CALCULUS AND DIFFERENTIAL EQUATIONS

quicksheets

State Space Solver

This QuickSheet illustrates how to use the solver **statespace** to solve a state space representation of a system of first-order ordinary differential equations (ODEs). State space representations are often used to describe problems in control theory and dynamical systems.

State Space Representations

A state space representation is a system of linear, first-order ODEs of the form

$$\frac{d}{d\tau}x(\tau) = A(\tau)\cdot x(\tau) + B(\tau)\cdot u(\tau)$$

where

- τ (tau) is the independent variable representing time.
- $x(\tau)$ is a vector of n states.
- $A(\tau)$ is an n-by-n *state matrix*.
- $B(\tau)$ is an n-by-k *input* matrix.
- $u(\tau)$ is an k-by-1 *input*, or *control*, vector.

In control theory, the input matrix and vector usually represent external forces that can be applied to the system to control its behavior. This QuickSheet gives an example in which the input matrix and vector represent the driving force in a forced harmonic oscillator.

State Space Representations for Higher Order ODEs

There is a standard method for writing a higher-order ODE as a system of first-order ODEs. You can use this method to write a state space representation for any higher order ODE. As an example, the following explains how to write a state space representation for damped harmonic oscillation, such as occurs when a mass, attached to a spring, has an external force, of the form $F_0 \cdot \cos(\omega_F \cdot \tau)$, applied to it.

$$m \cdot \frac{d^2}{d\tau^2} x(\tau) + b \cdot \frac{d}{d\tau} x(\tau) + k \cdot x(\tau) = F_0 \cdot \cos(\omega_F \cdot \tau)$$

Here m is the mass of the object, b is the damping constant, k is the spring constant, and $F_0 \cdot \cos(\omega_F \cdot \tau)$ represents an external force driving the system with forcing frequency ω_F.

You can simplify the above equation by dividing through by m and making the following substitutions:

$$(\omega_0)^2 = \frac{k}{m} \qquad A_0 = \frac{F_0}{m}$$

$$\frac{d^2}{d\tau^2} x(\tau) + \frac{b}{m} \cdot \frac{d}{d\tau} x(\tau) + (\omega_0)^2 \cdot x(\tau) = A_0 \cdot \cos(\omega_F \cdot \tau)$$

Here, $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural, or resonant, frequency of the system.

To write the state space representation for this second-order ODE, introduce new variables, x_0, x_1, and x_2 , corresponding to x and its first and second derivatives.

$$x_0 = x$$

$$x_1 = \frac{d}{d\tau} x_0$$

$$x_2 = \frac{d}{d\tau} x1$$

The following vector equation describes the relationships between the variables:

$$\frac{d}{d\tau}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that x_2 is the second derivative of x_0. You can rewrite the original equation in terms of these variables as

$$x_2 + \frac{b}{m}\cdot x_1 + \left(\omega_0\right)^2\cdot x_0 = A_0\cdot\cos\left(\omega_F\cdot\tau\right)$$

Solve for x_2 and substitute the result in the right-hand side of the preceding vector equation:

$$sol := x_2 + \frac{b}{m}\cdot x_1 + \left(\omega_0\right)^2\cdot x_0 = A_0\cdot\cos\left(\omega_F\cdot\tau\right) \text{ solve}, x_2 \rightarrow A_0\cdot\cos\left(\tau\cdot\omega_F\right) - x_0\cdot\left(\omega_0\right)^2 - \frac{b\cdot x_1}{m}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ substitute}, x_2 = sol \rightarrow \begin{bmatrix} x_1 \\ -\dfrac{m\cdot x_0\cdot\left(\omega_0\right)^2 + b\cdot x_1 - A_0\cdot m\cdot\cos\left(\tau\cdot\omega_F\right)}{m} \end{bmatrix}$$

You can then write the ODE in matrix form by setting

$$x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \qquad A(\tau) = \begin{bmatrix} 0 & 1 \\ -\left(\omega_0\right)^2 & \dfrac{-b}{m} \end{bmatrix}$$

$$B(\tau) := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad u(\tau) = A_0\cdot\cos\left(\omega_F\cdot\tau\right)$$

The state space representation of the original ODE is the following:

$$\frac{d}{d\tau}x(\tau) = A(\tau)\cdot x(\tau) + B(\tau)\cdot u(\tau)$$

Note that the product of the input matrix and input vector, $B(\tau)\cdot u(\tau)$, represent the driving force in the system.

Using the statespace Solver

You can solve the state space representation of the forced harmonic oscillator using the solver **statespace**, which has the following syntax:

statespace(init, t1, t2, npoints, A, B, u)

where

- init is a vector of initial conditions, whose length is the number of states.
- t1 is a starting point of the integration interval.
- t2 is an ending point of the integration interval.
- npoints is the number of points at which to return results.
- A is an n-by-n state matrix, where n is the number of states.
- B is an optional n-by-k input matrix.
- u is an optional k-by-1 the input vector.

To illustrate **statespace**, start with the case of unforced harmonic oscillator, in which B and u are not present, and the right-hand side of the harmonic oscillation equation is 0:

$$x_2 + \frac{b}{m} \cdot x_1 + \left(\omega_0\right)^2 x_0 = 0$$

There are three cases for the solutions:

Overdamped: $\left(\frac{b}{m}\right)^2 - 4 \cdot \left(\omega_0\right)^2 > 0$

Critically damped: $\left(\frac{b}{m}\right)^2 - 4 \cdot \left(\omega_0\right)^2 = 0$

Underdamped: $\left(\frac{b}{m}\right)^2 - 4 \cdot \left(\omega_0\right)^2 < 0$

The following examples illustrate these three cases:

$m := 2 \qquad b := 1$

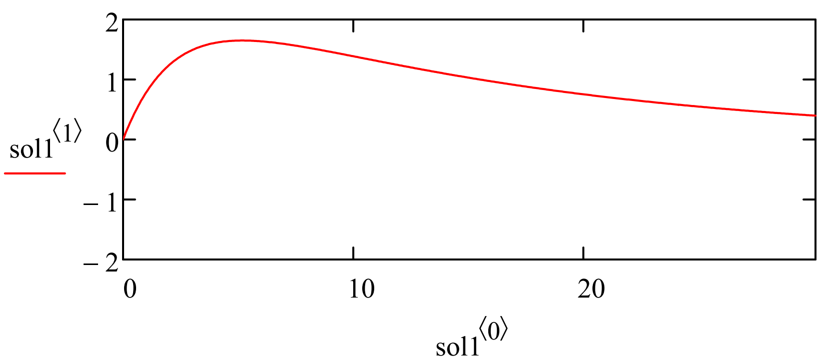
$init := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad t1 := 0 \qquad npoints := 5000$

Overdamped Solution

$\omega_0 := \frac{1}{6} \qquad \left(\frac{b}{m}\right)^2 - 4\cdot\left(\omega_0\right)^2 = 0.139 \qquad t2 := 30$

$$A(t) := \begin{bmatrix} 0 & 1 \\ -\left(\omega_0\right)^2 & \frac{-b}{m} \end{bmatrix}$$

$sol1 := statespace(init, t1, t2, 500, A)$

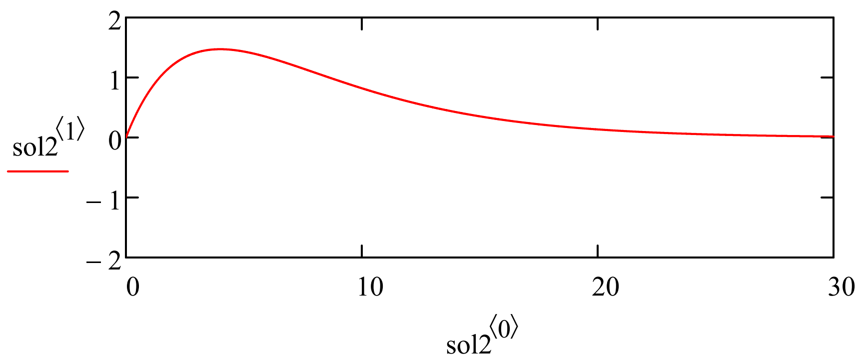


Critically Damped Solution

$$\omega_0 := \frac{1}{4} \qquad \left(\frac{b}{m}\right)^2 - 4\cdot\left(\omega_0\right)^2 = 0$$

$$A(t) := \begin{bmatrix} 0 & 1 \\ -\left(\omega_0\right)^2 & \frac{-b}{m} \end{bmatrix}$$

$$sol2 := statespace(init, t1, t2, npoints, A)$$

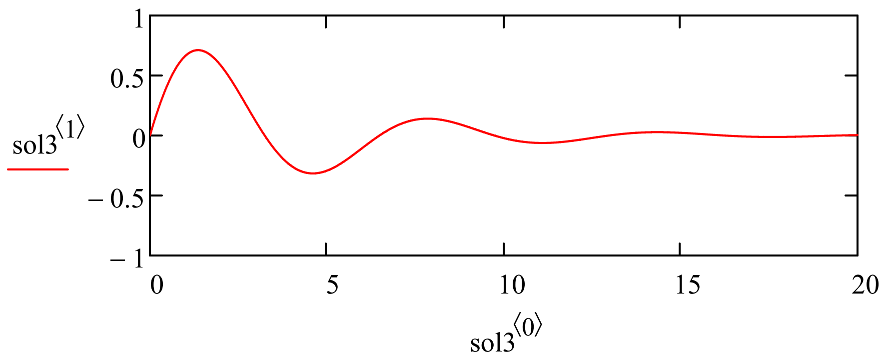


Underdamped Solution

$\omega_0 := 1$ $\left(\frac{b}{m}\right)^2 - 4\cdot(\omega_0)^2 = -3.75$ $t2 := 20$

$$A(t) := \begin{bmatrix} 0 & 1 \\ -(\omega_0)^2 & \frac{-b}{m} \end{bmatrix}$$

sol3 := statespace(init, t1, t2, npoints, A)



Forced Harmonic Oscillation

This section describes forced harmonic oscillation, in which the input matrix B and input vector u, corresponding to the external force, are present. The examples show the undamped case, in which the damping coefficient b = 0. The nature of the solutions depends on whether the forcing frequency ω_F is less than, equal to, or greater than the resonant frequency ω_0. The following examples illustrate the differences:

Case 1: Forcing frequency less than resonant frequency - $\omega_F < \omega_0$

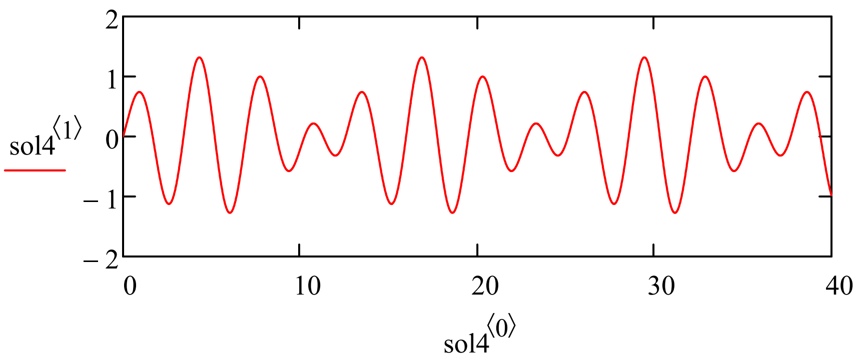
$\omega_F := 1.5 \qquad \omega_0 := 2$

$$b := 0 \qquad A(t) := \begin{bmatrix} 0 & 1 \\ -(\omega_0)^2 & \frac{-b}{m} \end{bmatrix}$$

$A_0 := 1 \qquad t2 := 40$

$$B(\tau) := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad u(\tau) := A_0 \cdot \cos(\omega_F \cdot \tau)$$

sol4 := statespace(init, t1, t2, npoints, A, B, u)



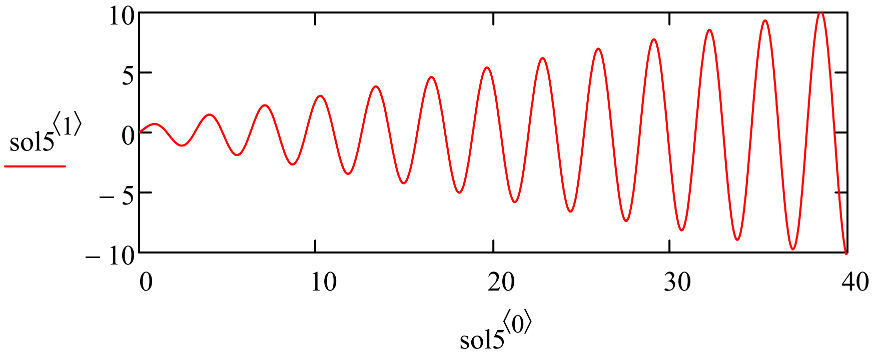
Case 2: Forcing frequency equal to resonant frequency - $\omega_F = \omega_0$

$\omega_F := 2 \qquad \omega_0 := 2$

$$A(t) := \begin{bmatrix} 0 & 1 \\ -(\omega_0)^2 & \frac{-b}{m} \end{bmatrix}$$

$A_0 := 1 \qquad t2 := 40 \qquad u(\tau) := A_0 \cdot \cos(\omega_F \cdot \tau)$

sol5 := statespace(init, t1, t2, npoints, A, B, u)



Case 3:: Forcing frequency greater than resonant frequency $\omega_F > \omega_0$

$\omega_F := 2.6$ $\omega_0 := 2$ $u(\tau) := A_0 \cdot \cos(\omega_F \cdot \tau)$

$u(\tau) := A_0 \cdot \cos(\omega_F \cdot \tau)$ $t2 := 40$

$u(\tau) := A_0 \cdot \cos(\omega_F \cdot \tau)$

sol6 := statespace(init , t1 , t2 , npoints , A , B , u)