MODELING BLACKBODY RADIATION

A Mathcad 11 Document by Roger L. Mansfield
http://mathcadwork.astroger.com/
2005 March 25 (updated 2011 February 2)

Preface

2005 is the "World Year of Physics" and it marks the 100th anniversary of Albert Einstein's publication of papers on the photoelectric effect, Brownian motion, and the special theory of relativity. All four papers on these three topics appeared in the year 1905.

But modern physics, i.e., modern quantum physics really began in 1901, when Max Planck propounded the notion that material bodies, but especially blackbodies [1], emit and absorb thermal radiation in discrete quanta of energy, rather than continuously.

Planck's Radiation Law

Planck's hypothesis of quantized absorption and emission of radiation made it possible for him to derive a radiation law that applies to blackbody emission at all wavelengths and all frequencies, a universal law that succeeds in spectral regions where the prior radiation laws of Rayleigh, Jeans and Wien had failed. Planck received the Nobel Prize in physics in 1918 for his quantum theory of radiation.

The Mathcad 11 worksheet, "Modeling Blackbody Radiation," revisits how Max Planck integrated the blackbody radiation curve for an arbitrary Kelvin temperature, T, over all possible wavelengths of thermal emission, to arrive at the Stefan-Boltzmann law. The Maple symbolic processing capability of Mathcad is invoked at key points of the derivation and Bernoulli numbers are used to evaluate the infinite series that is crucial to the derivation. Finally, Mathcad's X-Y Plot capability is used to plot the blackbody radiation curve for 2.725 degrees Kelvin.

Discovery of the Cosmic Microwave Background

Planck's radiation law is not just of historical interest. In 1964 Arno Penzias and Robert Wilson discovered radio noise emanating from all directions of the sky that is consistent with thermal emission from a blackbody at an equilibrium temperature of just a few degrees Kelvin. They deduced in 1965 that this radio noise is the cosmic microwave background (CMB). For this they were awarded a Nobel Prize in 1978 [2].

By the 1960s there were two competing theories of the origin of the cosmos, the "steady state" theory and the "Big Bang" theory. Existence of the CMB was predicted by the Big Bang theory, but not by the steady state theory. So when the CMB was found by Penzias and Wilson, most physicists and astronomers came to accept the Big Bang theory and to reject the steady state theory.

Definitive Measurements by the Cosmic Background Explorer

More recently, the Cosmic Background Explorer (COBE) mission measured the CMB in all directions of space, from space (i.e., from Earth orbit). Its measurements of energy density vs. frequency fit almost perfectly on Planck's radiation curve for a blackbody at 2.725 degrees Kelvin. But small "ripples" in energy density were in fact found; these are believed to be evidence of variations in the early universe's energy density. Since these variations are thought to have seeded star and galaxy formation, it would have been a setback for the Big Bang theory had they not been found.

NOTE AND REFERENCE

[1] A blackbody is an ideal body that absorbs all incident radiation and re-emits it as light energy distributed over the entire electromagnetic spectrum.

[2] Mather, John C. and Boslough, John, The Very First Light, Basic Books, New York, 1996; pp. 49-50 and 64. John Cromwell Mather was the original proposer and project scientist for the COBE mission. The COBE satellite was launched on November
18, 1989.

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In The Theory of Heat Radiation [1], Max Planck derives his remarkable formula for the space density of energy of uniform, monochromatic, non-polarized radiation of frequency ν, as emitted by a blackbody at Kelvin temperature T, as follows:

$$E_\nu = \frac{8\cdot\pi\cdot h\cdot\nu^3}{c^3}\cdot\left(e^{\frac{h\cdot\nu}{k\cdot T}} - 1\right)^{-1} \qquad (1)$$

Here h is Planck's constant, c is the velocity of light, and k is Boltzmann's constant. Planck goes on in [1] to integrate the space density of blackbody radiation energy over all frequencies, and thereby arrives at the Stefan-Boltzmann Law,

$$\int_0^\infty E_\nu\, d\nu = \int_0^\infty \frac{8\cdot\pi\cdot h\cdot\nu^3}{c^3}\cdot\left(e^{\frac{h\cdot\nu}{k\cdot T}} - 1\right)^{-1} d\nu = \sigma\cdot T^4 \qquad (2)$$

What I propose to do in this worksheet is to

(a) determine the Stefan-Boltzmann constant, σ, as a function of h, c, and k, using Mathcad;

(b) explore some of the mathematics hidden in the rather facile evaluation of the definite integral in Eq. 2 that can be obtained via Mathcad's Maple symbolic engine.

But before proceeding, I'd like to quote the book that I used in my study of physical chemistry [2],

"The derivation by Planck in 1901 of a radiation law which covers the entire frequency range [of blackbody radiation] is regarded as the beginning of modern physics,

$$\rho_\lambda = 8\pi hc\lambda^{-5}(e^{hc/\lambda kT} - 1)^{-1}.\text{"} \qquad (3)$$

(Below we will plot below Planck's equation in terms of wavelength, λ, rather than frequency, ν.)

Determination of the Stefan-Boltzmann Constant

If we make the change of variable x = hν/kT, then ν = (kT/h)x, and dν = (kT/h)dx. As x goes from 0 to infinity, ν also goes from 0 to infinity, so we have

$$\int_0^{\infty} E_{\nu}\, d\nu = \int_0^{\infty} \frac{8\cdot\pi\cdot h\cdot\nu^3}{c^3}\cdot\left(e^{\frac{h\cdot\nu}{k\cdot T}} - 1\right)^{-1} d\nu = \frac{8\cdot\pi\cdot k^4\cdot T^4}{h^3\cdot c^3}\cdot\int_0^{\infty} \frac{x^3}{e^x - 1}\, dx \qquad (4)$$

When we use Mathcad's Maple symbolic engine to evaluate the definite integral on the right, we get

$$\int_0^{\infty} \frac{x^3}{e^x - 1}\, dx \rightarrow \frac{1}{15}\cdot\pi^4 \qquad (5)$$

So the Stefan-Boltzmann constant, σ, is given by

$$\sigma = \frac{8\cdot\pi^5\cdot k^4}{15\cdot h^3\cdot c^3} \qquad (6)$$

If we were to stop now, we would fail to appreciate a most satisfying interplay between mathematics and physics: evaluation of an important infinite series using Bernoulli numbers.

An Exploration in Mathematical Physics, Using Mathcad

How did Maple arrive at its evaluation of the definite integral in Eq. 5? To answer this question, we must evaluate the integral by hand. First of all, we expand the denominator of the integrand in powers of e^{-x}. We get the following

$$\frac{1}{e^x - 1} = e^{-x} + e^{-2\cdot x} + e^{-3\cdot x} + e^{-4\cdot x} + ... \qquad (7)$$

Therefore,

$$\frac{x^3}{e^x - 1} = x^3 \cdot e^{-x} + x^3 e^{-2 \cdot x} + x^3 \cdot e^{-3 \cdot x} + x^3 \cdot e^{-4 \cdot x} + \dots \quad (8)$$

We can integrate each term on the right using the technique of integration by parts. E.g., starting with the integral

$$\int x^3 \cdot e^{-x} \, dx \quad (9)$$

we let

$$u = x^3 \qquad du = 3 \cdot x^2 \cdot dx$$
$$dv = e^{-x} \cdot dx \qquad v = -e^{-x} \quad (10)$$

then the integration-by-parts formula

$$\int u \, dv = u \cdot v - \int v \, du \quad (11)$$

is applied to the integral (9) to determine that

$$\int x^3 \cdot e^{-x} \, dx = -x^3 \cdot e^{-x} + \int 3 \cdot x^2 \cdot e^{-x} \, dx \quad (12)$$

Two more applications of the integration-by-parts recipe yield

$$\int x^3 \cdot e^{-x} \, dx = -x^3 \cdot e^{-x} - 3 \cdot x^2 \cdot e^{-x} - 6 \cdot x \cdot e^{-x} - 6 \cdot e^{-x} \quad (13)$$

If we integrate all terms of Eq. 8 in the same manner as the first, we obtain the same results as Maple:

$$\int x^3 \cdot e^{-x}\, dx \rightarrow -x^3 \cdot \exp(-x) - 3 \cdot x^2 \cdot \exp(-x) - 6 \cdot x \cdot \exp(-x) - 6 \cdot \exp(-x) \qquad (14)$$

$$\int x^3 \cdot e^{-2 \cdot x}\, dx \rightarrow \frac{-1}{2} \cdot x^3 \cdot \exp(-2 \cdot x) - \frac{3}{4} \cdot x^2 \cdot \exp(-2 \cdot x) - \frac{3}{4} \cdot x \cdot \exp(-2 \cdot x) - \frac{3}{8} \cdot \exp(-2 \cdot x)$$

$$\int x^3 \cdot e^{-3 \cdot x}\, dx \rightarrow \frac{-1}{3} \cdot x^3 \cdot \exp(-3 \cdot x) - \frac{1}{3} \cdot x^2 \cdot \exp(-3 \cdot x) - \frac{2}{9} \cdot x \cdot \exp(-3 \cdot x) - \frac{2}{27} \cdot \exp(-3 \cdot x)$$

$$\int x^3 \cdot e^{-4 \cdot x}\, dx \rightarrow \frac{-1}{4} \cdot x^3 \cdot \exp(-4 \cdot x) - \frac{3}{16} \cdot x^2 \cdot \exp(-4 \cdot x) - \frac{3}{32} \cdot x \cdot \exp(-4 \cdot x) - \frac{3}{128} \cdot \exp(-4 \cdot x)$$

We subract the antiderivatives evaluated at x = 0 from the antiderivatives evaluated at x = infinity. All that remains is the sequence of values

$$6 + \frac{3}{8} + \frac{2}{27} + \frac{3}{128} + ... = 6 \cdot \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...\right) \qquad (15)$$

The infinite series on the right is not trivial to evaluate, but it can be shown to be

$$\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...\right) = \frac{\pi^4}{90} \qquad (16)$$

(It is, in fact, shown in Knopp [3] that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^{2 \cdot p}}\right) = \frac{(-1)^{p-1} \cdot (2 \cdot \pi)^{2 \cdot p}}{2 \cdot (2 \cdot p)!} \cdot B_{2p} \qquad (17)$$

where B_{2p} is the 2p-th Bernoulli number. The first six Bernoulli numbers are B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, and B_6 = 1/42. With p = 2, the expression on the right of Eq. 17 yields the constant on the right of Eq. 16. See also Arfken [4], Courant [5], and Jeffrey [6] for related material on Bernoulli polynomials and Bernoulli numbers.)

We thus have that

$$\int_0^{\infty} \frac{x^3}{e^x - 1}\,dx = 6\cdot\frac{\pi^4}{90} = \frac{\pi^4}{15} \qquad (18)$$

which is what Mathcad's Maple symbolic engine "knew" all along.

<u>Transformation from Frequency to Wavelength</u>

It is customary in physical chemistry to plot blackbody radiation curves as a function of wavelength, not frequency. Since wavelength is related to frequency by

$$c = \nu\cdot\lambda \qquad (19)$$

we have

$$d\nu = \frac{-c}{\lambda^2}\cdot d\lambda \qquad (20)$$

so that the integral in the Stefan-Boltzmann law, Eq. 2, transforms to

$$\int_0^{\infty} \frac{8\cdot\pi\cdot h\cdot\nu^3}{c^3}\cdot\left(e^{\frac{h\cdot\nu}{k\cdot T}} - 1\right)^{-1} d\nu = \int_0^{\infty} \frac{8\cdot\pi\cdot h\cdot c}{\lambda^5}\cdot\left(e^{\frac{h\cdot c}{\lambda\cdot k\cdot T}} - 1\right)^{-1} d\lambda \qquad (21)$$

Therefore, Planck's radiation law, Eq. 1, transforms to

$$E_\nu\cdot d\nu = \frac{8\cdot\pi\cdot h\cdot\nu^3}{c^3}\cdot\left(e^{\frac{h\cdot\nu}{k\cdot T}} - 1\right)^{-1}\cdot d\nu = \frac{8\cdot\pi\cdot h\cdot c}{\lambda^5}\cdot\left(e^{\frac{h\cdot c}{\lambda\cdot k\cdot T}} - 1\right)^{-1}\cdot d\lambda = \rho_\lambda\cdot d\lambda \qquad (22)$$

It might seem strange to express Planck's radiation law with differentials. But if we don't, we can arrive at an erroneous expression for the radiation distribution law embodied in Eq. 3. (Planck deals with this in [1, p. 16].) There is no minus sign in front of the λ-based radiation formula of Eq. 22 because integration of ν from 0 to infinity is like integration of λ from infinity to zero, i.e., we omit the minus sign with the understanding that the $d\lambda$-integration will be from zero to infinity.

Plotting Blackbody Radiation Curves

With Planck's radiation law expressed in terms of wavelength, λ, we can now plot blackbody radiation curves. But first we need to set h, c, and k. We use the values from the second edition of Physics Vade Mecum [7].

$h := 6.6260755 \cdot 10^{-27}$ erg sec

$c := 2.99792458 \cdot 10^{10}$ cm sec^{-1}

$k := 1.380658 \cdot 10^{-16}$ erg $Kelvin^{-1}$

These values allow us to calculate the Stefan-Boltzmann constant, σ.

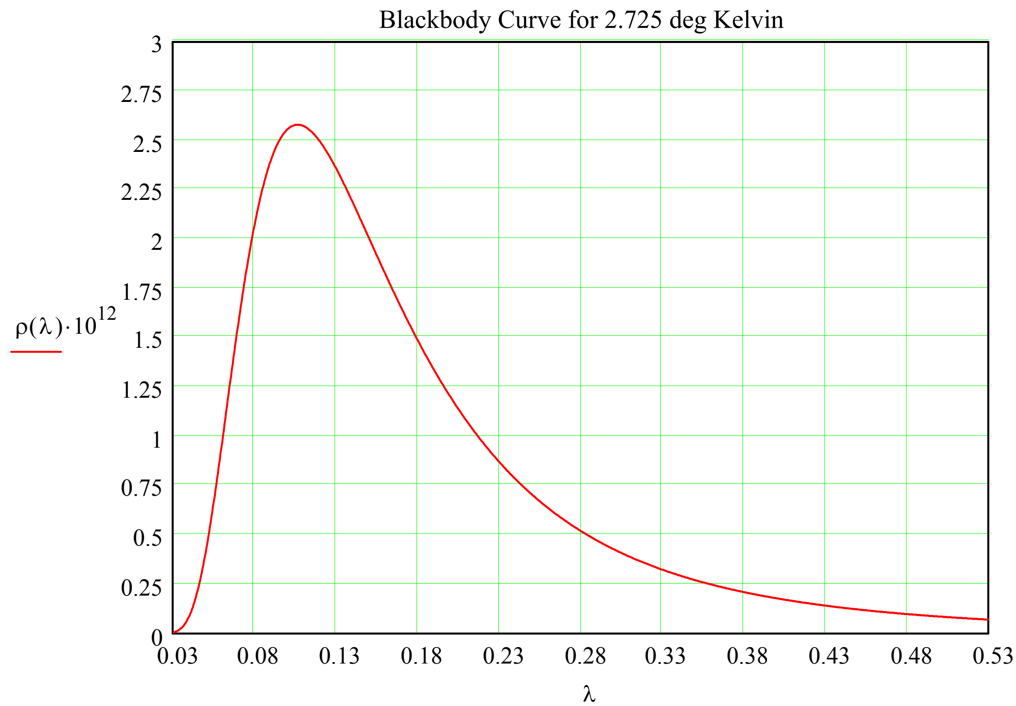
$$\sigma := \frac{8 \cdot \pi^5 \cdot k^4}{15 \cdot h^3 \cdot c^3} \qquad \sigma = 7.566 \times 10^{-15}$$

What we want to do now is to plot a blackbody radiation curve for a particular value of T. Let's plot at T = 2.725 Kelvins, since that is the temperature of the cosmic background radiation remaining from the Big Bang, as measured by John C. Mather et al. [8] during the Cosmic Background Explorer (COBE) mission.

$T := 2.725$

$$\rho(\lambda) := \frac{8 \cdot \pi \cdot h \cdot c}{\lambda^5} \cdot \left(e^{\frac{h \cdot c}{\lambda \cdot k \cdot T}} - 1 \right)^{-1}$$

In the plot below, note that the wavelength, λ, is in cm, and that the values of $\rho(\lambda)$ are plotted over a range of values from 0 to $3\text{x}10^{-12}$ erg per cm^3, per cm interval of wavelength.



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[1] Planck, Max, The Theory of Heat Radiation, Dover, New York, 1959 [translation by Morton Masius of the 2nd German edition of Waermestrahlung (1913)], Part IV, Chapter IV.

[2] Eggers, D.F. Jr.; Gregory, N.W.; Halsey, G.D. Jr.; and Rabinovitch. B.S., Physical Chemistry, John Wiley, New York, 1964, Chapter 1.

[3] Knopp, Konrad, Infinite Sequences and Series, Dover, New York, 1956 (translation by F. Bagemihl), Section 7.3 , p. 173.

[4] Arfken, George B. and Weber, Hans J., Mathematical Methods for Physicists, Academic Press, San Diego, 1995 (Fourth Edition), Section 5.9 (pp. 337-342).

[5] Courant, R., Differential and Integral Calculus, Volume I, Interscience Publishers, 2nd revised edition, 1937, Appendix to Chapter VIII, Section 4, "Series Involving Bernoulli's Numbers," pp. 422-424 (shows how to obtain the Bernoulli numbers and cites Knopp as a "more detailed treatise").

[6] Jeffrey, Alan, Handbook of Mathematical Formulas and Integrals, Academic Press, San Diego, 1995, Section 1.3 (pp. 36-43).

[7] Anderson, Herbert L. (Editor in Chief), A Physicist's Desk Reference, American Institute of Physics, New York, 1989, p. 4.

[8] Mather, John C. and Boslough, John, The Very First Light, Basic Books, New York, 1996. Fig. 9, p. 235 shows the cosmic microwave background (CMB) spectrum. To arrive at that graph, with the same scale (0 - 20 cm^{-1}) on the horizontal axis, we need to do a change of variables. Let

$$\nu = \frac{k \cdot T}{h \cdot c} \cdot x$$

with units of cm^{-1}, so that

$$x = \frac{h \cdot c}{k \cdot T} \cdot \nu$$

Planck's law for blackbody radiation, Eq. 1, transforms to

$$E(\nu) := \frac{8 \cdot \pi \cdot k^3 \cdot T^3 \left(\frac{h \cdot c}{k \cdot T} \cdot \nu \right)^3}{h^3 \cdot c^3 \cdot \left(e^{\frac{h \cdot c}{k \cdot T} \cdot \nu} - 1 \right)} \cdot \frac{\pi}{2}$$

(The factor $\pi/2$ converts the space density units to MegaJanskys per steradian, i.e., MJy/sr.)

and the CMB spectrum becomes