5.2.3 Ben-Tal *et al.* (1994) Problems : General Formulation

Ben-Tal et al. (1994) proposed a more general formulation for pooling problems which can accomodate any number of feed streams, pools and products, and in which any feed stream may reach any pool and product. A graphical representation of a system involving three feeds, one pool and two products is shown in Figure 5.2. The $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ vectors represent the flows between different units. Ben-Tal et al. (1994) suggested the substitution of flowrate x_{il}, denoting the flow from feed i to pool l, with a fractional flowrate, q_{il}, denoting the fraction of flow to pool l coming from feed i.

<u>Objective function</u>

$$\max_{\boldsymbol{q},\boldsymbol{y},\boldsymbol{z}} \sum_{j\in\mathcal{J}}\sum_{l\in\mathcal{L}}(d_j - \sum_{i\in\mathcal{I}} c_i q_{il}) y_{lj} + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}(d_j - c_i) z_{ij}$$

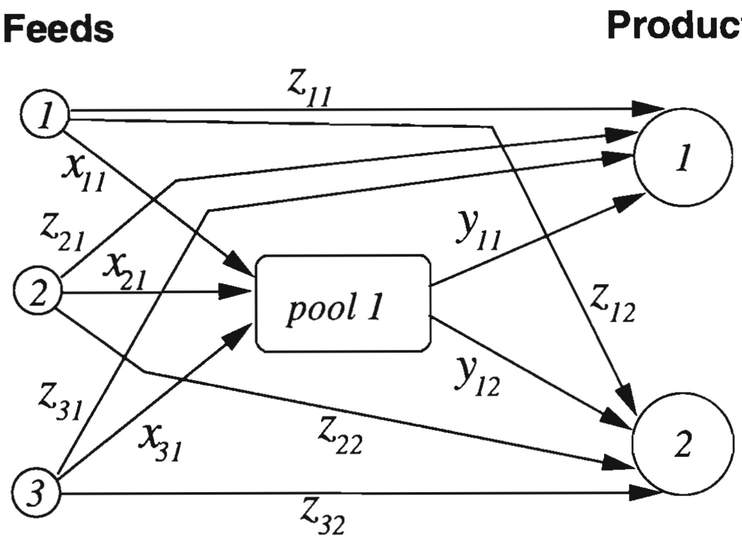

Figure 5.2: Schematic of a general pooling problem.

Constraints

$$
\begin{aligned}
\sum_{l\in\mathcal{L}}\sum_{j\in\mathcal{J}} q_{il}y_{lj} + \sum_{j\in\mathcal{J}} z_{ij} &\leq A_i, \forall i \in \mathcal{I} \\
\sum_{j\in\mathcal{J}} y_{lj} &\leq S_l, \forall l \in \mathcal{L} \\
\sum_{l\in\mathcal{L}} y_{lj} + \sum_{i\in\mathcal{I}} z_{ij} &\leq D_j, \forall j \in \mathcal{J} \\
\sum_{l\in\mathcal{L}} \left(\sum_{i\in\mathcal{I}} C_{ik}q_{il} - P_{jk}\right) y_{lj} + \sum_{i\in\mathcal{I}}(C_{ik} - P_{jk})z_{ij} &\leq 0, \forall j \in \mathcal{J}, \forall k \in \mathcal{K} \\
\sum_{i\in\mathcal{I}} q_{il} &= 1, \forall l \in \mathcal{L}
\end{aligned}
$$

Variable bounds

$$
\begin{aligned}
&0 \leq q_{il} \leq 1, && \forall i \in \mathcal{I}, \forall l \in \mathcal{L} \\
&0 \leq y_{lj} \leq D_j, && \forall l \in \mathcal{L}, \forall j \in \mathcal{J} \\
&0 \leq z_{ij} \leq D_j, && \forall i \in \mathcal{I}, \forall j \in \mathcal{J}
\end{aligned}
$$

Variable definitions

The sets are defined as follows. $\mathcal{I}$ is the set of feeds, $\mathcal{J}$ is the set of products, $\mathcal{L}$ is the set of pools and $\mathcal{K}$ is the set of components whose quality is being monitored.

The variables are q_{il}, the fractional flow from feed i to pool l; y_{lj}, the flow from pool l to product j; and z_{ij}, the flow from feed i directly to product j.

The parameters are A_i, the maximum available flow of feed i; D_j, the maximum demand for product j; S_l, the size of pool l; C_{ik}, the percentage of component k in feed i; P_{jk}, the maximum allowable percentage of component k in product j; c_i, the unit price of feed i; and d_j, the unit price of product j.

5.2.5 Ben-Tal *et al.* (1994) Problems : Test Problem 2

The second case suggested by Ben-Tal et al. (1994) consists of five feeds, three pools and five products. The quality of two components is monitored. The following restrictions are imposed on the problem: feeds 1, 2, 4 and 5 can be sent to the pool and feed 3 cannot. Thus, there are twelve q variables, and three z variables. The corresponding flowsheet is shown in Figure 5.4.

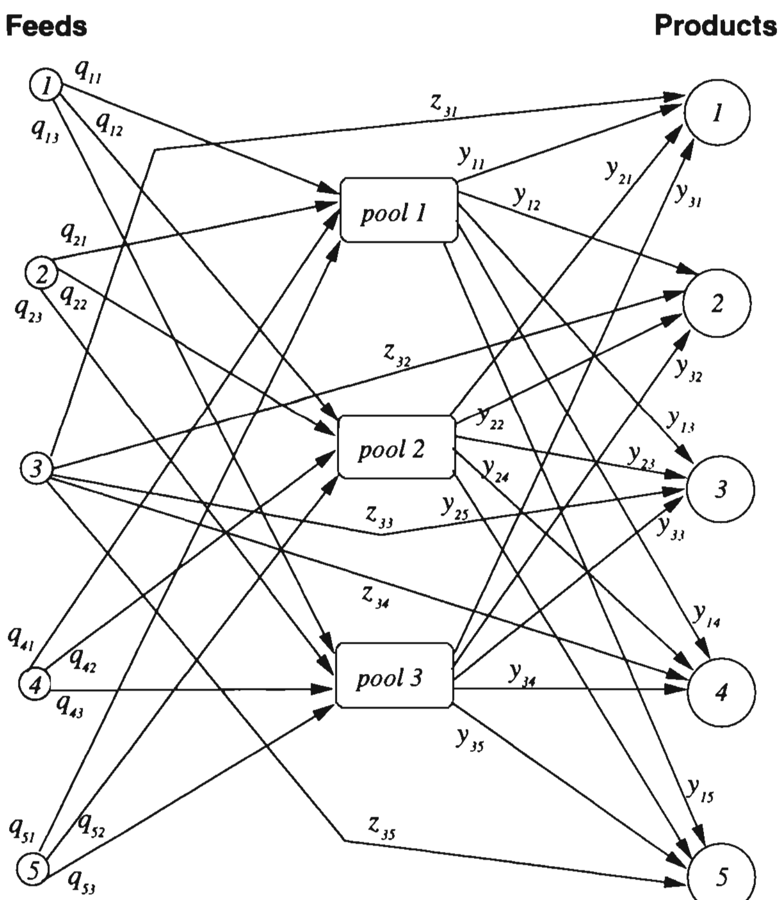

Figure 5.4: Second instance of the Ben-Tal *et al.* (1994) pooling problems

Formulation

Objective function

$$
\begin{aligned}
\max_{q,y,z}\quad & (18 - 6q_{11} - 16q_{21} - 15q_{41} - 12q_{51})y_{11} \\
& +(18 - 6q_{12} - 16q_{22} - 15q_{42} - 12q_{52})y_{21} \\
& +(18 - 6q_{13} - 16q_{23} - 15q_{43} - 12q_{53})y_{31} \\
& +(15 - 6q_{11} - 16q_{21} - 15q_{41} - 12q_{51})y_{12} \\
& +(15 - 6q_{12} - 16q_{22} - 15q_{42} - 12q_{52})y_{22} \\
& +(15 - 6q_{13} - 16q_{23} - 15q_{43} - 12q_{53})y_{32} \\
& +(19 - 6q_{11} - 16q_{21} - 15q_{41} - 12q_{51})y_{13} \\
& +(19 - 6q_{12} - 16q_{22} - 15q_{42} - 12q_{52})y_{23} \\
& +(19 - 6q_{13} - 16q_{23} - 15q_{43} - 12q_{53})y_{33} \\
& +(16 - 6q_{11} - 16q_{21} - 15q_{41} - 12q_{51})y_{14} \\
& +(16 - 6q_{12} - 16q_{22} - 15q_{42} - 12q_{52})y_{24} \\
& +(16 - 6q_{13} - 16q_{23} - 15q_{43} - 12q_{53})y_{34} \\
& +(14 - 6q_{11} - 16q_{21} - 15q_{41} - 12q_{51})y_{15} \\
& +(14 - 6q_{12} - 16q_{22} - 15q_{42} - 12q_{52})y_{25} \\
& +(14 - 6q_{13} - 16q_{23} - 15q_{43} - 12q_{53})y_{35} \\
& +8z_{31} + 5z_{32} + 9z_{33} + 6z_{34} + 4z_{35}
\end{aligned}
$$

Constraints

$$
\begin{aligned}
& q_{41}y_{11} + q_{41}y_{12} + q_{41}y_{13} + q_{41}y_{14} + q_{41}y_{15} + q_{42}y_{21} + q_{42}y_{22} + q_{42}y_{23} \\
& \quad +q_{42}y_{24} + q_{42}y_{25} + q_{43}y_{31} + q_{43}y_{32} + q_{43}y_{33} + q_{43}y_{34} + q_{43}y_{35} \le 50 \\
& y_{11} + y_{21} + y_{31} + z_{31} \le 100 \\
& y_{12} + y_{22} + y_{32} + z_{32} \le 200 \\
& y_{13} + y_{23} + y_{33} + z_{33} \le 100 \\
& y_{14} + y_{24} + y_{34} + z_{34} \le 100 \\
& y_{15} + y_{25} + y_{35} + z_{35} \le 100 \\
& (3q_{11} + q_{21} + q_{41} + 1.5q_{51} - 2.5)y_{11} \\
& \quad +(3q_{12} + q_{22} + q_{42} + 1.5q_{52} - 2.5)y_{21} \\
& \quad +(3q_{13} + q_{23} + q_{43} + 1.5q_{53} - 2.5)y_{31} - 0.5z_{31} \le 0 \\
& (q_{11} + 3q_{21} + 2.5q_{41} + 2.5q_{51} - 2)y_{11} \\
& \quad +(q_{12} + 3q_{22} + 2.5q_{42} + 2.5q_{52} - 2)y_{21} \\
& \quad +(q_{13} + 3q_{23} + 2.5q_{43} + 2.5q_{53} - 2)y_{31} + 0.5z_{31} \le 0 \\
& (3q_{11} + q_{21} + q_{41} + 1.5q_{51} - 1.5)y_{12} \\
& \quad +(3q_{12} + q_{22} + q_{42} + 1.5q_{52} - 1.5)y_{22} \\
& \quad +(3q_{13} + q_{23} + q_{43} + 1.5q_{53} - 1.5)y_{32} + 0.5z_{32} \le 0 \\
& (q_{11} + 3q_{21} + 2.5q_{41} + 2.5q_{51} - 2.5)y_{12} \\
& \quad +(q_{12} + 3q_{22} + 2.5q_{42} + 2.5q_{52} - 2.5)y_{22} \\
& \quad +(q_{13} + 3q_{23} + 2.5q_{43} + 2.5q_{53} - 2.5)y_{32} \le 0 \\
& (3q_{11} + q_{21} + q_{41} + 1.5q_{51} - 2)y_{13} \\
& \quad +(3q_{12} + q_{22} + q_{42} + 1.5q_{52} - 2)y_{23} \\
& \quad +(3q_{13} + q_{23} + q_{43} + 1.5q_{53} - 2)y_{33} \le 0
\end{aligned}
$$

$$
\begin{aligned}
&(q_{11} + 3q_{21} + 2.5q_{41} + 2.5q_{51} - 2.6)y_{13} \\
&\quad +(q_{12} + 3q_{22} + 2.5q_{42} + 2.5q_{52} - 2.6)y_{23} \\
&\quad +(q_{13} + 3q_{23} + 2.5q_{43} + 2.5q_{53} - 2.6)y_{33} - 0.1z_{33} &\leq 0 \\
&(3q_{11} + q_{21} + q_{41} + 1.5q_{51} - 2)y_{14} \\
&\quad +(3q_{12} + q_{22} + q_{42} + 1.5q_{52} - 2)y_{24} \\
&\quad +(3q_{13} + q_{23} + q_{43} + 1.5q_{53} - 2)y_{34} &\leq 0 \\
&(q_{11} + 3q_{21} + 2.5q_{41} + 2.5q_{51} - 2)y_{14} \\
&\quad +(q_{12} + 3q_{22} + 2.5q_{42} + 2.5q_{52} - 2)y_{24} \\
&\quad +(q_{13} + 3q_{23} + 2.5q_{43} + 2.5q_{53} - 2)y_{34} + 0.5z_{34} &\leq 0 \\
&(3q_{11} + q_{21} + q_{41} + 1.5q_{51} - 2)y_{15} \\
&\quad +(3q_{12} + q_{22} + q_{42} + 1.5q_{52} - 2)y_{25} \\
&\quad +(3q_{13} + q_{23} + q_{43} + 1.5q_{53} - 2)y_{35} &\leq 0 \\
&(q_{11} + 3q_{21} + 2.5q_{41} + 2.5q_{51} - 2)y_{15} \\
&\quad +(q_{12} + 3q_{22} + 2.5q_{42} + 2.5q_{52} - 2)y_{25} \\
&\quad +(q_{13} + 3q_{23} + 2.5q_{43} + 2.5q_{53} - 2)y_{35} + 0.5z_{35} &\leq 0 \\
&q_{11} + q_{21} + q_{41} + q_{51} &= 1 \\
&q_{12} + q_{22} + q_{42} + q_{52} &= 1 \\
&q_{13} + q_{23} + q_{43} + q_{53} &= 1
\end{aligned}
$$

Variable bounds

$$
\begin{array}{lll}
0 \leq q_{11} \leq 1 & 0 \leq q_{12} \leq 1 & 0 \leq q_{13} \leq 1 \\
0 \leq q_{21} \leq 1 & 0 \leq q_{22} \leq 1 & 0 \leq q_{23} \leq 1 \\
0 \leq q_{41} \leq 1 & 0 \leq q_{42} \leq 1 & 0 \leq q_{43} \leq 1 \\
0 \leq q_{51} \leq 1 & 0 \leq q_{52} \leq 1 & 0 \leq q_{53} \leq 1 \\
0 \leq y_{11} \leq 100 & 0 \leq y_{21} \leq 100 & 0 \leq y_{31} \leq 100 \\
0 \leq y_{12} \leq 200 & 0 \leq y_{22} \leq 200 & 0 \leq y_{32} \leq 200 \\
0 \leq y_{13} \leq 100 & 0 \leq y_{23} \leq 100 & 0 \leq y_{33} \leq 100 \\
0 \leq y_{14} \leq 100 & 0 \leq y_{24} \leq 100 & 0 \leq y_{34} \leq 100 \\
0 \leq y_{15} \leq 100 & 0 \leq y_{25} \leq 100 & 0 \leq y_{35} \leq 100 \\
0 \leq z_{31} \leq 100 & 0 \leq z_{32} \leq 200 & 0 \leq z_{33} \leq 100 \\
0 \leq z_{34} \leq 100 & 0 \leq z_{35} \leq 100 &
\end{array}
$$

Data

$\boldsymbol{A} = (\infty, \infty, \infty, 50, \infty)^T$ $\quad \boldsymbol{D} = (100, 200, 100, 100, 100)^T$ $\quad \boldsymbol{S} = (\infty, \infty, \infty)^T$

$\boldsymbol{c} = (6, 16, 10, 15, 12)^T$ $\quad \boldsymbol{d} = (18, 15, 19, 16, 14)^T$

$$
\boldsymbol{C} = \begin{pmatrix} 3.0 & 1.0 \\ 1.0 & 3.0 \\ 2.0 & 2.5 \\ 1.5 & 2.5 \end{pmatrix} \quad \boldsymbol{P} = \begin{pmatrix} 2.5 & 2.0 \\ 1.5 & 2.5 \\ 2.0 & 2.6 \\ 2.0 & 2.0 \\ 2.0 & 2.0 \end{pmatrix}
$$

Problem Statistics

No. of continuous variables	32
No. of linear equalities	3
No. of convex inequalities	5
No. of nonconvex inequalities	11
No. of known solutions	40

Global Solution

- Objective function: 3500
- Continuous variables: the variable values at the solution with the greatest objective function are reported here. The next best solution is very close, with a relative difference of only 1.5×10^{-8}.

$q_{11} = 1$	$q_{21} = 0$	$q_{41} = 0$	$q_{51} = 0$	
$q_{12} = 0$	$q_{22} = 0$	$q_{42} = 0$	$q_{52} = 1$	
$q_{13} = 0.2757$	$q_{23} = 0$	$q_{43} = 0$	$q_{53} = 0.7243$	
$y_{11} = 51.5527$	$y_{12} = 0$	$y_{13} = 0$	$y_{14} = 7.9609$	$y_{15} = 15.7866$
$y_{21} = 0$	$y_{22} = 200$	$y_{23} = 0$	$y_{24} = 0$	$y_{25} = 20.5622$
$y_{31} = 17.9504$	$y_{32} = 0$	$y_{33} = 0$	$y_{34} = 92.0391$	$y_{35} = 63.6512$
$z_{31} = 30.4969$	$z_{32} = 0$	$z_{33} = 100$	$z_{34} = 0$	$z_{35} = 0$