Formula of the ellipse in **rectangular** coordinates with center at origin and solve for y

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad y(x,a,b):=b\cdot\sqrt{1-\frac{x^2}{a^2}}$$ with semiaxis a (major) and b (minor)

$$e(a,b):=\sqrt{1-\left(\frac{b}{a}\right)^2}$$ Eccentricity of ellipse

Formula for the length of a curved line

$$ds=\sqrt{dx^2+dy^2}=\sqrt{1+\left(\frac{d}{dx}y\right)^2}\,dx \quad => \quad s=4\int_0^a \sqrt{1+\left(\frac{d}{dx}y\right)^2}\,dx$$

with numerical derivative

$$s_{rect\_D}(a,b):=4\cdot\int_0^a \sqrt{1+\left(\frac{d}{dx}y(x,a,b)\right)^2}\,dx$$

with symbolic derived derivative

$$\frac{d}{dx}y(x,a,b)\rightarrow\frac{-(b\cdot x)}{a^2\cdot\sqrt{-\frac{x^2}{a^2}+1}} \qquad s_{rect}(a,b):=4\cdot\int_0^a \sqrt{1+\left(\frac{b\cdot x}{a^2\cdot\sqrt{1-\frac{x^2}{a^2}}}\right)^2}\,dx$$

in terms of eccentricity

$$\sqrt{1+\left(\frac{b\cdot x}{a^2\cdot\sqrt{1-\frac{x^2}{a^2}}}\right)^2}=\frac{\sqrt{1-e(a,b)^2\cdot\left(\frac{x}{a}\right)^2}}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \quad => \quad s_{rect\_e}(a,b):=4\cdot\int_0^a \frac{\sqrt{1-e(a,b)^2\cdot\left(\frac{x}{a}\right)^2}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx$$

with τ variable transformation

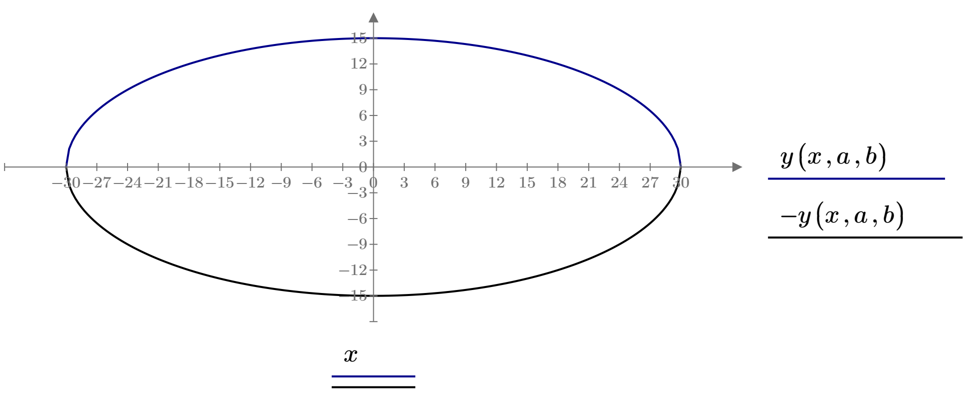
let $\tau=\frac{x}{a} \qquad \frac{d}{dx}\tau=\frac{1}{a} \qquad dx=a\cdot d\tau$ => $s_{rect\_\tau}(a,b):=4\cdot a\int_0^1 \frac{\sqrt{1-e(a,b)^2\cdot\tau^2}}{\sqrt{1-\tau^2}}\,d\tau$

Using Mathcad complete elliptic integral of the second kind where $m=e(a,b)^2$

$$\text{ellipticE}(m)=\int_0^1 \frac{\sqrt{1-m\cdot\tau^2}}{\sqrt{1-\tau^2}}\,d\tau=\int_0^{\frac{\pi}{2}} \sqrt{1-m\cdot sin(\theta)^2}\,d\theta \quad => \quad s_E(a,b):=4\cdot a\cdot\text{ellipticE}\left(e(a,b)^2\right)$$

$a:=30 \quad b:=15 \qquad e(a,b)\rightarrow\frac{\sqrt{3}}{2} \qquad e(a,b)^2\rightarrow\frac{3}{4}$

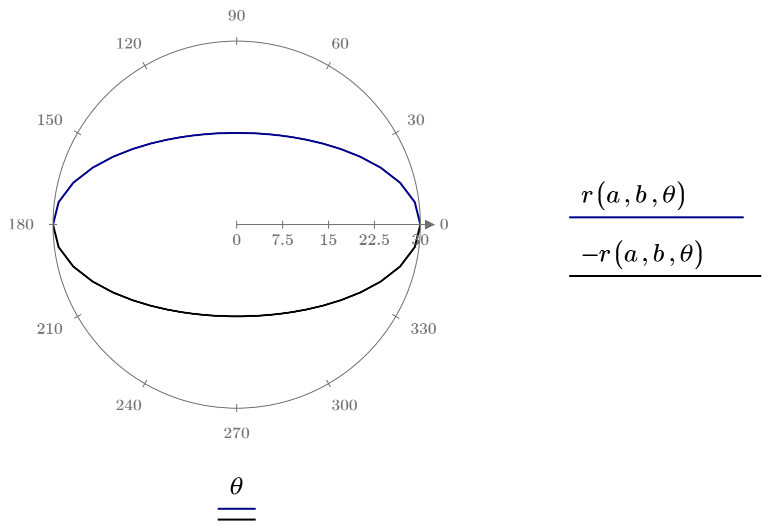
Create a range along major axis to plot ellipse $x:=-a,-0.99\,a..a$



Alternate method - transform to **polar** coordinates $x=a\cdot cos(\theta) \qquad y=b\cdot sin(\theta)$

$$\left(\frac{r\cdot cos(\theta)}{a}\right)^2+\left(\frac{r\cdot sin(\theta)}{b}\right)^2=1 \quad => \quad r=\frac{a\cdot b}{\sqrt{b^2\cdot cos(\theta)^2+a^2\cdot sin(\theta)^2}}=\frac{b}{\sqrt{1-e^2\ cos(\theta)^2}}$$

$$r(a,b,\theta):=\frac{b}{\sqrt{1-e(a,b)^2\ cos(\theta)^2}} \qquad \theta:=0,\frac{\pi}{25}..\pi$$



$$ds = \sqrt{dx^2 + dy^2} \quad dx = \frac{d}{d\theta} x \cdot d\theta \quad dy = \frac{d}{d\theta} y \cdot d\theta \quad ds = \sqrt{\left(\frac{d}{d\theta} x \cdot d\theta\right)^2 + \left(\frac{d}{d\theta} y \cdot d\theta\right)^2} = \sqrt{\left(\frac{d}{d\theta} x\right)^2 + \left(\frac{d}{d\theta} y\right)^2} \cdot d\theta$$

$$\frac{d}{d\theta} x = \frac{d}{d\theta} a \cdot cos(\theta) = -(a \cdot \sin(\theta)) \qquad \frac{d}{d\theta} y = \frac{d}{d\theta} b \cdot sin(\theta) = b \cdot \cos(\theta)$$

$$ds = \sqrt{(a \cdot \sin(\theta))^2 + (b \cdot \cos(\theta))^2} \cdot d\theta = a \cdot \sqrt{\sin(\theta)^2 + \left(\frac{b}{a}\right)^2 \cdot \cos(\theta)^2} \cdot d\theta$$ but $$e = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow \frac{b^2}{a^2} = 1 - e^2$$

$$ds = a \cdot \sqrt{\sin(\theta)^2 + (1 - e^2) \cdot (\cos(\theta))^2} \cdot d\theta = a \cdot \sqrt{1 - e^2 \cdot \cos(\theta)^2} \cdot d\theta$$

$$s = 4 \cdot a \cdot \int_0^{\frac{\pi}{2}} \sqrt{1 - e(a,b)^2 \cdot cos(\theta)^2} \, d\theta$$

with change in variable θ

let $\theta = \frac{\pi}{2} - \varphi \quad d\theta = -d\varphi \quad \theta = 0 \Rightarrow \varphi = \frac{\pi}{2} \quad \theta = \frac{\pi}{2} \Rightarrow \varphi = 0$

$$cos(\theta) = \cos\left(\frac{\pi}{2} - \varphi\right) = \cos\left(\frac{\pi}{2}\right) \cdot \cos(\varphi) + \sin\left(\frac{\pi}{2}\right) \cdot \sin(\varphi) = sin(\varphi)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - e(a,b)^2 \cdot \cos(\theta)^2} \, d\theta = -\int_{\frac{\pi}{2}}^{0} \sqrt{1 - e(a,b)^2 \cdot sin(\varphi)^2} \, d\varphi = \int_0^{\frac{\pi}{2}} \sqrt{1 - e(a,b)^2 \cdot sin(\varphi)^2} \, d\varphi = \text{ellipticE}\left(e(a,b)^2\right)$$

$$s_{polar}(a,b) := 4 \cdot a \cdot \int_0^{\frac{\pi}{2}} \sqrt{1 - e(a,b)^2 \cdot \sin(\varphi)^2} \, d\varphi$$

numerical solutions

$s_{rect\_D}(a,b) = 145.326723332463 \qquad s_{rect}(a,b) = 145.32672333447$

$s_{rect\_e}(a,b) = 145.326723334493 \qquad s_{rect\_\tau}(a,b) = 145.326723334482$

good approximations to 7 decimals

$s_{polar}(a,b) = 145.326723308215$ good approximation to 12 decimals

symbolic solutions

$s_E(a,b) \rightarrow 120 \cdot \text{ellipticE}\left(\frac{3}{4}\right) \xrightarrow{float} 145.32672330821518436$ assumed exact

$s_{rect\_D}(a,b) \xrightarrow{float} 145.33 \qquad s_{rect}(a,b) \xrightarrow{float} 145.33$ good approximation to 2 decimals

$s_{rect\_e}(a,b) \xrightarrow{float} 145.32672330821518436 \qquad s_{rect\_\tau}(a,b) \xrightarrow{float} 145.32672330821518436$ exact

$s_{polar}(a,b) \xrightarrow{float} 145.32672330821518436$ exact

$\mathbf{clear}(r)$ for circle of radius r=a=b

$s_{rect\_\tau}(r,r) \rightarrow 2 \cdot r \cdot \pi$ $\quad s_E(r,r) \rightarrow 2 \cdot r \cdot \pi$ $\quad s_{polar}(r,r) \rightarrow 2 \cdot r \cdot \pi$

$s_{rect\_e}(r,r) \rightarrow 8 \cdot \sqrt{r^2} \cdot \text{atan}\left(\frac{r}{\sqrt{r^2}}\right)$

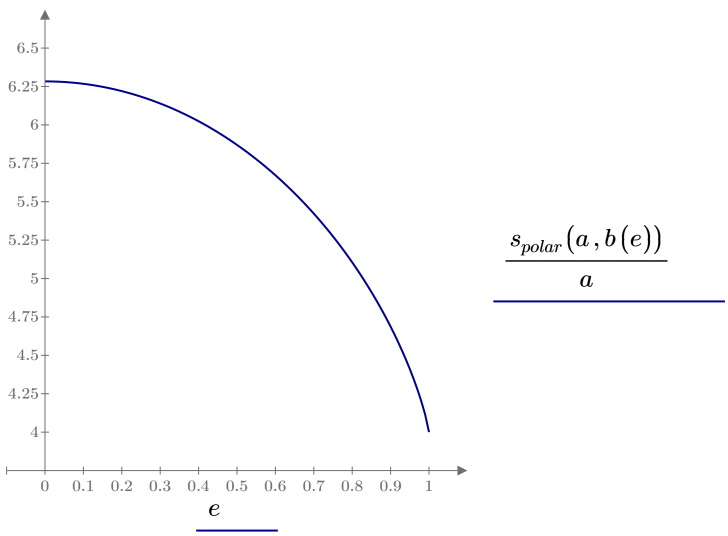
$s_{rect\_D}(r,r) \rightarrow \begin{Vmatrix} \text{if } \neg(r \geq 0 \vee -(2 \cdot r) \geq 0) \vee r \leq 0 \\ \quad \| 2 \cdot \pi \cdot \sqrt{r^2} \\ \text{else} \\ \quad \| \mathit{undefined} \end{Vmatrix}$

$s_{rect}(r,r) \rightarrow \begin{Vmatrix} \text{if } \neg(r \geq 0 \vee -(2 \cdot r) \geq 0) \vee r \leq 0 \\ \quad \| 2 \cdot \pi \cdot \sqrt{r^2} \\ \text{else} \\ \quad \| \mathit{undefined} \end{Vmatrix}$

$r := 1$ $\quad s_{rect}(r,r) \rightarrow 2 \cdot \pi$ $\quad s_{rect\_e}(r,r) \rightarrow 2 \cdot \pi$ $\quad s_{rect\_D}(r,r) \rightarrow 2 \cdot \pi$

$e := 0, 0.01 .. 1$ Eccentricity of ellipse

$a := 30$ $\quad b(e) := a \cdot \sqrt{1 - e^2}$ $\quad \frac{s_{polar}(a, b(0))}{a} \rightarrow 2 \cdot \pi$ $\quad \frac{s_{polar}(a, b(1))}{a} \rightarrow 4$



$\frac{s_{polar}\left(a, b\left(\frac{\sqrt{3}}{2}\right)\right)}{a} \rightarrow 4 \cdot \text{ellipticE}\left(\frac{3}{4}\right) \xrightarrow{float} 4.8442241102738394787$ $\quad \frac{s_{polar}\left(a, b\left(\frac{\sqrt{3}}{2}\right)\right)}{a} = 4.84422411027384$

$s_{polar}\left(a, b\left(\frac{\sqrt{3}}{2}\right)\right) \xrightarrow{float} 145.32672330821518436$ $\quad s_{polar}\left(a, b\left(\frac{\sqrt{3}}{2}\right)\right) = 145.326723308215$