An Algorithm for Stability of Takagi–Sugeno Fuzzy Logic Controller

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Abstract: This paper presents the design of fuzzy logic controllers (FLC's) for nonlinear systems. Fuzzy logic control systems consist of a plant and a fuzzy logic controller. The output of the FLC is made by defuzzification method. So, the output of the FLC is a function of the degrees of membership of the fuzzy rules, and these degrees of membership are function of input variables that are the system states. Therefore, the control system become highly non-linear and the analysis of system stability for this system are very difficult. In this paper, it is proved that if each fuzzy logic control systems consist of a plant and each fuzzy logic rule is stable in the sense of Lyapunov under common Lyapunov function, the overall system is also stable in the sense of Lyapunov.

Keywords: fuzzy logic controller, Lyapunov stability.

1 Introduction

Conventional automatic control system design methods involve the construction of mathematical models describing the dynamic system to be controlled and the application of analytical techniques to the model to derive control laws. These models work well provided the system does meet the requirements and assumptions of synthesis techniques. Although application of fuzzy logic to industrial problems has often produced results superior to classical control, the design procedures are limited by the heuristic rules of the system. This implicit assumption limits the application of fuzzy logic controller. Moreover, the majority of FLCs to date have been static and based upon knowledge derived from imprecise heuristic knowledge of experienced operators. The fuzzy logic-based approach to solving problems in control has been found to excel in those systems which are very complex, highly nonlinear and with parameter uncertainty. We may view a fuzzy logic controller as a real time expert system that employs fuzzy logic to analyse input to output performance. Indeed, they provide a means of converting a linguistic control strategy derived from expert knowledge into automatic control strategies and give us a means of interrogating the control system evolution and system performance.

A fuzzy logic system consist of a plant and a fuzzy logic controller (FLC) as shown in Fig. 1.

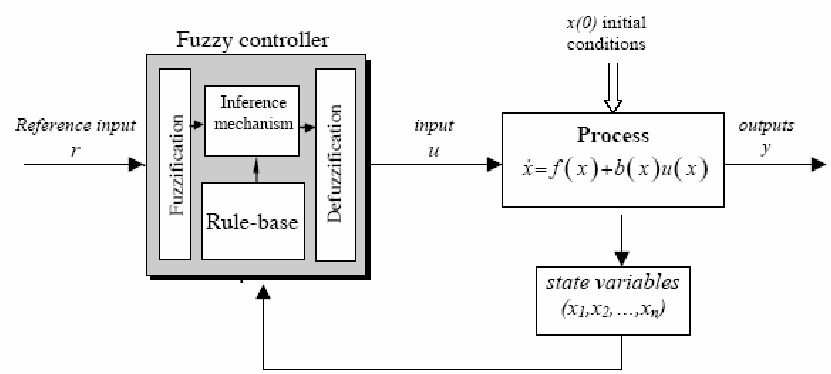

Figure 1

Fuzzy logic control system

Tanaka and Sugeno proposed a stability design approach [1] and [8] which first modelled the plant by a Takagi–Sugeno (TS) fuzzy model. This fuzzy model represents the plant as a weighted sum of a set of linear state equations. An FLC is designed based on this fuzzy plant model. Then, Lyapunov's direct method can be applied to each fuzzy subsystem that is formed by each rule of the fuzzy plant model and the FLC. The stability of the whole system can be ensured if a required positive-definite matrix exists. Other stability design approaches related to this fuzzy-model-based approach can also be found in [1]–[8].

2 Fuzzy Logic Control System

Let X be a universe of discourse. Consider a single-input nth-order nonlinear system of the following form:

$$\dot{x} = f(x) + b(x)u \tag{1}$$

where $x \in X$, $x = [x_1, x_2, ..., x_n]^T$, is the state vector,

$f(x) = [f_1(x), f_2(x), ..., f_n(x)]^T$, $b(x) = [b_1(x), b_2(x), ..., b_n(x)]^T$ are functions describing the dynamics of the plant.

u - is the control input of which the value is determined by an FLC.

The i-th IF–THEN rule in the fuzzy rule base of the FLC is of the following form:

Rule i:

IF x_i is X_{i1} AND x_2 is X_{i2} AND … AND x_n is X_{in} THEN $u = u_i(x)$, $i = \overline{1, r}$ (2)

were X_{i1}, X_{i2} … X_{in} are fuzzy sets which describe the linguistics terms (LT) of input variables, and u_i describes the linguistics terms of output variables.

Each fuzzy rule generate an activation degree: $\alpha_i \in [0,1], i = 1,2,..., r,$ $\alpha_i(x(t)) = \min(\mu_{i,1}(x_1(t)), \mu_{i,2}(x_2(t)) \dots \mu_{i,n}(x_n(t)))$. u_i can be a single value or a function of states vector, *x(t)*. It is assumed that for any $x \in X$ in the input universe of discourse *X*, there exists at least one $\alpha_i \in [0,1], i = 1,2,..., r,$ among all rules that is not equal to zero. The control signal *u*, which must be applied to PC, is a function of α_i and u_i. By applying the weighted sum defuzzification method, the output of the FLC is given by:

$$u = \frac{\sum_{i=1}^{r} \alpha_i u_i}{\sum_{i=1}^{r} \alpha_i} \tag{3}$$

where *r* is the total number of rules.

Property 1: For any input $x_0 \in X$, exist two rules p, q such that $u_{\min}(x_0) = u_p(x_0) \le u(x_0) \le u_q(x_0) = u_{\max}(x_0)$.

Proof:

Let $x_0 \in X$ than among all rules, can be found two rules *p* and *q* such that $u_p(x_0) = u_{\min}(x_0)$ and $u_q(x_0) = u_{\max}(x_0)$, where $u_{\min}(x_0) = \min_{i=1..r}(u_i(x_0))$ and $u_{\max}(x_0) = \max_{i=1..r}(u_i(x_0))$. Therefore, then:

$$u_{\min}(x_0) = \frac{\sum_{i=1}^{r} \alpha_i(x_0) \cdot u_{\min}(x_0)}{\sum_{i=1}^{r} \alpha_i(x_0)} \le \frac{\sum_{i=1}^{r} \alpha_i(x_0) \cdot u_i(x_0)}{\sum_{i=1}^{r} \alpha_i(x_0)} \le \frac{\sum_{i=1}^{r} \alpha_i(x_0) \cdot u_{\max}(x_0)}{\sum_{i=1}^{r} \alpha_i(x_0)} = u_{\max}(x_0)$$

and $u_{\min}(x_0) \le u(x_0) \le u_{\max}(x_0) \Rightarrow u_{\min}(x) \le u(x) \le u_{\max}(x), \forall x \in X$. (4)

In conclusion, for all RG-F of type Takagi- Sugeno, release 4 hold.

2 Design of Stable Fuzzy Logic Controllers

The method for stability analysis proposed in this paper is based on the following theorem. Each subsystem consist from one fuzzy rule and the process described by equation (1). It is proved that if each subsystem is stable in the sense of Lyapunov, under a common Lyapunov function, the overall system is also stable in sense of Lyapunov.

Theorem 1 [4 pag. 431]: If exist a quadric and positive-definite P matrix and:

1 $V(x) = x^T Px \to \infty$ as $\|x\| \to \infty$, $V(0) = 0$,

2 $\dot{V}(x) < 0, \forall x \neq 0$, $\dot{V}(0) = 0$ in respect with any rule,

then the system composes by FLC of type Takagi-Sugeno and the process described by equation (1), is globally asymptotically stable in at the origin.

Based on this theorem we can be found the algorithm for designed stable fuzzy logic controllers. In the next paragraph we show this.

Definition 1: A *fuzzy subsystem* associated with fuzzy rule *i* is a system with a plant of form (1) controlled by only u_i , which is the output of fuzzy rule *i* of the form (2).

2.1 The Algorithm for Designed Stable Fuzzy Logic Controllers

The idea of the proposed stability analysis algorithm is to break down the problem of analyzing the stability of the whole fuzzy logic control system into analyzing the stability of the fuzzy subsystems individually. The complexity of the analysis is drastically decreased as it is easier to check whether every fuzzy subsystem can give a negative-definite $\dot{V}$ for a given Lyapunov function *V*.

The steps of the algorithm are:

Step 1: is given the equation of the dynamics of the plant;

Step 2: we given the membership functions of the input linguistics terms;

Step 3: we given fuzzy rules database;

Step 4: define the Lyapunov function and we find the derivative of this;

Step 5: analyzing the stability of the fuzzy subsystems individually, and determine the control signal u_i such that each fuzzy subsystems to be stabilising.

Next, from the theorem 2 results that the whole fuzzy logic control system is stable. The stability proof of fuzzy logic control systems can be carried out by first applying Lyapunov direct method to each rule. If every rule individually applying to the plant of (1) results in a stable sub-system in the sense of Lyapunov subject to a common Lyapunov function, the whole fuzzy logic control system is stable.

3 Illustrative Example

Consider a nonlinear inverted pendulum system (Fig. 2) in the form of (1):

$$-(m+M)\cdot l^2\cdot\ddot{\theta}+(m+M)\cdot l\cdot g\cdot\sin(\theta)=u\ , \qquad (5)$$

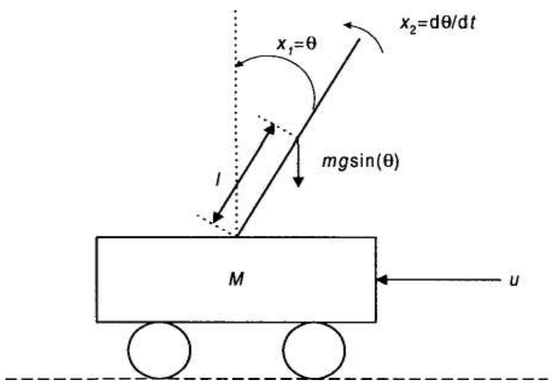

Figure 2
A car-pole inverted pendulum

where:

m – the mass of the pendulum;

M – the mass of the cart;

l – the length of the pendulum;

g – the acceleration due to gravity;

θ – the angle of the pendulum from the vertical;

u – is the force applied to the cart (in N).

Step 1: We choice as the state variables (input linguistics terms), the angle of the pendulum from the vertical $x_1=\theta$ (in grade) and the angular velocity $x_2=\dot{\theta}$ (in grade/s). The system to be controlled by:

$$\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=\frac{g}{l}\sin(x_1)-\frac{1}{(m+M)\cdot l^2}u\ ,\end{aligned} \qquad x_1\in[-80,80],\ x_2\in[-30,30] \qquad (6)$$

Step 2: The membership functions of the input linguistics terms, are:

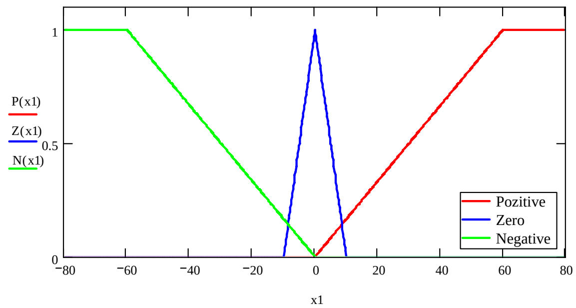

Figure 3

Membership function of the input linguistics term x_1

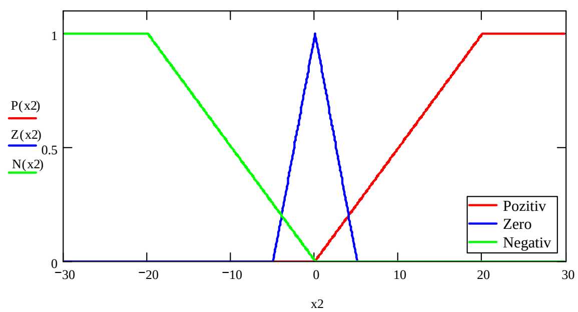

Figure 4

Membership function of the input linguistics term x_2

Step 3: We given fuzzy rules database in table 1:

x1 \ x2	**P**	**N**	**Z**
P	?	?	?
N	?	?	?
Z	?	?	?

Table 1

The control signals u_i must be determinate by applying the theorem 2.

Step 4: The Lyapunov function is: $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$. Then

$$\dot{V}(x_1, x_2) = x_1\dot{x}_2 + x_2\dot{x}_2 = x_2 \cdot \left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right).$$

Step 5: analyzing the stability of the fuzzy subsystems individually, and determine the control signals u_i such that each fuzzy subsystems to be stabilising.

Rule 1: x_1 is P and x_2 is P, that is $x_1 \in [0,80]$, $x_2 \in [0,30]$. Then

$$\dot{V}(x_1, x_2) = x_1\dot{x}_2 + x_2\dot{x}_2 = x_2 \cdot \left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) < 0 \text{ if}$$

$\left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) < 0$ for $x \neq 0$. Therefore the control variable must be satisfying the next condition: $u > (m+M) \cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$. We take $u = (m+M) \cdot l \cdot (x_1 \cdot l + g) > (m+M) \cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$.

Rule 2: x_1 is N and x_2 is N, that is $x_1 \in [-80,0]$, $x_2 \in [-30,0]$. Then

$$\dot{V}(x_1, x_2) = x_1\dot{x}_2 + x_2\dot{x}_2 = x_2 \cdot \left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) < 0 \text{ if}$$

$\left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) > 0$ for $x \neq 0$. Therefore the control variable must be satisfying the next condition: $u < (m+M) \cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$. We take $u = (m+M) \cdot l \cdot (x_1 \cdot l - g) < (m+M) \cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$.

Rule 3: x_1 is P and x_2 is N, that is $x_1 \in [0,80]$, $x_2 \in [-30,0]$. Then

$$\dot{V}(x_1, x_2) = x_1\dot{x}_2 + x_2\dot{x}_2 = x_2 \cdot \left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) < 0 \text{ if}$$

$\left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) > 0$ for $x \neq 0$.

Therefore the control variable must be satisfying the next condition: $u < (m+M) \cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$.

We take $u = (m+M) \cdot l \cdot (x_1 \cdot l - g) < (m+M) \cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$.

Rule 4: x_1 is N and x_2 is P, that is $x_1 \in [-80,0], x_2 \in [0,30]$. Then $\dot{V}(x_1, x_2) = x_1\dot{x}_2 + x_2\dot{x}_2 = x_2 \cdot \left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) < 0$ if $\left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) < 0$ for $x \neq 0$. Therefore the control variable must be satisfying the next condition: $u > (m+M)\cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$.

We take $u = (m+M)\cdot l \cdot (x_1 \cdot l + g) > (m+M)\cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$.

Rule 5: x_1 is P and x_2 is Z, that is $x_1 \in [0,80], x_2 \in [-5,5]$. A solution would be bringing $\dot{V}(x_1, x_2) = x_2 \cdot \left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right)$ to $\dot{V}(x_1, x_2) = x_1 x_2 < 0$ for $x \neq 0$. That is happened if $u = \left(x_1 + x_2 + \frac{g}{l}\cdot\sin(x_1) \right)(m+M)\cdot l^2$.

Rule 6: x_1 is N and x_2 is Z, that is $x_1 \in [-80,0], x_2 \in [-5,5]$. The result, in this case, is identically with result from rule 5.

Rule 7: x_1 is Z and x_2 is P, that is $x_1 \in [-10,10], x_2 \in [0,30]$. Then $\dot{V}(x_1, x_2) = x_1\dot{x}_2 + x_2\dot{x}_2 = x_2 \cdot \left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) < 0$ if $\left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) < 0$ for $x \neq 0$. Therefore the control variable must be satisfying the next condition: $u > (m+M)\cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$.

We take $u = (m+M)\cdot l \cdot (x_1 \cdot l + g) > (m+M)\cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$.

Rule 8: x_1 is Z and x_2 is N, that is $x_1 \in [-10,10]$, $x_2 \in [-30,0]$. Then

$\dot{V}(x_1, x_2) = x_1\dot{x}_2 + x_2\dot{x}_2 = x_2 \cdot \left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) < 0$ if $\left(x_1 + \frac{g}{l}\sin(x_1) - \frac{1}{(m+M)l^2}u \right) > 0$ for $x \neq 0$. Therefore the control variable must be satisfying the next condition: $u < (m+M)\cdot l \cdot (x_1 \cdot l + g \cdot \sin(x_1))$.

Rule 9: x_1 is Z and x_2 is Z, that is $x_1 \in [-10,10]$, $x_2 \in [-5,5]$. The result, in this case, is identically with result from rule 5.

Finally, after analysing all rules, we get now the fuzzy rules database in table 2:

Rule	**Premise**		**Consequently**
	x_1	x_2	u
1	**P**	**P**	$(m+M)\cdot l\cdot(x_1\cdot l+g)$
2	**N**	**N**	$(m+M)\cdot l\cdot(x_1\cdot l-g)$
3	**P**	**N**	$(m+M)\cdot l\cdot(x_1\cdot l-g)$
4	**N**	**P**	$(m+M)\cdot l\cdot(x_1\cdot l+g)$
5	**P**	**Z**	$\left(x_1+x_2+\frac{g}{l}\cdot\sin(x_1)\right)(m+M)\cdot l^2$
6	**N**	**Z**	$\left(x_1+x_2+\frac{g}{l}\cdot\sin(x_1)\right)(m+M)\cdot l^2$
7	**Z**	**P**	$(m+M)\cdot l\cdot(x_1\cdot l+g)$
8	**Z**	**N**	$(m+M)\cdot l\cdot(x_1\cdot l-g)$
9	**Z**	**Z**	$\left(x_1+x_2+\frac{g}{l}\cdot\sin(x_1)\right)(m+M)\cdot l^2$

Table 2

3.1 The Simulation Example

The designed FLC is applied to the process described by equation (6), for m=0.5 kg, M=9 kg, l=5 m, g=9.8 m/s^{-1}. The initial state is $x_1(0)=10$, $x_2(0)=-3$. The response of x_1 and x_2 are shown in Fig. 5.

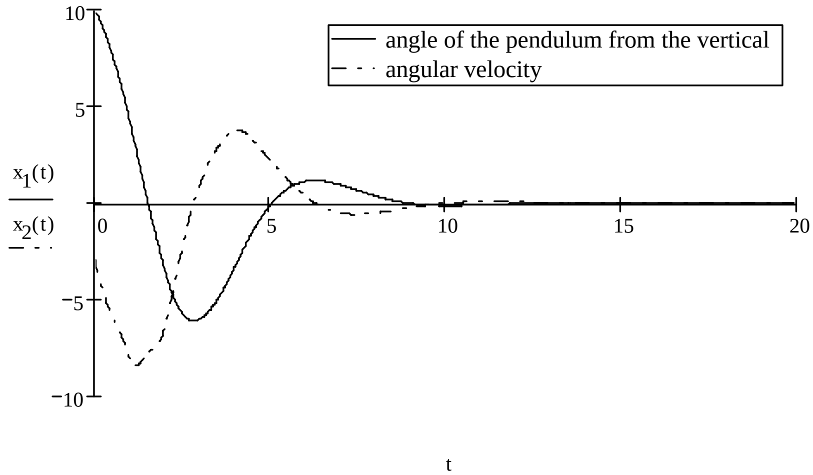

Figure 5
Response of x_1 and x_2 in the illustrated example, with FLC

The designed FLC is applied to the plant of (6), and simulation results of system responses are obtained. The stability of fuzzy logic control system is verified.

Conclusions

An approach for designing stable heuristic FLC's has been proposed in this paper. It has been shown that a fuzzy logic control system is stable in the sense of Lyapunov, if every individual rule applying to the plant gives a stable subsystem in the sense of Lyapunov under a common Lyapunov function. Therefore, the stability of the fuzzy logic control systems can be guaranteed by examining each individual rule in the FLC, which is much simpler than the existing approaches. The stability of a nonlinear car-pole inverted pendulum system controlled by an FLC has been analyzed based on the proposed approach as an illustrative example.

The algorithm for designed stable fuzzy logic controllers assures sufficient conditions of stability for the systems of automatic control with fuzzy FLC of Takagi-Sugeno type. The method has advantages because it needs to verify one condition: the derivate of Lyapunov function should be negative in respect to each fuzzy rule.

By using the proposed design approach, adding of new fuzzy rules become very easy because this needs only the fulfilment of the stability criterion, given by theorem 1. This stability analysis can be applied to other types of defuzzyfication.

The disadvantage of this type of analysis consists in the heavy calculus of the Lyapunov function.

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