"Of all the planes tangent to the ellipsoid

Ellipsoid equation:

The general equation of an ellipsoid centered at the origin is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

one of them cuts the pyramid of least possible volume from the first octant $x \geq 0$, $y \geq 0$, $z \geq 0$. Show that the point of tangency of that plane is the centroid of the face ABC."

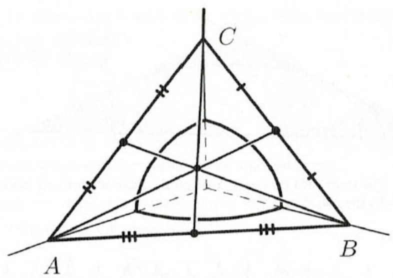

Figure 3.8. If the tangent plane minimizes the volume of the pyramid, then the point of tangency is the centroid of ΔABC.

The pyramid in this situation has four sides. One of the corners is at the origin (0,0,0). The centroid is this case means the intersection of its medians. (Source: "The Mathematical Mechanic" by Mark Levi, section 3.5.)

Normalize to a Sphere

$x` = \frac{x}{a}$ $y` = \frac{y}{b}$ $z` = \frac{z}{c}$

Ellipsoid becomes a sphere

$$x`^2 + y`^2 + z`^2 = 1$$

Equation of the Plane

$$A \cdot x` + B \cdot y` + C \cdot z` = D$$

For the plane to be tangent to the sphere, the distance from the origin to the plane must be 1 (the radius of the sphere), hence: D=1

The volume V of the pyramid with base ABC and vertex at the origin is:
V = 1 /3 × Area(ABC) × height.

Using Lagrange multipliers, we maximize the function f(x` , y′, z′) = 3 x ′y ′z′ subject to the constraint g(x′, y′, z′) = x ′2 + y ′2 + z ′2 − 1 = 0.

$$\zeta(x`,y`,z`,\lambda):=x`\cdot y`\cdot z`+\lambda\cdot\left(1-x`^{2}-y`^{2}-z`^{2}\right)$$

$$\frac{\partial}{\partial x`}\zeta(x`,y`,z`,\lambda)=0\rightarrow y`\cdot z`-2\cdot\lambda\cdot x`=0$$

$$\frac{\partial}{\partial y`}\zeta(x`,y`,z`,\lambda)=0\rightarrow x`\cdot z`-2\cdot\lambda\cdot y`=0$$

$$\frac{\partial}{\partial z`}\zeta(x`,y`,z`,\lambda)=0\rightarrow -(2\cdot\lambda\cdot z`)+x`\cdot y`=0$$

$$\frac{\partial}{\partial \lambda}\zeta(x`,y`,z`,\lambda)=0\rightarrow -z`^{2}+\left(1-x`^{2}-y`^{2}\right)=0$$

Solving the Inequalities

$$y`\cdot z`=2\cdot\lambda\cdot x`$$

$$x`\cdot z`=2\cdot\lambda\cdot y`$$

$$x`\cdot y`=2\cdot\lambda\cdot z`$$

We deduce

$$x`=y`=z`$$

Substitute into the Constraint Equation

$$x`:=3\cdot x`^{2}=1\xrightarrow{solve,x`}\begin{bmatrix}\frac{\sqrt{3}}{3}\\ \frac{-\sqrt{3}}{3}\end{bmatrix}$$

x` = y` = z`

Select Positive - As in First Quadrant

$$x` := (x`)_0 = 0.577$$

$$y` := x`$$
$$z` := x`$$

Convert to Original Co-ordinates

$$x = a \cdot x` \rightarrow x = 0.57735026918962573 \cdot a$$

$$y = b \cdot y` \rightarrow y = 0.57735026918962573 \cdot b$$

$$z = c \cdot z` \rightarrow z = 0.57735026918962573 \cdot c$$

The plane touches the ellipsoid at

$$\left[\frac{a}{\sqrt{3}} \quad \frac{b}{\sqrt{3}} \quad \frac{c}{\sqrt{3}} \right]$$

The Centroid of a Triangle the centroid of a triangle in three-dimensional space with vertices on the coordinate axes is the **average (1/3)** of the coordinates.

$$\left[\frac{a}{\sqrt{3}} \quad \frac{b}{\sqrt{3}} \quad \frac{c}{\sqrt{3}} \right] \cdot \frac{1}{3} \xrightarrow{substitute, a = \sqrt{3} \cdot x, b = \sqrt{3} \cdot y, c = \sqrt{3} \cdot z} \left[\frac{x}{3} \quad \frac{y}{3} \quad \frac{z}{3} \right]$$

Thus the point of tangency is at the centroid of the triangle.