58056674 + 6.94205086i \end{bmatrix}$$

$$\lambda_{error}\left(\overline{\lambda}, T\right) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 1 \cdot 10^{-13} + 1i \cdot 10^{-14} \\ -2 \cdot 10^{-14} - 1i \cdot 10^{-14} \\ 9 \cdot 10^{-14} - 8i \cdot 10^{-15} \\ 5 \cdot 10^{-14} + 4i \cdot 10^{-14} \\ 7 \cdot 10^{-14} + 4i \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$rs(\lambda) = \begin{bmatrix} 13.14066209 - 4.93688069i \\ 13.14066209 + 4.93688069i \\ 4.8798093 \\ -4.58056674 + 6.94205086i \\ -4.58056674 - 6.94205086i \end{bmatrix} \qquad rs\left(\overline{\lambda}\right) = \begin{bmatrix} 13.14066209 + 4.93688069i \\ 13.14066209 - 4.93688069i \\ 4.8798093 \\ -4.58056674 - 6.94205086i \\ -4.58056674 + 6.94205086i \end{bmatrix}$$

have to get same order between $diag(T)$ and λ for error comparison

Schur Decomposition of Square Matrices

$$A := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & -3 & 4 & 5 & 6 & 7 \\ 3 & 4 & -5 & 6 & 7 & 8 \\ 4 & 5 & 6 & -7 & 8 & 9 \\ 5 & 6 & 7 & 8 & -9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{bmatrix} \qquad \lambda := \mathrm{eigenvals}(A) = \begin{bmatrix} 31.40835272 \\ -1.47300448 \\ -4.98718606 \\ -8.00500708 \\ -12.28750334 \\ -16.65565176 \end{bmatrix}$$

$$|\mathrm{tr}(A)| = 12 \qquad |||A||| = 378000 \qquad Sym(A) = \text{"yes"}$$

$$Normality_{check}(A) = \text{"yes"} \qquad \mathrm{conde}(A) = 28.197$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 121 \qquad rs(\mathrm{diag}(T)) = \begin{bmatrix} 31.40835272 \\ -1.47300448 \\ -4.98718606 \\ -8.00500708 \\ -12.28750334 \\ -16.65565176 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 3 \cdot 10^{-14} \\ 3 \cdot 10^{-14} \\ 0 \\ -1 \cdot 10^{-13} \\ 2 \cdot 10^{-13} \\ -1 \cdot 10^{-13} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 56.00057 \qquad rs(\mathrm{diag}(T)) = \begin{bmatrix} 31.40835272 \\ -1.47300448 \\ -4.98718606 \\ -8.00500708 \\ -12.28750334 \\ -16.65565176 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -3 \cdot 10^{-14} \\ -4 \cdot 10^{-13} \\ -2 \cdot 10^{-14} \\ 4 \cdot 10^{-14} \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$Q \cdot T \cdot \overline{Q}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & -3 & 4 & 5 & 6 & 7 \\ 3 & 4 & -5 & 6 & 7 & 8 \\ 4 & 5 & 6 & -7 & 8 & 9 \\ 5 & 6 & 7 & 8 & -9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{bmatrix} \qquad T = \begin{bmatrix} 31.40835272 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8.00500708 & 0 & 0 & 0 & 0 \\ 0 & 0 & -16.65565176 & 0 & 0 & 0 \\ 0 & 0 & 0 & -12.28750334 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4.98718606 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.47300448 \end{bmatrix}$$

$$V := E_{Vec}(Q, T) = \begin{bmatrix} -0.2927 & 0.1439 & 0.0759 & 0.1056 & 0.0033 & 0.9363 \\ -0.308 & 0.6253 & 0.1162 & 0.2059 & 0.6377 & -0.2272 \\ -0.3345 & -0.6795 & 0.182 & 0.5646 & 0.261 & -0.0795 \\ -0.3563 & -0.2432 & 0.4464 & -0.7524 & 0.2186 & -0.0261 \\ -0.3746 & -0.1561 & -0.8624 & -0.225 & 0.2023 & 0.0014 \\ -0.6635 & 0.2076 & 0.068 & 0.1044 & -0.6607 & -0.2543 \end{bmatrix} \qquad Q = \begin{bmatrix} -0.2927 & 0.1439 & 0.0759 & 0.1056 & 0.0033 & 0.9363 \\ -0.308 & 0.6253 & 0.1162 & 0.2059 & 0.6377 & -0.2272 \\ -0.3345 & -0.6795 & 0.182 & 0.5646 & 0.261 & -0.0795 \\ -0.3563 & -0.2432 & 0.4464 & -0.7524 & 0.2186 & -0.0261 \\ -0.3746 & -0.1561 & -0.8624 & -0.225 & 0.2023 & 0.0014 \\ -0.6635 & 0.2076 & 0.068 & 0.1044 & -0.6607 & -0.2543 \end{bmatrix}$$

$$V^{-1} \cdot A \cdot V = \begin{bmatrix} 31.4084 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8.005 & 0 & 0 & 0 & 0 \\ 0 & 0 & -16.6557 & 0 & 0 & 0 \\ 0 & 0 & 0 & -12.2875 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4.9872 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.473 \end{bmatrix}$$

Note that V=Q and T is diagonal
since the A-matrix is normal

$$Z := M_p\left(\mathrm{diag}(T), V, \frac{1}{3}\right) = \begin{bmatrix} 0.8099+0.9348i & 0.2915+0.0122i & 0.2559-0.0917i & 0.2323-0.1675i & 0.2138-0.2288i & 0.5246-0.1526i \\ 0.2915+0.0122i & 1.1332+1.4444i & 0.2136-0.193i & 0.2042-0.2462i & 0.1951-0.2926i & 0.4824-0.2812i \\ 0.2559-0.0917i & 0.2136-0.193i & 1.2868+1.6173i & 0.2049-0.2965i & 0.1994-0.3394i & 0.5072-0.3344i \\ 0.2323-0.1675i & 0.2042-0.2462i & 0.2049-0.2965i & 1.4086+1.7459i & 0.2007-0.3819i & 0.5241-0.3844i \\ 0.2138-0.2288i & 0.1951-0.2926i & 0.1994-0.3394i & 0.2007-0.3819i & 1.5102+1.8488i & 0.5355-0.431i \\ 0.5246-0.1526i & 0.4824-0.2812i & 0.5072-0.3344i & 0.5241-0.3844i & 0.5355-0.431i & 1.8603+0.8162i \end{bmatrix}$$

$$Z^3 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & -3 & 4 & 5 & 6 & 7 \\ 3 & 4 & -5 & 6 & 7 & 8 \\ 4 & 5 & 6 & -7 & 8 & 9 \\ 5 & 6 & 7 & 8 & -9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{bmatrix}$$

check

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 0 \qquad rs(\mathrm{diag}(T)) = \begin{bmatrix} 31.40835272 \\ -1.47300448 \\ -4.98718606 \\ -8.00500708 \\ -12.28750334 \\ -16.65565176 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -5 \cdot 10^{-14} \\ -2 \cdot 10^{-14} \\ -5 \cdot 10^{-14} \\ 0 \\ -4 \cdot 10^{-14} \\ 2 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$B := \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

2 real distinct eigenvalues and 2 pairs of complex conjugate eigenvalues

$|\mathrm{tr}(B)| = 0 \quad \||B|\| = 4.5$

$Normality_{check}(B) = \text{“no”} \quad \mathrm{conde}(B) = 10.4003$

$$\lambda := \mathrm{eigenvals}(B) = \begin{bmatrix} 1 \\ 0.47473445 - 1.43725651i \\ 0.47473445 + 1.43725651i \\ -0.38126774 + 1.2285915i \\ -0.38126774 - 1.2285915i \\ -1.18693341 \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(B, -n, d) \quad i = 1000000 \quad rs(\mathrm{diag}(T)) = \begin{bmatrix} 1 \\ 0.75207205 \\ 0.19739684 \\ -0.14182049 \\ -0.62071499 \\ -1.18693341 \end{bmatrix}$$

Bad

$$\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} -8 \cdot 10^{-14} \\ -8 \cdot 10 - 5i \cdot 10 \\ -10 \cdot 10 + 1i \cdot 10 \\ -10 \cdot 10 + 1i \cdot 10 \\ -9 \cdot 10 - 5i \cdot 10 \\ 2 \cdot 10^{-13} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(B, n, d) \quad i = 107.00108$$

$$rs\left(\mathrm{diag}\left(\overline{T}\right)\right) = \begin{bmatrix} 1 \\ 0.47473445 + 1.43725651i \\ 0.47473445 - 1.43725651i \\ -0.38126774 + 1.2285915i \\ -0.38126774 - 1.2285915i \\ -1.18693341 \end{bmatrix}$$

$$\lambda_{error}\left(\lambda, \overline{T}\right) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 2 \cdot 10^{-13} - 2i \cdot 10^{-14} \\ 2 \cdot 10^{-13} - 2i \cdot 10^{-13} \\ -10 \cdot 10^{-14} + 1i \cdot 10^{-13} \\ -3 \cdot 10^{-13} + 5i \cdot 10^{-14} \\ 10 \cdot 10^{-14} - 2i \cdot 10^{-13} \\ 0 \end{bmatrix} \end{bmatrix}$$

$$Q \cdot T \cdot \overline{Q}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0 \end{bmatrix}$$

check

$$T = \begin{bmatrix} 0.4747 - 1.4373i & -0.0976 - 0.5501i & -1.0003 - 0.2864i & -0.4529 + 0.3076i & -0.1389 - 0.3291i & 0.0036 - 0.7203i \\ 0 & -1.1869 & -0.1552 + 0.3172i & -0.3069 + 0.1461i & -0.3732 + 0.1642i & 0.3643 + 0.3748i \\ 0 & 0 & -0.3813 - 1.2286i & -0.2774 - 0.6089i & 0.1618 - 0.2217i & 0.2953 + 0.5449i \\ 0 & 0 & 0 & 1 & 0.2026 - 0.0346i & 0.8724 + 0.3533i \\ 0 & 0 & 0 & 0 & -0.3813 + 1.2286i & 1.1212 + 0.2942i \\ 0 & 0 & 0 & 0 & 0 & 0.4747 + 1.4373i \end{bmatrix}$$

$$V := E_{Vec}(Q, T) = \begin{bmatrix} -0.4847 - 0.1775i & -0.1303 - 0.3226i & -0.6149 - 0.0374i & 0.1669 - 0.5629i & -0.0214 - 0.4246i & -0.6487 - 0.7439i \\ -0.0471 - 0.3404i & 0.2096 + 0.5188i & 0.3953 - 0.5272i & 0.2226 - 0.7505i & -0.3667 + 0.269i & -0.6399 + 0.1488i \\ 0.1348 + 0.2413i & -0.1866 - 0.4618i & 0.3568 + 0.7835i & 0.1113 - 0.3753i & 0.5381 + 0.2519i & 0.523 + 0.0755i \\ -0.1924 + 0.4759i & 0.1472 + 0.3644i & -0.6074 + 0.4529i & 0 & 0.3169 - 0.4159i & 0.7179 - 0.6692i \\ -0.4074 + 0.3193i & -0.134 - 0.3317i & -0.1398 - 0.1868i & -0.1113 + 0.3753i & -0.1279 - 0.0978i & 0.2911 - 0.9458i \\ 0.0269 - 0.0837i & 0.1185 + 0.2932i & 0.1836 + 0.2473i & -0.0556 + 0.1876i & 0.1693 + 0.1285i & -0.131 + 0.1052i \end{bmatrix}$$

$$V^{-1} \cdot B \cdot V = \begin{bmatrix} 0.4747 - 1.4373i & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.1869 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.3813 - 1.2286i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.3813 + 1.2286i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4747 + 1.4373i \end{bmatrix}$$

$$Z := M_p\left(\mathrm{diag}(T), V, \frac{1}{3}\right) = \begin{bmatrix} 0.972 + 0.075i & 0.146 - 0.146i & 0.292 + 0.099i & 0.329 - 0.192i & 0.257 + 0.036i & 0.571 - 0.234i \\ 0.262 - 0.12i & 0.801 + 0.235i & -0.2 - 0.159i & -0.026 + 0.308i & 0.022 - 0.058i & -0.451 + 0.377i \\ -0.424 + 0.107i & 0.414 - 0.209i & 0.995 + 0.141i & -0.124 - 0.274i & 0.218 + 0.051i & -0.064 - 0.335i \\ -0.256 - 0.084i & -0.039 + 0.165i & 0.44 - 0.111i & 0.903 + 0.217i & -0.02 - 0.04i & -0.007 + 0.265i \\ -0.28 + 0.077i & 0.004 - 0.15i & 0.01 + 0.101i & 0.371 - 0.197i & 0.817 + 0.037i & -0.436 - 0.241i \\ -0.034 - 0.068i & 0.084 + 0.133i & 0.056 - 0.09i & -0.181 + 0.174i & 0.221 - 0.033i & 0.905 + 0.213i \end{bmatrix}$$

$$Z^3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0 \end{bmatrix}$$

check

$$rs(\lambda) = \begin{bmatrix} 1 \\ 0.47473445 + 1.43725651i \\ 0.47473445 - 1.43725651i \\ -0.38126774 + 1.2285915i \\ -0.38126774 - 1.2285915i \\ -1.18693341 \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(B, n, d) \quad i = 0$$

$$rs(\mathrm{diag}(T)) = \begin{bmatrix} 1 \\ 0.47473445 + 1.43725651i \\ 0.47473445 - 1.43725651i \\ -0.38126774 + 1.2285915i \\ -0.38126774 - 1.2285915i \\ -1.18693341 \end{bmatrix}$$

$$\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} -2 \cdot 10^{-14} \\ 6 \cdot 10^{-14} + 1i \cdot 10^{-14} \\ 8 \cdot 10^{-14} - 3i \cdot 10^{-14} \\ -1 \cdot 10^{-13} + 2i \cdot 10^{-14} \\ -6 \cdot 10^{-14} - 4i \cdot 10^{-14} \\ 4 \cdot 10^{-14} - 5i \cdot 10^{-15} \end{bmatrix} \end{bmatrix}$$

Schur Decomposition of Square Matrices

Example from MATLAB help files.

Note that Mathcad eigenvalue routine does not obtain accurate numerical values in this case

$$A := \begin{bmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{bmatrix} \qquad \lambda := \text{eigenvals}(A) = \begin{bmatrix} 2.99999999999724 \\ 1.99999999999276 \\ 1.00000000000991 \end{bmatrix}$$

$|\text{tr}(A)| = 6 \qquad |||A||| = 6 \qquad Sym(A) = \text{“no”}$

$Normality_{check}(A) = \text{“no”} \qquad \text{conde}(A) = 275850.1039$

Real distinct eigenvalues

$$\lambda_{exact} := \text{EV}(A) \rightarrow \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 85 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 2 \cdot 10^{-10} \\ 1 \cdot 10^{-9} \\ -3 \cdot 10^{-9} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 8.00009 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 2 \cdot 10^{-10} \\ 1 \cdot 10^{-9} \\ -3 \cdot 10^{-9} \end{bmatrix} \end{bmatrix}$$

$$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{bmatrix} \qquad T = \begin{bmatrix} 3 & 2.40979759 & 803.59808556 \\ 0 & 2 & -151.4836947 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 8.00009 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 2 \cdot 10^{-10} \\ 1 \cdot 10^{-9} \\ -3 \cdot 10^{-9} \end{bmatrix} \end{bmatrix}$$

Second example from lecture notes.

Note that Mathcad eigenvalue routine does not obtain accurate numerical values in this case

$$A := \begin{bmatrix} 13 & 8 & 8 \\ -1 & 7 & -2 \\ -1 & -2 & 7 \end{bmatrix} \qquad \lambda := \text{eigenvals}(A) = \begin{bmatrix} 9.00000012 \\ 9 \\ 8.99999988 \end{bmatrix}$$

$|\text{tr}(A)| = 27 \qquad |||A||| = 729$

$Normality_{check}(A) = \text{“no”}$

$Sym(A) = \text{“no”} \qquad \text{conde}(A) = 5$

$$\lambda_{exact} := \text{EV}(A) \rightarrow \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}$$

Repeated real roots

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 1000000 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 9.000009 \\ 9 \\ 8.999991 \end{bmatrix} \qquad \lambda_{error}(\lambda_{exact}, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 1 \cdot 10^{-4} \\ 0 \\ -1 \cdot 10^{-4} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 3.00004 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 9.00000016 \\ 9 \\ 8.99999984 \end{bmatrix} \qquad \lambda_{error}(\lambda_{exact}, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 2 \cdot 10^{-6} \\ 0 \\ -2 \cdot 10^{-6} \end{bmatrix} \end{bmatrix}$$

$$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} 13 & 8 & 8 \\ -1 & 7 & -2 \\ -1 & -2 & 7 \end{bmatrix}$$ check $$T = \begin{bmatrix} 9 & -12.0432 & 4.1183 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 0 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 9.00000015 \\ 9 \\ 8.99999985 \end{bmatrix} \qquad \lambda_{error}(\lambda_{exact}, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 2 \cdot 10^{-6} \\ 2 \cdot 10^{-14} \\ -2 \cdot 10^{-6} \end{bmatrix} \end{bmatrix}$$

Schur Decomposition of Square Matrices

$$A := \begin{bmatrix} -1 & -0.5 & -0.5 & -0.5 \\ -0.5 & -1 & -0.5 & -0.5 \\ -0.5 & -0.5 & -1 & -0.5 \\ -0.5 & -0.5 & -0.5 & -1 \end{bmatrix} \qquad \lambda := \text{eigenvals}(A) = \begin{bmatrix} -0.5 \\ -0.5 \\ -0.5 \\ -2.5 \end{bmatrix}$$

$|\text{tr}(A)| = 4$ $\quad |||A||| = 0.3125$ $\quad Sym(A) = \text{“yes”}$

$Normality_{check}(A) = \text{“yes”}$ $\quad \text{conde}(A) = 9.2261$

3 Repeated real roots

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 22 \qquad rs(\text{diag}(T)) = \begin{bmatrix} -0.5 \\ -0.5 \\ -0.5 \\ -2.5 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} -1 \cdot 10^{-14} \\ 0 \\ 2 \cdot 10^{-14} \\ 0 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 1.00002 \qquad rs(\text{diag}(T)) = \begin{bmatrix} -0.5 \\ -0.5 \\ -0.5 \\ -2.5 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} -1 \cdot 10^{-14} \\ 2 \cdot 10^{-14} \\ 2 \cdot 10^{-14} \\ 2 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} -1 & -0.5 & -0.5 & -0.5 \\ -0.5 & -1 & -0.5 & -0.5 \\ -0.5 & -0.5 & -1 & -0.5 \\ -0.5 & -0.5 & -0.5 & -1 \end{bmatrix} \quad \text{check} \quad T = \begin{bmatrix} -2.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 0 \qquad rs(\text{diag}(T)) = \begin{bmatrix} -0.5 \\ -0.5 \\ -0.5 \\ -2.5 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 0 \\ 2 \cdot 10^{-14} \\ 7 \cdot 10^{-14} \\ 5 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$A := \begin{bmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \\ 4 & 4 & 4 \end{bmatrix} \qquad \lambda := \text{eigenvals}(A) = \begin{bmatrix} 9.65685425 \\ -1.65685425 \\ -4 \end{bmatrix}$$

$|\text{tr}(A)| = 4$ $\quad |||A||| = 64$ $\quad Sym(A) = \text{“yes”}$

$Normality_{check}(A) = \text{“yes”}$ $\quad \text{conde}(A) = 7$

Real distinct eigenvalues

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 43 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 9.65685425 \\ -1.65685425 \\ -4 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} -2 \cdot 10^{-13} \\ -5 \cdot 10^{-14} \\ 6 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 5.00006 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 9.65685425 \\ -1.65685425 \\ -4 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} -7 \cdot 10^{-14} \\ -8 \cdot 10^{-14} \\ -7 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \\ 4 & 4 & 4 \end{bmatrix} \quad \text{check} \quad T = \begin{bmatrix} 9.65685425 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1.65685425 \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 0 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 9.65685425 \\ -1.65685425 \\ -4 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 2 \cdot 10^{-14} \\ -8 \cdot 10^{-14} \\ -1 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

Schur Decomposition of Square Matrices

$$A := \begin{bmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \\ 4 & 4 & 0 \end{bmatrix} \qquad \lambda := \text{eigenvals}(A) = \begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix}$$

$|\text{tr}(A)| = 0$ $|||A||| = 128$ $Sym(A) = \text{"yes"}$

$Normality_{check}(A) = \text{"yes"}$ $\text{conde}(A) = 3.6742$

2 Repeated real roots

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 57 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -9 \cdot 10^{-14} \\ -2 \cdot 10^{-14} \\ -4 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 2.00003 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -3 \cdot 10^{-14} \\ -1 \cdot 10^{-14} \\ -4 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \\ 4 & 4 & 0 \end{bmatrix}$$ check $$T = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 0 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 3 \cdot 10^{-14} \\ -1 \cdot 10^{-14} \\ -2 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$A := \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \lambda := \text{eigenvals}(A) = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$$

$|\text{tr}(A)| = 10$ $|||A||| = 0$ $Sym(A) = \text{"no"}$

$Normality_{check}(A) = \text{"no"}$

2 Repeated real roots and singular

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 1000000 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 5.000005 \\ 4.999995 \\ 0 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Difference"} \\ \begin{bmatrix} 5 \cdot 10^{-6} \\ -5 \cdot 10^{-6} \\ 0 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 2.00003 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 5.00000002 \\ 4.99999998 \\ 0 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Difference"} \\ \begin{bmatrix} 2 \cdot 10^{-8} \\ -2 \cdot 10^{-8} \\ 0 \end{bmatrix} \end{bmatrix}$$

$$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ check $$T = \begin{bmatrix} 5.00000002 & 1.01902034 & 0.96079966 \\ 0 & 4.99999998 & -0.19611616 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 2.00003 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 5.00000002 \\ 4.99999998 \\ 0 \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Difference"} \\ \begin{bmatrix} 2 \cdot 10^{-8} \\ -2 \cdot 10^{-8} \\ 0 \end{bmatrix} \end{bmatrix}$$

Schur Decomposition of Square Matrices

$A := \begin{bmatrix} 7 & -2 \\ 12 & -3 \end{bmatrix}$ $\lambda := \text{eigenvals}(A) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $|\text{tr}(A)| = 4$ $|||A||| = 3$ $Sym(A) = \text{"no"}$ $Normality_{check}(A) = \text{"no"}$ $\text{conde}(A) = 68.6667$

Real distinct eigenvalues

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d)$ $i = 30$ $rs(\text{diag}(T)) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 0 \\ -3 \cdot 10^{-13} \end{bmatrix} \end{bmatrix}$ $rs(\lambda) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d)$ $i = 1.00002$ $rs(\text{diag}(T)) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -1 \cdot 10^{-14} \\ 1 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, -d)$ $i = 1.00002$ $rs(\text{diag}(T)) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -1 \cdot 10^{-14} \\ 1 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$

$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} 7 & -2 \\ 12 & -3 \end{bmatrix}$ check $T = \begin{bmatrix} 3 & -14 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d)$ $i = 0$ $rs(\text{diag}(T)) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 6 \cdot 10^{-14} \\ -7 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$

$A := \begin{bmatrix} 3 & 2 & 1 \\ 4 & 2 & 1 \\ 4 & 4 & 0 \end{bmatrix}$ $\lambda := \text{eigenvals}(A) = \begin{bmatrix} 6.60555128 \\ -0.60555128 \\ -1 \end{bmatrix}$ $|\text{tr}(A)| = 5$ $|||A||| = 4$ $Sym(A) = \text{"no"}$ $Normality_{check}(A) = \text{"no"}$ $\text{conde}(A) = 24.9787$

Real distinct eigenvalues

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d)$ $i = 67$ $rs(\text{diag}(T)) = \begin{bmatrix} 6.60555128 \\ -0.60555128 \\ -1 \end{bmatrix}$ $\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 1 \cdot 10^{-14} \\ -4 \cdot 10^{-13} \\ 1 \cdot 10^{-13} \end{bmatrix} \end{bmatrix}$ $rs(\lambda) = \begin{bmatrix} 6.6056 \\ -0.6056 \\ -1 \end{bmatrix}$

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d)$ $i = 7.00008$ $rs(\text{diag}(T)) = \begin{bmatrix} 6.60555128 \\ -0.60555128 \\ -1 \end{bmatrix}$ $\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -5 \cdot 10^{-14} \\ -1 \cdot 10^{-13} \\ 7 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$

$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 2 & 1 \\ 4 & 4 & 0 \end{bmatrix}$ check $T = \begin{bmatrix} 6.60555128 & -4.499975 & -0.77440274 \\ 0 & -1 & 1.07262546 \\ 0 & 0 & -0.60555128 \end{bmatrix}$

$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d)$ $i = 0$ $rs(\text{diag}(T)) = \begin{bmatrix} 6.60555128 \\ -0.60555128 \\ -1 \end{bmatrix}$ $\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -8 \cdot 10^{-14} \\ -6 \cdot 10^{-14} \\ 9 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$

Schur Decomposition of complex matrix with complex eigenvalues.

Complex distinct eigenvalues

$$A := \begin{bmatrix} 1+12i & 3 & 7+5i \\ 3 & 5-7i & 9 \\ 7 & 9 & 24-6i \end{bmatrix} \qquad \lambda := \text{eigenvals}(A) = \begin{bmatrix} 28.57661407-4.2687316i \\ 1.43853697-6.85468943i \\ -0.01515104+10.12342103i \end{bmatrix}$$

$|\text{tr}(A)| = 30.0167$ $\quad \||A\|| = 2048.6957$

$Normality_{check}(A) = \text{"no"}$ $\quad Sym(A) = \text{"no"}$

$\text{conde}(A) = 6.185$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 91 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 28.57661407-4.2687316i \\ 1.43853697-6.85468943i \\ -0.01515104+10.12342103i \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -6\cdot10^{-14}+6i\cdot10^{-14} \\ -8\cdot10^{-14}-6i\cdot10^{-14} \\ 9\cdot10^{-14}-1i\cdot10^{-13} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 5.00006 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 28.57661407-4.2687316i \\ 1.43853697-6.85468943i \\ -0.01515104+10.12342103i \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -6\cdot10^{-14}+5i\cdot10^{-14} \\ -4\cdot10^{-14}+1i\cdot10^{-14} \\ -1\cdot10^{-13}-2i\cdot10^{-14} \end{bmatrix} \end{bmatrix}$$

check

$$Q\cdot T\cdot \overline{Q}^{T} = \begin{bmatrix} 1+12i & 3 & 7+5i \\ 3 & 5-7i & 9 \\ 7 & 9 & 24-6i \end{bmatrix} \qquad T = \begin{bmatrix} 28.57661407-4.2687316i & 11.56907626-3.0177873i & -0.64674693-0.883885705i \\ 0 & -0.01515104+10.12342103i & 1.50326297-1.09780224i \\ 0 & 0 & 1.43853697-6.85468943i \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 0 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 28.57661407-4.2687316i \\ 1.43853697-6.85468943i \\ -0.01515104+10.12342103i \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 6\cdot10^{-14}+2i\cdot10^{-14} \\ -8\cdot10^{-14}+6i\cdot10^{-15} \\ 4\cdot10^{-14}+5i\cdot10^{-14} \end{bmatrix} \end{bmatrix}$$

$$rs(\lambda) = \begin{bmatrix} 28.57661407-4.2687316i \\ 1.43853697-6.85468943i \\ -0.01515104+10.12342103i \end{bmatrix}$$

$$A := \begin{bmatrix} 10 & 1i & -3 \\ -1 & 2 & 1 \\ 3i & 1 & 5i \end{bmatrix} \qquad \lambda := \text{eigenvals}(A) = \begin{bmatrix} 10.38695123-0.78386257i \\ 2.04330424+0.25488805i \\ -0.43025547+5.52897453i \end{bmatrix}$$

$|\text{tr}(A)| = 13$ $\quad \||A\|| = 118.9496$

Complex distinct eigenvalues

$Normality_{check}(A) = \text{"no"}$

$Sym(A) = \text{"no"}$ $\quad \text{conde}(A) = 6.808$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 57 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 10.38695123-0.78386257i \\ 2.04330424+0.25488805i \\ -0.43025547+5.52897453i \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -2\cdot10^{-13}-1i\cdot10^{-14} \\ 9\cdot10^{-14}-5i\cdot10^{-15} \\ -10\cdot10^{-14}+9i\cdot10^{-15} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 5.00006 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 10.38695123-0.78386257i \\ 2.04330424+0.25488805i \\ -0.43025547+5.52897453i \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -5\cdot10^{-14}-4i\cdot10^{-15} \\ 2\cdot10^{-14}+5i\cdot10^{-18} \\ -4\cdot10^{-15}-5i\cdot10^{-14} \end{bmatrix} \end{bmatrix}$$

check

$$Q\cdot T\cdot \overline{Q}^{T} = \begin{bmatrix} 10 & 1i & -3 \\ -1 & 2 & 1 \\ 3i & 1 & 5i \end{bmatrix} \qquad T = \begin{bmatrix} 10.38695123-0.78386257i & -1.38783261+0.04792714i & -1.20630511-0.79711915i \\ 0 & -0.43025547+5.52897453i & -1.18253726+1.4438569i \\ 0 & 0 & 2.04330424+0.25488805i \end{bmatrix}$$

$$V := E_{Vec}(Q, T) = \begin{bmatrix} 0.47674274-0.83425236i & 0.18308804+0.15093313i & 0.03102529+0.04643046i \\ -0.05416145+0.11993814i & -0.04829549-0.11822636i & -0.42601181+0.93892133i \\ 0.11650829+0.2141181i & 0.95412894+0.17122882i & -0.22674265-0.02149559i \end{bmatrix}$$

$$V^{-1}\cdot A\cdot V = \begin{bmatrix} 10.38695123-0.78386257i & 0 & 0 \\ 0 & -0.43025547+5.52897453i & 0 \\ 0 & 0 & 2.04330424+0.25488805i \end{bmatrix}$$

$$Z := M_p\left(\text{diag}(T), V, \frac{1}{3}\right) = \begin{bmatrix} 2.178+0.029i & 0.019+0.088i & -0.259+0.119i \\ -0.128-0.009i & 1.252+0.03i & 0.107-0.074i \\ 0.129+0.245i & 0.136-0.114i & 1.534+0.864i \end{bmatrix} \qquad Z^3 = \begin{bmatrix} 10 & 1i & -3 \\ -1 & 2 & 1 \\ 3i & 1 & 5i \end{bmatrix}$$

check

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 0 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 10.38695123-0.78386257i \\ 2.04330424+0.25488805i \\ -0.43025547+5.52897453i \end{bmatrix} \qquad \lambda_{error}(\lambda, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -3\cdot10^{-14}+9i\cdot10^{-15} \\ 2\cdot10^{-14}+3i\cdot10^{-15} \\ -2\cdot10^{-14}-2i\cdot10^{-14} \end{bmatrix} \end{bmatrix}$$

$$rs(\lambda) = \begin{bmatrix} 10.38695123-0.78386257i \\ 2.04330424+0.25488805i \\ -0.43025547+5.52897453i \end{bmatrix}$$

Schur Decomposition of Square Matrices

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 \\ 0.5 & 1 & 2 & 4 & -1 & 1 \\ 0 & 0.5 & 2 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0.5 & 0 & 0 \\ 10 & 5 & 3 & 8 & -1 & 0 \\ 9 & 7 & 4 & 3 & 1 & 0 \end{bmatrix} \qquad \lambda := \text{eigenvals}(A) = \begin{bmatrix} 6.70163752 \\ 1.85427699+1.02506954i \\ 1.85427699-1.02506954i \\ 0.2945476 \\ -2.7300129 \\ -4.47472621 \end{bmatrix}$$

4 real distinct eigenvalues and pair of complex conjugate eigenvalues

$|\text{tr}(A)| = 3.5 \qquad |||A||| = 108.25$

$Normality_{check}(A) = \text{“no”} \qquad Sym(A) = \text{“no”}$

$\text{conde}(A) = 106.444$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \qquad i = 1000000 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 6.70163752 \\ 2.34003476 \\ 1.36851922 \\ 0.2945476 \\ -2.7300129 \\ -4.47472621 \end{bmatrix}$$

Bad

$$\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 2\cdot10^{-13} \\ -3-5i\cdot10 \\ -4\cdot10+3i\cdot10 \\ -4\cdot10^{-14} \\ -3\cdot10^{-13} \\ 2\cdot10^{-13} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \qquad i = 54.00055 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 6.70163752 \\ 1.85427699+1.02506954i \\ 1.85427699-1.02506954i \\ 0.2945476 \\ -2.7300129 \\ -4.47472621 \end{bmatrix}$$

$$\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} 2\cdot10^{-13}+10i\cdot10^{-15} \\ 2\cdot10^{-13}+7i\cdot10^{-14} \\ -4\cdot10^{-14}-2i\cdot10^{-13} \\ -9\cdot10^{-13}-8i\cdot10^{-14} \\ -1\cdot10^{-13}-2i\cdot10^{-14} \\ 6\cdot10^{-14}-1i\cdot10^{-13} \end{bmatrix} \end{bmatrix}$$

check

$$Q\cdot T\cdot \overline{Q}^{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 \\ 0.5 & 1 & 2 & 4 & -1 & 1 \\ 0 & 0.5 & 2 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0.5 & 0 & 0 \\ 10 & 5 & 3 & 8 & -1 & 0 \\ 9 & 7 & 4 & 3 & 1 & 0 \end{bmatrix} \qquad rs(\lambda) = \begin{bmatrix} 6.70163752 \\ 1.85427699-1.02506954i \\ 1.85427699+1.02506954i \\ 0.2945476 \\ -2.7300129 \\ -4.47472621 \end{bmatrix} \qquad rs(\overline{\lambda}) = \begin{bmatrix} 6.70163752 \\ 1.85427699+1.02506954i \\ 1.85427699-1.02506954i \\ 0.2945476 \\ -2.7300129 \\ -4.47472621 \end{bmatrix}$$

$$T = \begin{bmatrix} 6.7016 & -3.7708+6.7774i & 1.4994+8.6941i & -3.7104+7.607i & -4.8667-5.2639i & 0.7601-1.3672i \\ 0 & -4.4747 & -3.2212+2.4841i & -4.0664+0.3111i & 0.0137-0.0417i & 0.445-0.0002i \\ 0 & 0 & 1.8543-1.0251i & -1.7149-1.1313i & 1.6873-0.2605i & 0.7684+1.2994i \\ 0 & 0 & 0 & 1.8543+1.0251i & -0.6461-0.3509i & 0.3244-0.5356i \\ 0 & 0 & 0 & 0 & -2.73 & -0.705-2.1474i \\ 0 & 0 & 0 & 0 & 0 & 0.2945 \end{bmatrix}$$

$$V := E_{Vec}(Q, T) = \begin{bmatrix} -0.0492+0.1825i & -0.2739-0.2661i & 0.5662-0.2668i & -0.397+0.7922i & -0.1638+0.0859i & -0.1107-0.1074i \\ -0.0499+0.1853i & 0.2327+0.2261i & -0.5339+0.1212i & 0.1898-0.7514i & 0.5735-0.3008i & 0.4821+0.4681i \\ -0.0786+0.2918i & -0.1545-0.1501i & 0.3983+0.1981i & 0.2668+0.5704i & 0.1298-0.0681i & -0.6074-0.5897i \\ -0.0128+0.0475i & -0.0708-0.0688i & -0.5377+0.0258i & 0.0549-0.7602i & -0.2684+0.1408i & 0.0711+0.069i \\ -0.1402+0.5203i & 0.7498+0.7284i & 0.1051-0.4042i & -0.5756+0.1349i & 0.3055-0.1602i & 0.039+0.0379i \\ -0.1918+0.7117i & 0.2049+0.199i & 0.8446-0.1193i & -0.1978+1.1913i & -0.9377+0.4917i & 0.6847+0.6647i \end{bmatrix}$$

$$V^{-1}\cdot A\cdot V = \begin{bmatrix} 6.70163752 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4.47472621 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.85427699-1.02506954i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.85427699+1.02506954i & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.7300129 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2945476 \end{bmatrix}$$

$$Z := M_p\left(\text{diag}(T), V, \frac{1}{3}\right) = \begin{bmatrix} 1.446+0.597i & 0.111+0.155i & -0.073+0.03i & 0.109+0.418i & 0.199-0.302i & -0.039-0.012i \\ -0.232-0.187i & 0.943+0.352i & 0.435+0.029i & 0.507-0.77i & -0.012+0.25i & 0.101-0.187i \\ 0.243+0.5i & 0.306+0.335i & 1.079+0.045i & 0.03+0.024i & -0.002-0.174i & 0.188-0.108i \\ -0.18-0.012i & 0.134-0.212i & 0.201-0.021i & 1.095+0.324i & -0.044-0.074i & 0.013+0.099i \\ 0.791-1.738i & 0.397-0.583i & 0.349-0.1i & 0.895-1.008i & 1.136+0.83i & 0.039+0.098i \\ 1.057-1.226i & 0.568-1.296i & 0.619-0.155i & 0.562+0.698i & 0.368+0.244i & 0.906+0.49i \end{bmatrix}$$

$$Z^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 \\ 0.5 & 1 & 2 & 4 & -1 & 1 \\ 0 & 0.5 & 2 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0.5 & 0 & 0 \\ 10 & 5 & 3 & 8 & -1 & 0 \\ 9 & 7 & 4 & 3 & 1 & 0 \end{bmatrix}$$

check

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \qquad i = 0 \qquad rs(\text{diag}(T)) = \begin{bmatrix} 6.70163752 \\ 1.85427699+1.02506954i \\ 1.85427699-1.02506954i \\ 0.2945476 \\ -2.7300129 \\ -4.47472621 \end{bmatrix}$$

$$\lambda_{error}(\lambda, T) = \begin{bmatrix} \text{“Percent error”} \\ \begin{bmatrix} -1\cdot10^{-14} \\ 6\cdot10^{-14}+4i\cdot10^{-14} \\ -8\cdot10^{-15}-3i\cdot10^{-14} \\ 1\cdot10^{-13}-2i\cdot10^{-14} \\ -2\cdot10^{-13}+5i\cdot10^{-15} \\ 8\cdot10^{-14}+2i\cdot10^{-14} \end{bmatrix} \end{bmatrix}$$

Schur Decomposition of Square Matrices

Note that Mathcad eigenvalue routine does not obtain accurate numerical values in this case

Repeated real roots

$$A := \begin{bmatrix} -5 & -9 & -7 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \lambda := \text{eigenvals}(A) = \begin{bmatrix} -0.99998755 \\ -1.00000623 + 0.00001078i \\ -1.00000623 - 0.00001078i \\ -2 \end{bmatrix} \quad |\text{tr}(A)| = 5 \quad |||A||| = 2 \quad \lambda_{exact} := \text{EV}(A) \rightarrow \begin{bmatrix} -1 \\ -1 \\ -1 \\ -2 \end{bmatrix}$$

$$Normality_{check}(A) = \text{"no"}$$

$$Sym(A) = \text{"no"} \quad \text{conde}(A) = 82.4864$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \quad i = 1000000 \quad rs(\text{diag}(T)) = \begin{bmatrix} -0.99998752 \\ -0.99999686 \\ -1.00001562 \\ -2 \end{bmatrix} \quad \lambda_{error}(\lambda_{exact}, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -1 \cdot 10^{-3} \\ -3 \cdot 10^{-4} \\ 2 \cdot 10^{-3} \\ 4 \cdot 10^{-13} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \quad i = 29.0003 \quad rs(\text{diag}(T)) = \begin{bmatrix} -0.99998694 + 0.00000011i \\ -1.00000643 - 0.00001137i \\ -1.00000663 + 0.00001125i \\ -2 \end{bmatrix} \quad \lambda_{error}(\lambda_{exact}, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -1 \cdot 10^{-3} - 1i \cdot 10^{-5} \\ 6 \cdot 10^{-4} + 1i \cdot 10^{-3} \\ 7 \cdot 10^{-4} - 1i \cdot 10^{-3} \\ 5 \cdot 10^{-13} + 2i \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

check

$$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} -5 & -9 & -7 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad T = \begin{bmatrix} -2 & 1.3975175 - 0.059156i & 3.15019411 + 0.83312345i & -7.74083912 + 6.72040631i \\ 0 & -1.00000663 + 0.00001125i & -1.75653009 - 0.54497628i & 3.08525217 - 2.45766563i \\ 0 & 0 & -1.00000643 - 0.00001137i & 2.41247002 - 3.54681074i \\ 0 & 0 & 0 & -0.99998694 + 0.00000011i \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \quad i = 29.0003 \quad rs(\text{diag}(T)) = \begin{bmatrix} -0.99998694 + 0.00000011i \\ -1.00000643 - 0.00001137i \\ -1.00000663 + 0.00001125i \\ -2 \end{bmatrix} \quad \lambda_{error}(\lambda_{exact}, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -1 \cdot 10^{-3} - 1i \cdot 10^{-5} \\ 6 \cdot 10^{-4} + 1i \cdot 10^{-3} \\ 7 \cdot 10^{-4} - 1i \cdot 10^{-3} \\ 5 \cdot 10^{-13} + 2i \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$rs(\lambda) = \begin{bmatrix} -0.99998755 \\ -1.00000623 - 0.00001078i \\ -1.00000623 + 0.00001078i \\ -2 \end{bmatrix}$$

Note that Mathcad eigenvalue routine does not obtain accurate numerical values in this case

$$A := \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 4 \\ -1 & -2 & 5 \end{bmatrix} \quad \lambda := \text{eigenvals}(A) = \begin{bmatrix} 2 + 0.00000002i \\ 2 - 0.00000002i \\ 1 \end{bmatrix} \quad |\text{tr}(A)| = 5 \quad |||A||| = 4 \quad \lambda_{exact} := \text{EV}(A) \rightarrow \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Repeated real roots

$$Normality_{check}(A) = \text{"no"}$$

$$Sym(A) = \text{"no"} \quad \text{conde}(A) = 19.615$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, -n, d) \quad i = 1000000 \quad rs(\text{diag}(T)) = \begin{bmatrix} 2.000002 \\ 1.999998 \\ 1 \end{bmatrix} \quad \lambda_{error}(\lambda_{exact}, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 10 \cdot 10^{-5} \\ -10 \cdot 10^{-5} \\ -4 \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_S(A, n, d) \quad i = 4.00005 \quad rs(\text{diag}(T)) = \begin{bmatrix} 2 + 0.00000003i \\ 2 - 0.00000003i \\ 1 \end{bmatrix} \quad \lambda_{error}(\lambda_{exact}, T) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} -2 \cdot 10^{-14} + 1i \cdot 10^{-6} \\ -2 \cdot 10^{-14} - 1i \cdot 10^{-6} \\ -1 \cdot 10^{-13} + 8i \cdot 10^{-14} \end{bmatrix} \end{bmatrix}$$

check

$$Q \cdot T \cdot \overline{Q}^{T} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 4 \\ -1 & -2 & 5 \end{bmatrix} \quad T = \begin{bmatrix} 2 - 0.00000003i & -1.22474487 - 0.00000012i & 6.36396103 + 0.00000062i \\ 0 & 1 & 1.73205081 \\ 0 & 0 & 2 + 0.00000003i \end{bmatrix}$$

$$\begin{bmatrix} i \\ Q \\ T \end{bmatrix} := Schur_V(A, n, d) \quad i = 0 \quad rs(\text{diag}(\overline{T})) = \begin{bmatrix} 2 + 0.00000002i \\ 2 - 0.00000002i \\ 1 \end{bmatrix} \quad \lambda_{error}(\lambda_{exact}, \overline{T}) = \begin{bmatrix} \text{"Percent error"} \\ \begin{bmatrix} 8i \cdot 10^{-7} \\ -8i \cdot 10^{-7} \\ -6 \cdot 10^{-14} - 6i \cdot 10^{-23} \end{bmatrix} \end{bmatrix}$$