Intervals

	Inequality	Interval
closed intervals	$a \le x \le b$	$[a, b]$
	$a \le x$	$[a, \infty)$
	$x \le a$	$(-\infty, a]$
Open intervals	$a < x < b$	(a, b)
	$a < x$	(a, ∞)
	$x < a$	$(-\infty, a)$
Half-open(or half-closed)	$a \le x < b$	$[a, b)$
	$a < x \le b$	$(a, b]$

a and b are real numbers

Example

For each of the following inequalities, write down the corresponding interval and describe it as closed, open or half - open (half- closed):

(a) $0 < x < 1$
(b) $-3 \le x \le 2$
(c) $x \ge 0$

Solution:

(a) (0,1) (open)
(b) [-3,2] (closed)
(c) $[0, \infty)$ (closed)

Functions

A function is specified by giving:

1. a set of allowed input values, called the ***domain*** of the function.
2. a process, called the ***rule*** of the function, for converting each input value into a unique output value.

Example:

$$f(x) = x^2 + 1 \qquad -1 \le x < 3$$

So

$f(1) = 2$, $f(2) = 5$, $f(0) = 1$

And the domain is:

[-1,3)

Domain convention

When a function is specified by just a rule, it is understood that the domain of the function is the largest possible set of real numbers for which the rule is applicable.

Example

Specify the domain of each of the following functions, as given by the domain convention.

(a) $f(x) = \sqrt{x - 1}$

(b) $f(x) = \dfrac{1}{x-2} + \dfrac{1}{x+3}$

Solution

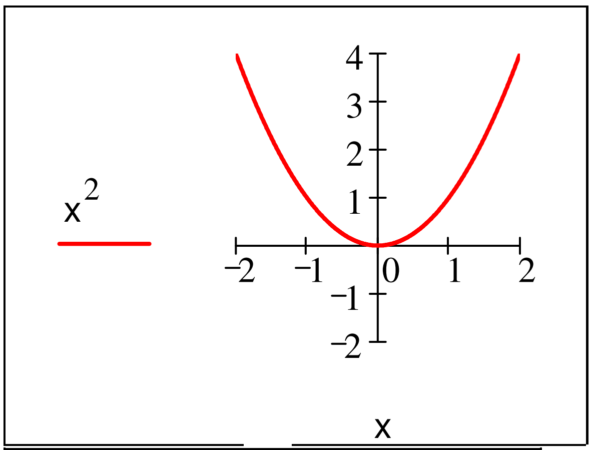
(a) $[1, \infty)$

(b) All real numbers except 2 and -3.

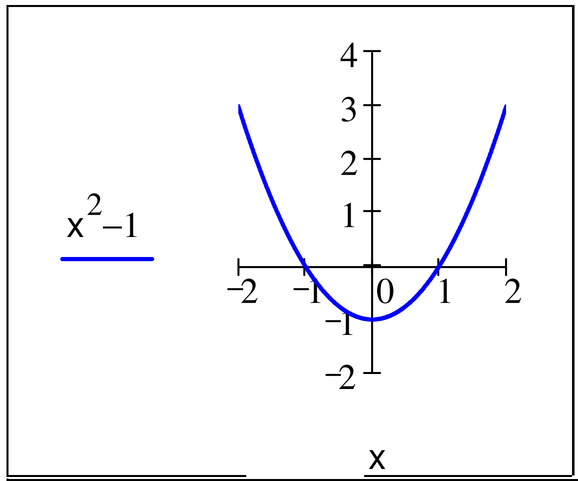
by symbols

R - { 2 , -3 }

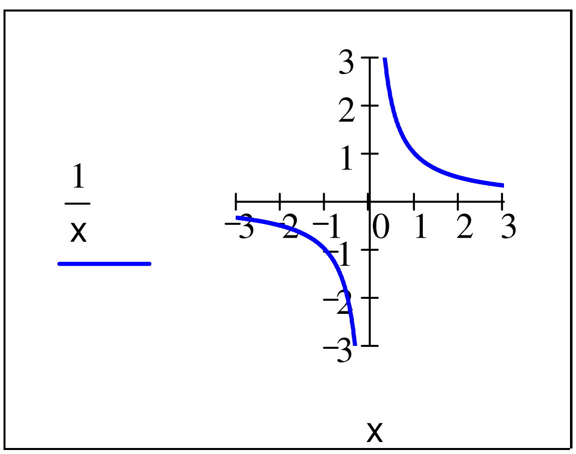
Graphs of the functions



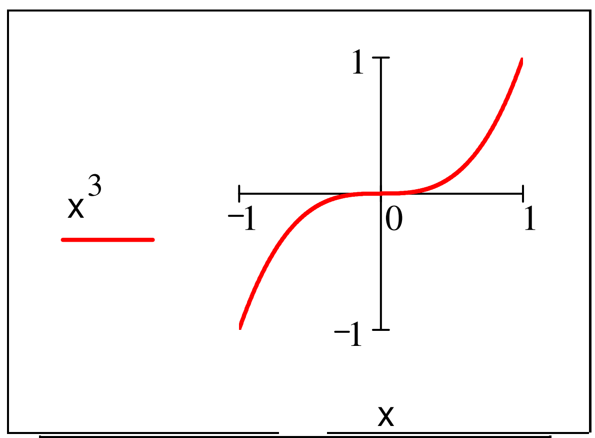
Graph of $y = x^2, -2 \le x \le 2$



Graph of $y = x^2 - 1, -2 \le x \le 2$



Graph of $y = \frac{1}{x}$



Graph of $y = x^3, -1 \le x \le 1$

Image set (range)

The image set is the complete set of output values of the function. For example the image set of the function $f(x)=x^2$ is the interval [0,∞), and the image set of the function g(x)= 1/x ($1 \le x \le 2$) is the interval [1/2 , 1].

Problem

Find the range of the functions:

(1) $f(x) = x^2$, (2) $h(x) = \frac{1}{x}$, (3) $g(x) = x^3$, (4) $f(x) = \sqrt{x}$

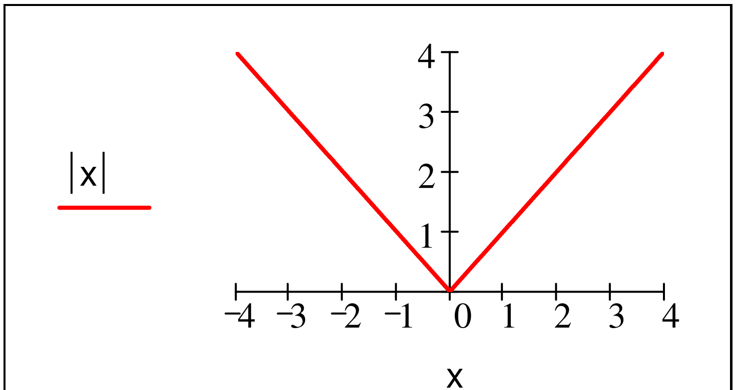
Absolute value

The absolute value or modulus, of real number x is the magnitude of x, regardless of its sign; it is denoted by | x |.

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example

$|-3| = 3$, $\left|\frac{2}{3}\right| = \frac{2}{3}$, $|0| = 0$



Graph of | x |

Example (1)

Find the solutions of the following equation:

$|3x + 7| + 4 = 0$

Solution

$|3x + 7| = -4$, but

$|3x - 7|$ can never be negative, so the equation has no solutions

Example (2)

Find the solutions of the following equation:

(a) $|3x + 7| - 4 = 0$

Solution:

$|3x + 7| = 4$

this is equivalent to the two equations

$3x + 7 = 4$ or $3x + 7 = -4$,

giving the solutions

$$x = -1 \quad \text{or} \quad x = \frac{-11}{3}.$$

Example (3)

Find the solution of the inequality

$|x + 2| \le 4$

Solution

$$-4 \le x + 2 \le 4$$

$$-4 - 2 \le x \le 4 - 2$$

$$-6 \le x \le 2$$

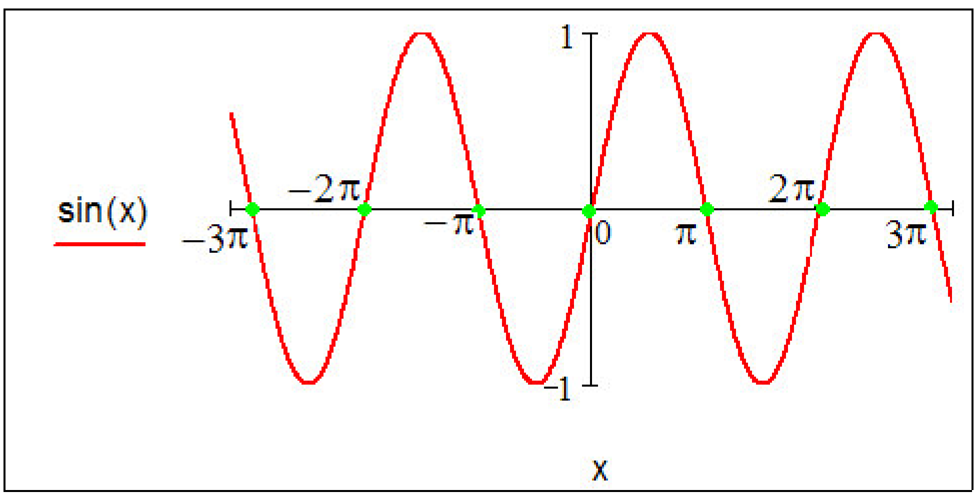
So, the solution of the inequality is the interval [-6,2]

Problem

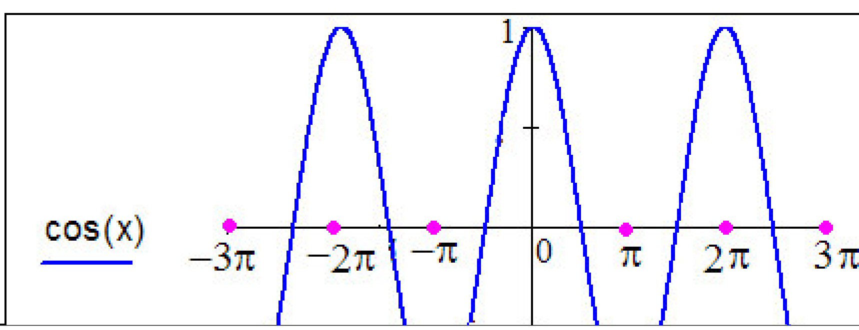
Find the solution of the inequality:

$$|x + 2| > 4$$

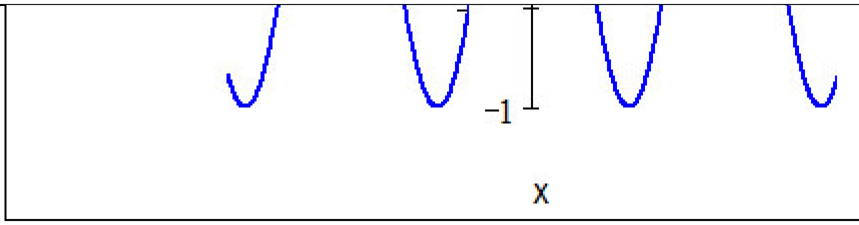
Trigonometric Functions



Graph of sin



Graph of cos



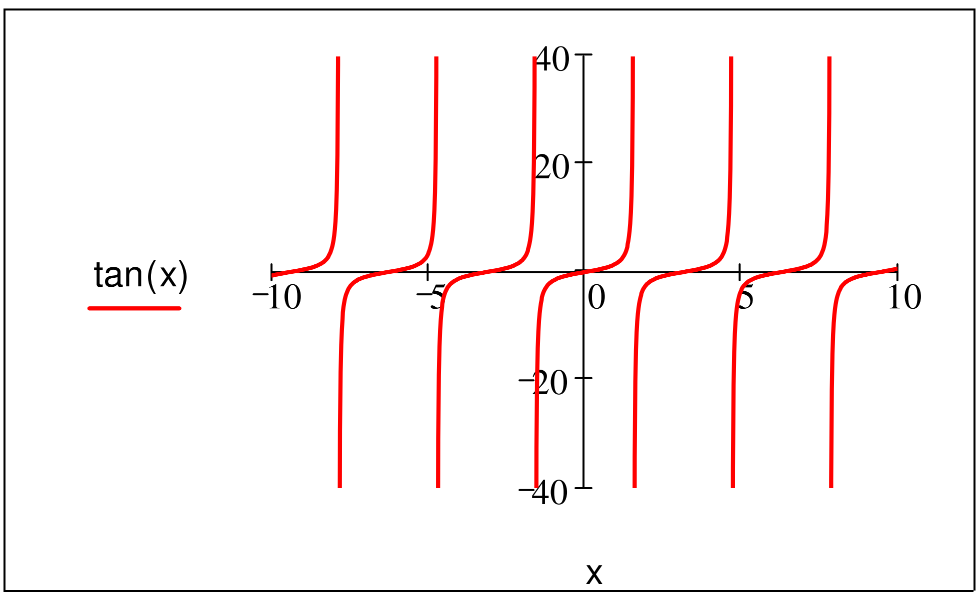
Periodic function

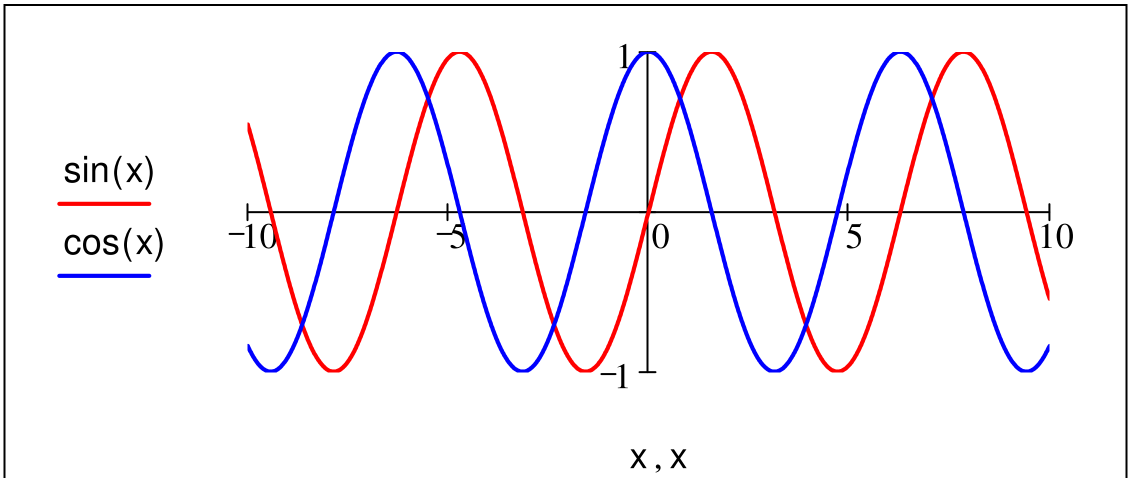
A function f is said to be periodic, with period p, if f(t+p) = f(t) for all t in the domain of f.

Example:

Cosine and Sine functions are both periodic, with period 2π

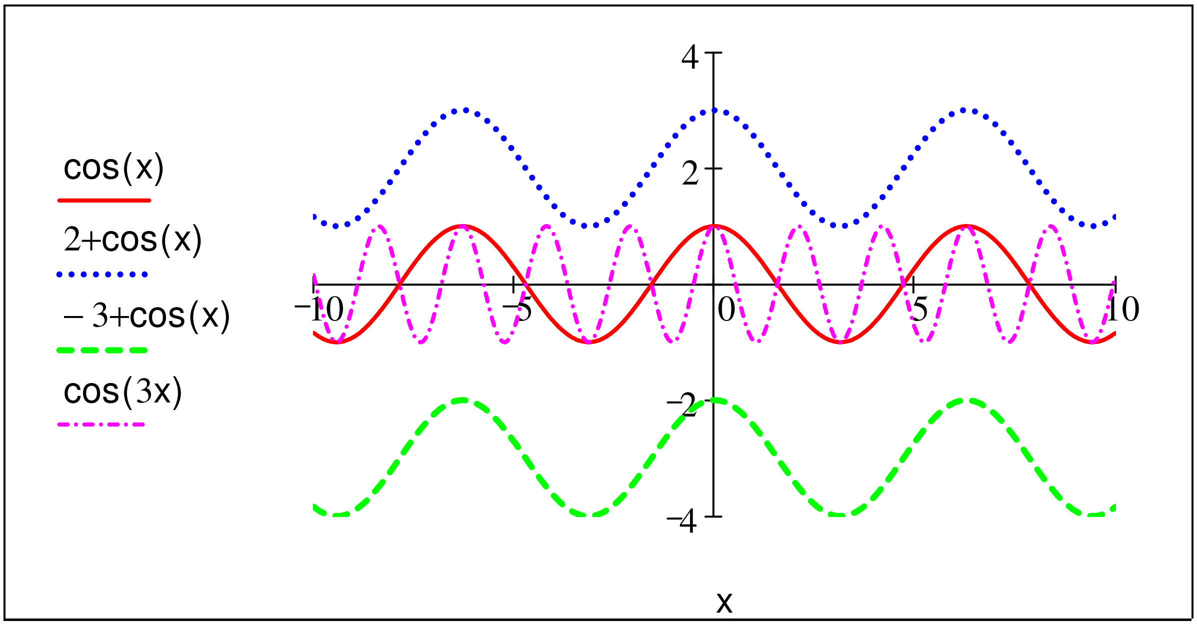
Graph of tan



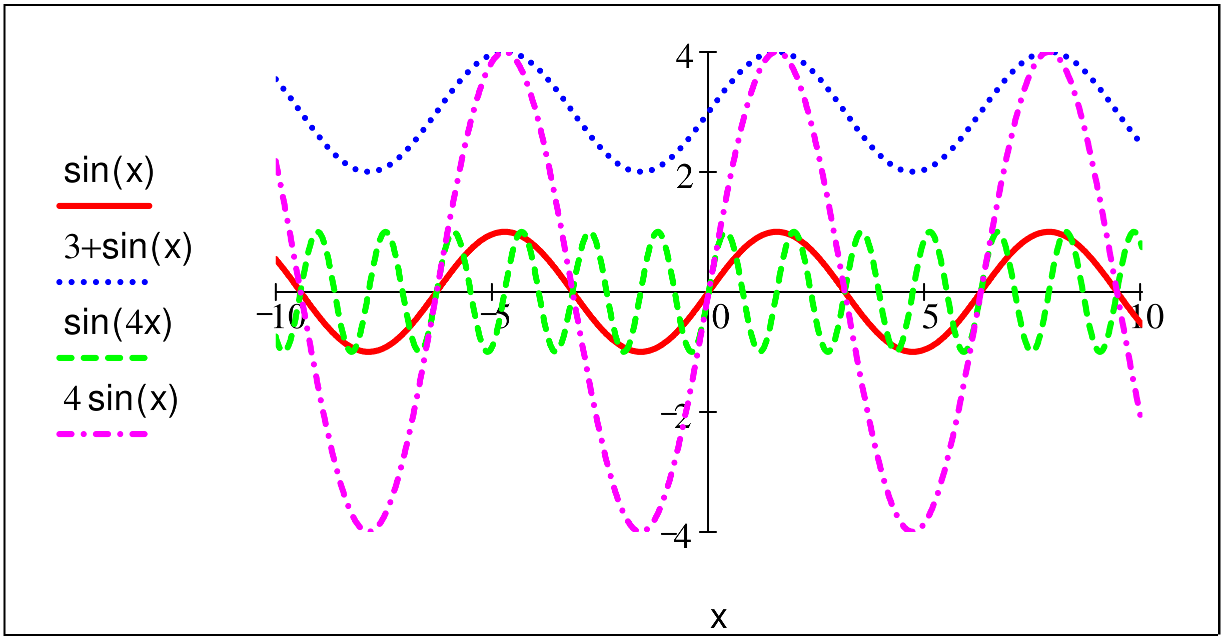


graphs of sin and cos on the same axes

Transforming the graph of cos



Transforming the graph of sin



Increasing, Decreasing and One to one functions

Definition (1)

A function f is said to be one - to- one ,if it has the following property:

for all x_1, x_2 in the domain of f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Definition (2)

A function f is said to be increasing if it has the following property:

for all x_1, x_2 in the domain of f, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.

Definition (3)

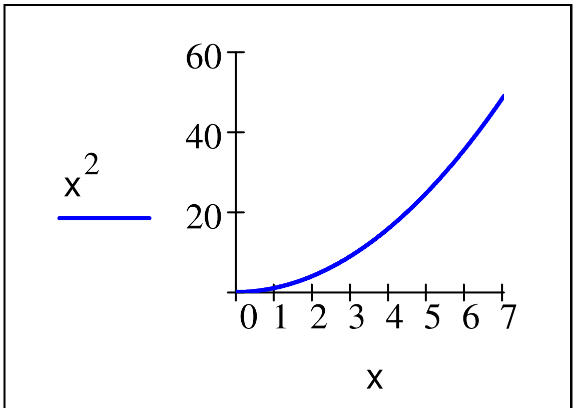
A function f is said to be decreasing ,if it has the following property:

for all x_1, x_2 in the domain of f, if $x_1 < x_2$, then $f(x_1) > f(x_2)$

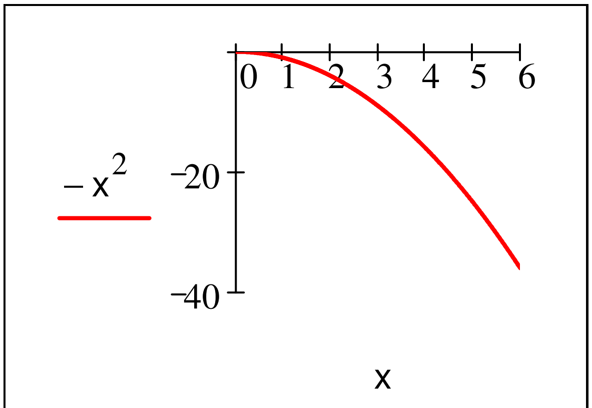
Note

If a function is either increasing or decreasing, then it is certainly one - one .

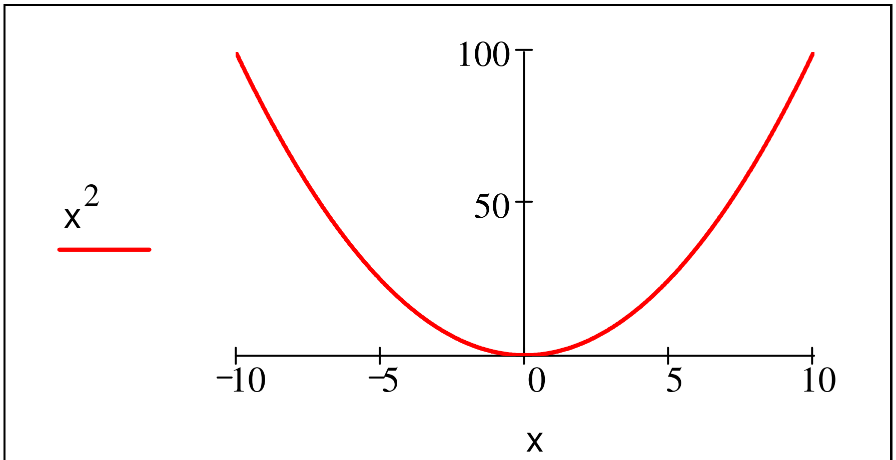
Examples



increasing



decreasing



neither increasing nor decreasing

Example

For each of the following functions, state whether the function is: increasing, decreasing, neither increasing nor decreasing, one to one, many to one

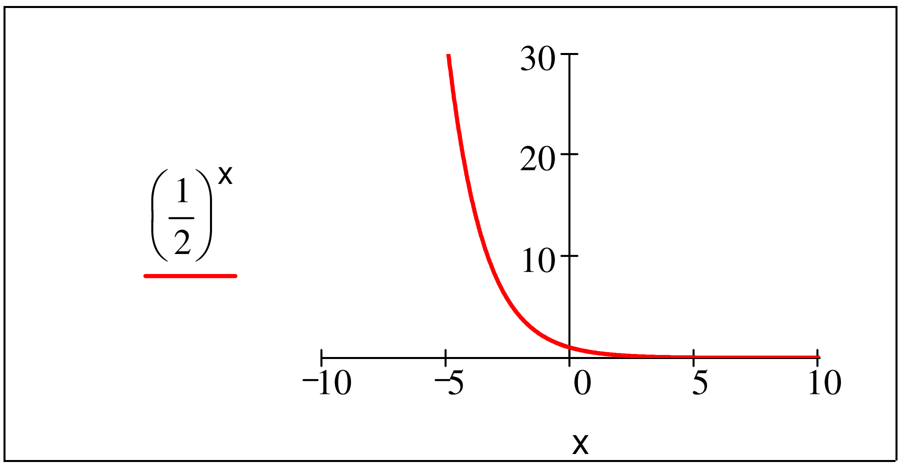
(a) $h(x) = \left(\frac{1}{2}\right)^x$

(b) $f(x) = \cos(x)$

(c) $f(x) = \sin(x) \qquad \left(\frac{-1}{2}\pi \le x \le \frac{1}{2}\pi\right)$

Solution:

(a)



The function $h(x) = \left(\frac{1}{2}\right)^x$ is decreasing, so it is one to one

Inverse Functions

For any one to one function f, it is possible to define an inverse function, denoted by f^{-1}. The domain of the inverse function f^{-1} is the range of f.

The inverse function f^{-1} undoes, the effect of f. Also f undoes the effect of f^{-1}.

Example (1)

Let $h(x) = \left(\frac{1}{2}\right)^x$

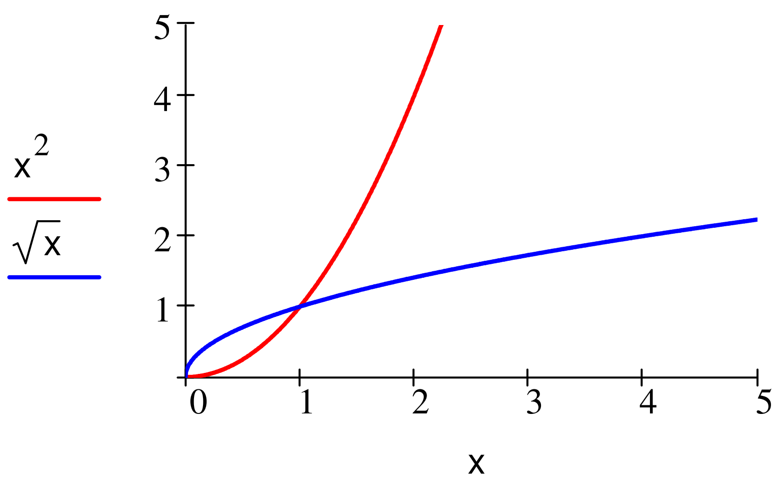
then

$h(1) = \frac{1}{2}$, so $h^{-1}\left(\frac{1}{2}\right) = 1$;

$h(2) = \frac{1}{4}$, so $h^{-1}\left(\frac{1}{4}\right) = 2$.

Example (2)

The function $g(x) = x^2$ (x in $[0,\infty]$), is one to one, and its inverse function $g^{-1}(x) = \sqrt{x}$ (x in $[0,\infty]$)



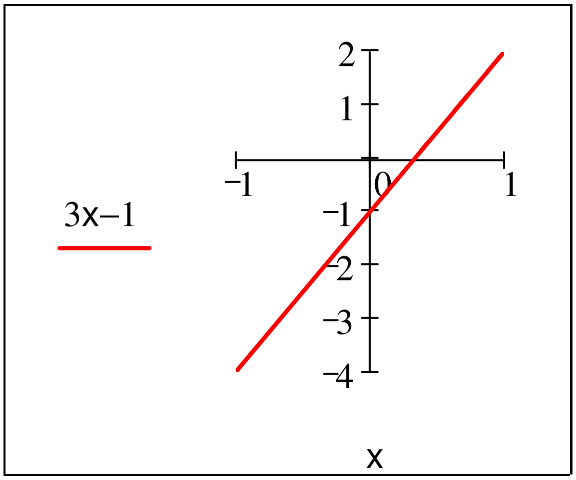
Finding Inverse Function

Show that the function

$f(x) = 3x - 1$ (x in [-1,1])

has an inverse function f^{-1}. Find the rule of f^{-1} and sketch its

graph.
Solution



From the graph we see that:

- the function f is increasing, and so one-one.
- the rang of f is [-4,2]

Therefore the function f has an inverse f^{-1} with domain [-4,2] and rang [-1,1].

We can find the rule of f^{-1} by solving the equation

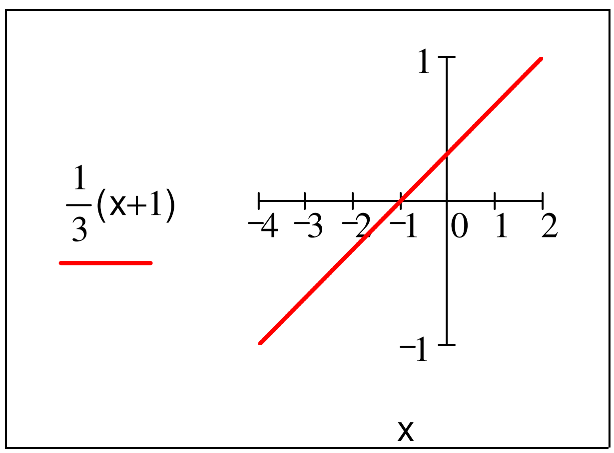
$y = f(x) = 3x - 1$

to obtain x in terms of y:

$y = 3x - 1$ so $x = \frac{1}{3}(y + 1)$

Thus the inverse function is

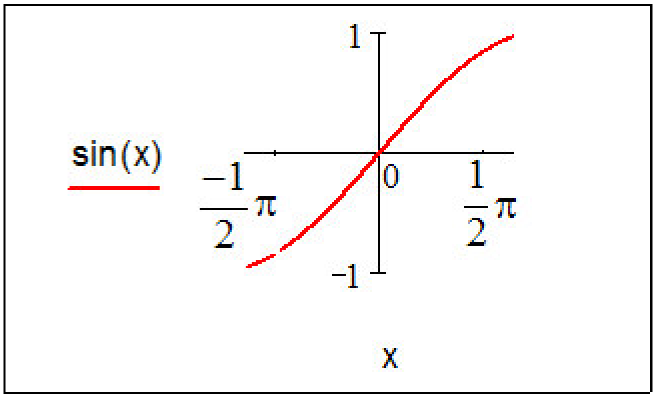
$f^{-1}(x) = \frac{1}{3}(x + 1)$ (x in [-4,2])



Inverse trigonometric functions

let

$f(x) = \sin(x) \quad (\frac{-1}{2}\pi \le x \le \frac{1}{2}\pi)$



From the graph :

- the function f is increasing, and so one to one ;
- the range (image set) of f is [-1,1].

Therefore f has inverse function (arcsine) with domain [-1,1] and range $[\frac{-1}{2}\pi, \frac{1}{2}\pi]$

In a similar way

if $h(x) = \cos(x) \qquad (0 \le x \le \pi)$

then the inverse h^{-1} is arccosine with domain [-1,1] and range $[0, \pi]$.

If $f(x) = \tan(x) \qquad (\frac{-1}{2}\pi < x < \frac{1}{2}\pi)$

then the inverse f^{-1} is arctangent with domain R (all real numbers) and range $(\frac{-1}{2}\pi, \frac{1}{2}\pi)$.

Logarithms

The function $f(x) = a^x$ for $a > 0$ and $a \neq 1$ has an inverse function with domain $(0, \infty)$ and range R, this inverse function is called logarithm to the base a, denoted by $\log_a$. Thus, for $y > 0$

$$x = \log_a y \quad \text{means that } y = a^x$$

Example:

$\log_2 32 = 5$ means that $2^5 = 32$

$\log_3 81 = 4$ means that ....

$\log_5 25 =$ ....

Properties of logarithms ($a > 0$, $a \neq 1$)

(a) $\log_a 1 = 0$ and $\log_a a = 1$.

(b) For $x > 0$ and $y > 0$,

- $\log_a (x \cdot y) = \log_a x + \log_a y$
- $\log_a \left(\dfrac{x}{y} \right) = \log_a x - \log_a y$.

(c) For $x > 0$ and p in R,

$$\log_a (x^p) = p \cdot \log_a x .$$

Example (1)

Verify the following equation:

$$\log_2\left(\frac{x^4 \cdot 4^{3x}}{2^{x^2}}\right) = 4\log_2 x + 6x - x^2$$

Solution:

By properties

$$\log_2\left(\frac{x^4 \cdot 4^{3x}}{2^{x^2}}\right) = \log_2\left(x^4\right) + \log_2\left(4^{3x}\right) - \log_2\left(2^{x^2}\right)$$

$$= 4\log_2 x + 3x \cdot \log_2 4 - x^2 \log_2 2$$

$$= 4\log_2 x + 3x \cdot 2 - x^2 1$$

$$= 4\log_2 x + 6x - x^2$$

Example (2)

find the value of:

$$\log_a 6 + \log_a 8 - \log_a 2 - \log_a 24$$

Solution:

By properties

$$\log_a 6 + \log_a 8 - \log_a 2 - \log_a 24 = \log_a\left(\frac{6 \times 8}{2 \times 24}\right)$$

$$= \log_a(1) = 0$$

Problem

Give the exact value of each of the following expression, with out using your calculator.

- $\log_3 1$

- $2^{3\log_2 5}$

Common logarithms and Natural Logarithms

- Logarithms to the base 10 are called ***common logarithms*** and denoted log x instead of $\log_{10} x$.
- Logarithms to the base e = 2.718 are called ***natural logarithms*** and denoted ln(x) instead of $\log_e x$.

It means that log(10) = 1 and ln(e) = 1.

Example

Verify the equation:

$$\ln\left(\frac{e^{x+1}}{x^3 + x^2}\right) = x + 1 - 2\ln x - \ln(x + 1)$$

Solution:

$$\ln\left(\frac{e^{x+1}}{x^3 + x^2}\right) = \ln\left(e^{x+1}\right) - \ln\left(x^3 + x^2\right)$$

$$= x + 1 - \ln\left[x^2(x + 1)\right]$$

$$= x + 1 - \ln\left(x^2\right) - \ln(x + 1)$$

$$= x + 1 - 2\ln(x) - \ln(x + 1)$$

Problem

Solve

(1) $\log\left(x^2 + 49\right) = 2$