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# Hermite and Laguerre polynomials with complex matrix arguments

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**Abstract**

This paper defines and discusses the complex Hermite and Laguerre polynomials associated with the complex matrix-variate normal and Wishart distributions, respectively. Various properties of these polynomials are investigated, including generating functions, Rodrigues formulae (differential and integral versions), and series expressions. These polynomials are also discussed from the viewpoint of the multivariate complex Meixner distributions. We present applications in asymptotic distribution theory on the complex Stiefel manifold. The theory of complex zonal polynomials is of great use in the derivations.


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## 1. Introduction and preliminary results

Statistical analysis on the complex matrix spaces is useful, in particular, in time series analysis. In this paper, we define and investigate the complex Hermite polynomials with Hermitian and complex matrix argument and the complex Laguerre polynomials with Hermitian matrix argument, which are associated with the Hermitian and complex matrix-variate normal distributions and the complex Wishart distribution, respectively. These unitary polynomials play important roles particularly

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in asymptotic complex distribution theory. This paper extends the results of Chikuse [4–6], in which Hermite and Laguerre polynomials with real matrix arguments were thoroughly investigated.

In Section 2, we define the (standard) complex normal $\tilde{N}_{mm}(0, I_m)$ distribution (see (2.1)) on the space of $m \times m$ Hermitian matrices

$$\tilde{S}_m = \{S(m \times m);\ S = S^*\}, \tag{1.1}$$

where $S^*$ indicates the conjugate transpose of an $m \times m$ complex matrix $S$; the condition $S = S^*$, i.e., $S_1 + \mathrm{i}S_2 = S_1' - \mathrm{i}S_2'$, indicates that $S_1$ and $S_2$ are $m \times m$ symmetric and skew-symmetric, respectively. Now, let $\tilde{O}(m)$ denote the unitary group, i.e., $\tilde{O}(m) = \{H(m \times m);\ H^*H = I_m\}$. Here, we note that the complex notions Hermitian and unitary correspond to the real symmetric and orthogonal, respectively, and that we will put $\tilde{}$ on real notations to denote the complex counterparts in this paper. A Hermitian matrix $S \in \tilde{S}_m$ can be expressed as $S = H^*DH$, where $H \in \tilde{O}(m)$ and $D = \mathrm{diag}(d_1, \ldots, d_m)$ with $d_i$s real latent roots of $S$. When all $d_i$s are positive, $S$ is positive definite, and we let $\tilde{S}_m^+$ denote the set of all $m \times m$ positive definite Hermitian matrices. The inverse matrix $S^{-1}$ and the determinant $|S|$ are defined similarly to those for the real case.

The complex Hermite polynomials $\tilde{H}_\lambda^{(m)}(S)$, $\lambda \vdash l = 0, 1, \ldots$, are defined as a complete system of unitary polynomials associated with the normal $\tilde{N}_{mm}(0, I_m)$ distribution. Here $\lambda \vdash l$ denotes that $\lambda$ is an ordered partition of an integer $l$ into not more than $m$ parts; $\lambda = (l_1, l_2, \ldots, l_m)$, $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_m \geqslant 0$, $\sum_{i=1}^m l_i = l$. We discuss various properties of the polynomials $\tilde{H}_\lambda^{(m)}(S)$, i.e., generating function (g.f.), Fourier transform, Rodrigues formulae, and series expressions for $\tilde{H}_\lambda^{(m)}(S)$ and for its differentials. In particular, the integral version of Rodrigues formulae is of great use in asymptotic distribution theory for complex matrix variates; see Section 6 for applications.

Section 3 is concerned with the (standard) complex normal $\tilde{N}_{m,k}(0; I_m, I_k)$ distribution (see (3.1)) on the space of $m \times k$ complex matrices

$$\tilde{R}_{m,k} = \{Z(m \times k) = Z_1 + \mathrm{i}Z_2;\ Z_1 \text{ and } Z_2\ m \times k \text{ real matrices}\}. \tag{1.2}$$

The complex Hermite polynomials $\tilde{H}_\lambda^{(mk)}(Z)$, $\lambda \vdash l = 0, 1, \ldots$, constitute a complete system of unitary polynomials associated with the normal $\tilde{N}_{m,k}(0; I_m, I_k)$ distribution. A similar line of discussion on the properties of the polynomials $\tilde{H}_\lambda^{(mk)}(Z)$ is carried out.

The complex Laguerre polynomials $\tilde{L}_\lambda^u(W)$, $\lambda \vdash l = 0, 1, \ldots$, are defined as a complete system of unitary polynomials associated with the complex Wishart distribution. Section 4 gives a discussion on the properties of the polynomials $\tilde{L}_\lambda^u(W)$, including a limit normal $\tilde{N}_{mm}(0, I_m)$ property of the Wishart distribution and a relationship of $\tilde{L}_\lambda^u(Z^*Z)$ with $\tilde{H}_\lambda^{(mk)}(Z)$ polynomials.

Here we note that Andersen et al. [1] gave a thorough discussion of the complex matrix-variate normal $\tilde{N}_{m,k}$ and Wishart $\tilde{W}_m$ distributions.

Section 5 gives a brief discussion of the multivariate complex Meixner classes of invariant distributions, which include the two kinds of normal and Wishart distributions considered in the previous sections.

In Section 6, we present applications of the previous results on the complex Stiefel manifold $\tilde{V}_{k,m}$. We define the complex matrix Langevin distribution on $\tilde{V}_{k,m}$, and derive large sample asymptotic distributions for the sample mean matrix using Rodrigues formulae.

In the rest of this section, we present some results, which are relevant for the derivations throughout this paper.

### *1.1. Theory of complex zonal polynomials*

We give a brief discussion of the theory of complex zonal polynomials (mainly due to James [13]), which play important roles in the multivariate complex distribution theory, corresponding to the more familiar one of real zonal polynomials (e.g., [10,13]).

The complex zonal polynomials $\tilde{C}_\lambda(S)$ are defined by the theory of group representation of the full linear group $Gl(m, C)$ of all $m \times m$ nonsingular complex matrices under the congruence transformation,

$$S \to LSL^* \quad \text{for } L \in Gl(m, C),$$

in the vector space of homogeneous polynomials of degree $l$ in a matrix $S \in \tilde{S}_m$, where $\lambda$ corresponds to the irreducible representation indexed by $[2\lambda]$, $\lambda \vdash l = 0, 1, \ldots$. The $\tilde{C}_\lambda(S)$, $\lambda \vdash l = 0, 1, \ldots$, constitute a basis of all homogeneous symmetric polynomials in the latent roots of $S$, having the property of invariance under the transformation $S \to HSH^*$, $H \in \tilde{O}(m)$.

The basic property is

$$\int_{\tilde{O}(m)} \tilde{C}_\lambda(HSH^*T)[\mathrm{d}H] = \frac{\tilde{C}_\lambda(S)\tilde{C}_\lambda(T)}{\tilde{C}_\lambda(I_m)}, \tag{1.3}$$

where $[\mathrm{d}H]$ denotes the normalized invariant measure (i.e., $\int_{\tilde{O}(m)}[\mathrm{d}H] = 1$) on $\tilde{O}(m)$. Chikuse [8] investigates the (normalized) invariant measures on the complex Stiefel manifold $\tilde{V}_{k,m}$, which includes $\tilde{O}_{(m)} = \tilde{V}_{m,m}$ as a special case and will be treated in Section 6.

The Laplace transform is

$$\int_{\tilde{S}_m^+} \mathrm{etr}(-S)|S|^{a-m}\tilde{C}_\lambda(AS)(\mathrm{d}S) = \tilde{\Gamma}_m(a)[a]_\lambda \tilde{C}_\lambda(A), \tag{1.4}$$

where $\mathrm{etr}\, A = \exp(\mathrm{tr}\, A)$, the Lebesgue measure $(\mathrm{d}S)$ is defined by (1.19), and we define the complex multivariate gamma function

$$\tilde{\Gamma}_m(a) = \int_{\tilde{S}_m^+} \mathrm{etr}(-S)|S|^{a-m}(\mathrm{d}S) = \pi^{m(m-1)/2}\prod_{i=1}^{m}\Gamma(a-i+1), \tag{1.5}$$

and the complex multivariate hypergeometric coefficient

$$[a]_\lambda = \prod_{i=1}^{p}(a-i+1)_{l_i} \quad \text{for } \lambda = (l_1, l_2, \ldots, l_p),$$
$$\text{with } (a)_l = a(a+1)\cdots(a+l-1).$$

Following a similar argument to that due to e.g., Davis and coworkers [9,11], we can express

$$\tilde{C}_\sigma(S)\tilde{C}_\phi(S) = \sum_{\lambda^* \in \sigma\cdot\phi} \tilde{g}^{\lambda^*}_{\sigma,\phi}\tilde{C}_{\lambda^*}(S), \tag{1.6}$$

and

$$\tilde{C}_\nu(S^2) = \sum_{\phi\in\nu\cdot\nu} \tilde{\eta}_{\nu,\phi}\tilde{C}_\phi(S), \tag{1.7}$$

for suitably defined coefficients $\tilde{g}$ and $\tilde{\eta}$. Here, $\lambda \in \sigma \cdot \phi$ indicates that the irreducible representation indexed by $[2\lambda]$ occurs in the decomposition of the Kronecker product $[2\sigma] \otimes [2\phi]$ of irreducible representations indexed by $[2\sigma]$ and $[2\phi]$, and $\Sigma_{\lambda^*}$ denotes the summation ignoring the multiplicity. The generalized binomial coefficients $\tilde{b}$ are defined by

$$\frac{\tilde{C}_\lambda(S+I_m)}{\tilde{C}_\lambda(I_m)} = \sum_{s=0}^{l}\sum_{\sigma\vdash s} \tilde{b}_{\lambda,\sigma}\frac{\tilde{C}_\sigma(S)}{\tilde{C}_\sigma(I_m)}. \tag{1.8}$$

We can readily show that

$$\tilde{b}_{\lambda,\sigma} = \sum_{\substack{\phi\vdash(l-s)\\(\lambda\in\sigma\cdot\phi)}} \frac{l!}{s!f!}\tilde{g}^{\lambda}_{\sigma,\phi}, \quad \text{for } \tilde{g} \text{ defined by (1.6)}, \tag{1.9}$$

and

$$\sum_{s\leqslant n\leqslant l}\sum_{\nu\vdash n}(-1)^{n+s}\tilde{b}_{\lambda,\nu}\tilde{b}_{\nu,\sigma} = \delta_{\lambda,\sigma}, \quad \text{for Kronecker's delta } \delta. \tag{1.10}$$

The complex hypergeometric functions with Hermitian matrix arguments are defined by

$${}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;S) = \sum_{l=0}^{\infty}\sum_{\lambda\vdash l}\frac{[a_1]_\lambda\cdots[a_p]_\lambda}{[b_1]_\lambda\cdots[b_q]_\lambda}\frac{\tilde{C}_\lambda(S)}{l!}, \tag{1.11}$$

and

$$\begin{aligned}
&{}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;S,T)\\
&\quad= \int_{\tilde{O}(m)} {}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;HSH^*T)[\mathrm{d}H]\\
&\quad= \sum_{l=0}^{\infty}\sum_{\lambda\vdash l}\frac{[a_1]_\lambda\cdots[a_p]_\lambda}{[b_1]_\lambda\cdots[b_q]_\lambda}\frac{\tilde{C}_\lambda(S)\tilde{C}_\lambda(T)}{l!\tilde{C}_\lambda(I_m)},
\end{aligned} \tag{1.12}$$

for $a_1, \dots, a_p, b_1, \dots, b_q$ complex constants. The special cases are

$$ {}_0\tilde{F}_0(S) = \mathrm{etr}(S), \tag{1.13} $$

and

$$ {}_0\tilde{F}_1\left(m; \frac{1}{4}Z^*Z\right) = \int_{\tilde{O}(m)} \mathrm{etr}[\mathrm{Re}(ZH)][\mathrm{d}H], \tag{1.14} $$

where $\mathrm{Re}(A) = A_1$, the real part of $A$ for $A = A_1 + \mathrm{i}A_2$.

*1.2. Differentials and measures*

Let us define the complex matrices of differential operators

$$ \partial S = \partial S_1 + \mathrm{i}\partial S_2, \quad \text{for } S = S_1 + \mathrm{i}S_2 \in \tilde{S}_m, \tag{1.15} $$

where

$$ \begin{cases} \partial S_1 = \left(\dfrac{1}{2}(1+\delta_{ij})\dfrac{\partial}{\partial s_{ij}^{(1)}}\right), & \text{with } S_1 = \left(s_{ij}^{(1)}\right) \text{ symmetric,} \\ \partial S_2 = \left(\dfrac{1}{2}(1-\delta_{ij})\dfrac{\partial}{\partial s_{ij}^{(2)}}\right), & \text{with } S_2 = \left(s_{ij}^{(2)}\right) \text{ skew-symmetric,} \end{cases} $$

and

$$ \partial Z = \partial Z_1 + \mathrm{i}\partial Z_2, \quad \text{for } Z = Z_1 + \mathrm{i}Z_2 \in \tilde{R}_{m,k}, \tag{1.16} $$

where

$$ \partial Z_l = \left(\frac{\partial}{\partial z_{ij}^{(l)}}\right), \quad \text{with } Z_l = \left(z_{ij}^{(l)}\right), \ \ l = 1, 2. $$

We can show the following properties for suitable analytic functions $f(\cdot)$:

$$ \text{(i)} \quad \begin{cases} f(\partial S)\,\mathrm{etr}(TS) = f(T)\,\mathrm{etr}(TS), \quad \text{for } S, T \in \tilde{S}_m, \\ f(\partial Z)\,\mathrm{etr}[\mathrm{Re}(T^*Z)] = f(T)\,\mathrm{etr}[\mathrm{Re}(T^*Z)], \\ \quad \text{for } Z, T \in \tilde{R}_{m,k}. \end{cases} \tag{1.17} $$

(ii) *Taylor's expansions*:

$$ \begin{aligned} &f(S+T) = \mathrm{etr}(T\partial S)f(S), \quad \text{for } S, T \in \tilde{S}_m, \\ &f(Z+T) = \mathrm{etr}[\mathrm{Re}(T^*\partial Z)]f(Z), \quad \text{for } Z, T \in \tilde{R}_{m,k}. \end{aligned} \tag{1.18} $$

The Lebesgue measures on $\tilde{S}_m$ and $\tilde{R}_{m,k}$ are

$$ (\mathrm{d}S) = \bigwedge_{\substack{1 \\ i\leqslant j}}^{m} \mathrm{d}s_{ij}^{(1)} \cdot \bigwedge_{\substack{1 \\ i<j}}^{m} \mathrm{d}s_{ij}^{(2)}, \quad \text{for } S = S_1 + \mathrm{i}S_2, $$

$$ \quad \text{with } S_l = \left(s_{ij}^{(l)}\right), \ \ l = 1, 2, $$

and

$$(\mathrm{d}Z) = \bigwedge_{l=1}^{2} \bigwedge_{j=1}^{k} \bigwedge_{i=1}^{m} \mathrm{d}z_{ij}^{(l)}, \quad \text{for } Z = Z_1 + \mathrm{i}Z_2, \quad \text{with } Z_l = \left(z_{ij}^{(l)}\right), \quad l = 1, 2, \tag{1.19}$$

respectively, where we use the symbol $\wedge$ to indicate the exterior product of differentials. Throughout this paper, the probability density functions (p.d.f.'s) of distributions on $\tilde{S}_m$ and $\tilde{R}_{m,k}$ are expressed with respect to these Lebesgue measures.

## 2. Hermite polynomials with Hermitian matrix argument

### 2.1. Associated normal distributions

An $m \times m$ random Hermitian matrix $S = S_1 + \mathrm{i}S_2$ is said to have the $m \times m$ Hermitian matrix-variate normal $\tilde{N}_{mm}(0, I_m)$ distribution if the p.d.f. is

$$\tilde{\varphi}^{(m)}(S) = \tilde{a}_m \operatorname{etr}(-S^2) = \tilde{a}_m \operatorname{etr}(-S_1' S_1 - S_2' S_2),$$
$$\text{with } \tilde{a}_m = 2^{m(m-1)/2}/\pi^{m^2/2}. \tag{2.1}$$

That is, $s_{ii}^{(1)}, s_{ij}^{(1)}$ and $s_{ij}^{(2)}$ $(i, j = 1, \ldots, m, i < j)$ are all independently distributed as normal $N(0, \frac{1}{2})$, $N(0, \frac{1}{4})$ and $N(0, \frac{1}{4})$, respectively. The moment generating function (m.g.f.) is given by

$$\tilde{M}_S(T) = E \operatorname{etr}(TS) = E \operatorname{etr}(S_1 T_1 - S_2 T_2) = \operatorname{etr}\left(\frac{1}{4} T^2\right), \tag{2.2}$$

for a Hermitian matrix $T = T_1 + \mathrm{i}T_2$.

The normal $\tilde{N}_{mm}(0, I_m)$ distribution is obtained as a limit of the complex Wishart distribution, which will be discussed in Section 4.

In general, for $m \times m$ matrices $\Sigma \in \tilde{S}_m^+$ and $M \in \tilde{S}_m$, and an $m \times m$ random Hermitian matrix $V$ distributed as normal $\tilde{N}_{mm}(0, I_m)$, the random matrix

$$S = \Sigma^{1/2} V \Sigma^{1/2} + M$$

may be said to have the $m \times m$ Hermitian matrix-variate normal $\tilde{N}_{mm}(M, \Sigma)$ distribution, whose p.d.f. is

$$\tilde{a}_m |\Sigma|^{-m} \operatorname{etr}[-(S - M)\Sigma^{-1}(S - M)\Sigma^{-1}].$$

Here, $A^{1/2}$ denotes the unique square root of a matrix $A \in \tilde{S}_m^+$.

### 2.2. Complex Hermite polynomials $\tilde{H}_\lambda^{(m)}(S)$

The complex Hermite polynomials $\tilde{H}_\lambda^{(m)}(S)$, $\lambda \vdash l = 0, 1, \ldots$, constitute the complete system of unitary polynomials associated with the normal $\tilde{N}_{mm}(0, I_m)$ distribution.

*Generating function*

The g.f. for the $\tilde{H}_\lambda^{(m)}(S)$ is

$$\sum_{l=0}^{\infty}\sum_{\lambda\vdash l}\frac{\tilde{H}_\lambda^{(m)}(S)\tilde{C}_\lambda(T)}{l!\tilde{C}_\lambda(I_m)} = \operatorname{etr}\left(-\frac{1}{4}T^2\right)\int_{\tilde{O}(m)}\operatorname{etr}(HSH^*T)[\mathrm{d}H]. \tag{2.3}$$

Now, various properties can be derived from (2.3).

*Fourier transform*

On multiplying by $\operatorname{etr}(\mathrm{i}YS)\tilde{\varphi}^{(m)}(S)$, for $Y \in \tilde{S}_m$, and integrating over $S \in \tilde{S}_m$ both sides of (2.3), we can evaluate the right-hand side of the resulting equation as

$$\operatorname{etr}\left(-\frac{1}{4}Y^2\right)\int_{\tilde{O}(m)}\operatorname{etr}\left(\frac{1}{2}\mathrm{i}HYH^*T\right)[\mathrm{d}H] = \operatorname{etr}\left(-\frac{1}{4}Y^2\right){}_0\tilde{F}_0\left(\frac{1}{2}\mathrm{i}Y, T\right),$$

from (1.12) and (1.13).

Expanding the ${}_0\tilde{F}_0$ function in terms of complex zonal polynomials using (1.12) and comparing the coefficients of $\tilde{C}_\lambda(T)$ in the resulting equation, we obtain the Fourier transform

$$\int_{\tilde{S}_m}\operatorname{etr}(\mathrm{i}YS)\tilde{\varphi}^{(m)}(S)\tilde{H}_\lambda^{(m)}(S)(\mathrm{d}S) = \operatorname{etr}\left(-\frac{1}{4}Y^2\right)\tilde{C}_\lambda\left(\frac{1}{2}\mathrm{i}Y\right). \tag{2.4}$$

*Rodrigues formulae*

On multiplying both sides of (2.3) by $\tilde{\varphi}^{(m)}(S)$, we can evaluate the right-hand side of the resulting equation as

$$\begin{aligned}
&\int_{\tilde{O}(m)}\tilde{\varphi}^{(m)}\left(\frac{1}{2}H^*TH - S\right)[\mathrm{d}H]\\
&\quad= \int_{\tilde{O}(m)}\operatorname{etr}\left(-\frac{1}{2}H^*TH\partial S\right)\tilde{\varphi}^{(m)}(S)[\mathrm{d}H], \quad \text{from (1.18)},\\
&\quad= {}_0\tilde{F}_0\left(-\frac{1}{2}\partial S, T\right)\tilde{\varphi}^{(m)}(S), \quad \text{from (1.12)},
\end{aligned}$$

which, on comparing the coefficients of $\tilde{C}_\lambda(T)$, gives the *differential version* of Rodrigues formulae

$$\tilde{H}_\lambda^{(m)}(S)\tilde{\varphi}^{(m)}(S) = \tilde{C}_\lambda\left(-\frac{1}{2}\partial S\right)\tilde{\varphi}^{(m)}(S). \tag{2.5}$$

Inverting (2.4) gives the *integral version* of Rodrigues formulae (inverse Fourier transform)

$$\tilde{H}_\lambda^{(m)}(S)\tilde{\varphi}^{(m)}(S) = \frac{2^{m(m-1)}}{(2\pi)^{m^2}}\int_{\tilde{S}_m}\operatorname{etr}\left(-\mathrm{i}ST - \frac{1}{4}T^2\right)\tilde{C}_\lambda\left(\frac{1}{2}\mathrm{i}T\right)(\mathrm{d}T). \tag{2.6}$$

*Series expression for $\tilde{H}_\lambda^{(m)}(S)$ in terms of the $\tilde{C}_\sigma(S)$ and vice versa*

Expanding the right-hand side of (2.3), that is, $\mathrm{etr}(-\frac{1}{4}T^2)_0\tilde{F}_0(S,T)$, in terms of complex zonal polynomials using (1.12) and (1.13), then the formulae (1.6) and (1.7) for the coefficients $\tilde{g}$ and $\tilde{\eta}$, respectively, and comparing the coefficients of $\tilde{C}_\lambda(T)$, we obtain

$$\frac{\tilde{H}_\lambda^{(m)}(S)}{l!\tilde{C}_\lambda(I_m)} = \sum_{s=0}^{l}\sum_{\sigma\vdash s}\left[\sum_{\nu\vdash n}\sum_{\substack{\phi\in\nu\cdot\nu \\ (\lambda\in\sigma\cdot\phi)}} \frac{\tilde{\eta}_{\nu,\phi}\tilde{g}_{\sigma,\phi}^{\lambda}}{(-4)^n n!}\right]\frac{\tilde{C}_\sigma(S)}{s!\tilde{C}_\sigma(I_m)}. \tag{2.7}$$

Hence, the term of the highest degree of $\tilde{H}_\lambda^{(m)}(S)$ is $\tilde{C}_\lambda(S)$, and for $l$ even (odd) only the terms of even (odd) degree in $S$ appear in the expansion.

Next, we multiply both sides of (2.3) by $\mathrm{etr}(\frac{1}{4}T^2)$ and carry out a similar procedure. We obtain the series expression of $\tilde{C}_\lambda(S)/l!\tilde{C}_\lambda(I_m)$ which is given by the right-hand side of (2.7) with $4^n$ and $\tilde{H}_\sigma^{(m)}(S)$ replacing $(-4)^n$ and $\tilde{C}_\sigma(S)$, respectively.

*Series expression for the differential*

Applying $\tilde{C}_\nu(\partial S)$ on both sides of (2.3), the resulting right-hand side becomes $\tilde{C}_\nu(T)\times$ [the left-hand side of (2.3)], where (1.17) is used. Comparing the coefficients of $\tilde{C}_\lambda(T)$, we obtain

$$\frac{\tilde{C}_\nu(\partial S)\tilde{H}_\lambda^{(m)}(S)}{l!\tilde{C}_\lambda(I_m)} = \sum_{\substack{\sigma\vdash(l-n) \\ (\lambda\in\nu\cdot\sigma)}} \frac{\tilde{g}_{\nu,\sigma}^{\lambda}}{(l-n)!\tilde{C}_\sigma(I_m)}\tilde{H}_\lambda^{(m)}(S). \tag{2.8}$$

## 3. Hermite polynomials with complex matrix argument

### 3.1. Associated normal distributions

An $m\times k$ random complex matrix $Z=Z_1+\mathrm{i}Z_2$ is said to have the $m\times k$ complex matrix-variate normal $\tilde{N}_{m,k}(0;I_m,I_k)$ distribution if the p.d.f. is

$$\tilde{\varphi}^{(mk)}(Z)=\pi^{-km}\,\mathrm{etr}(-Z^*Z)=\pi^{-km}\,\mathrm{etr}(-Z_1'Z_1-Z_2'Z_2). \tag{3.1}$$

For $T\in\tilde{R}_{m,k}$, the m.g.f. is given by

$$\tilde{M}_Z(T)=E\,\mathrm{etr}[\mathrm{Re}(T^*Z)]=\mathrm{etr}\left(\frac{1}{4}T^*T\right). \tag{3.2}$$

In general, for $\Sigma_1\in\tilde{S}_m^+$, $\Sigma_2\in\tilde{S}_k^+$ and $M\in\tilde{R}_{m,k}$, and an $m\times k$ random complex matrix $Y$ distributed as normal $\tilde{N}_{m,k}(0;I_m,I_k)$, the random matrix

$$Z=\Sigma_1^{1/2}Y\Sigma_2^{1/2}+M$$

may be said to have the $m\times k$ complex matrix-variate normal $\tilde{N}_{m,k}(M;\Sigma_1,\Sigma_2)$ distribution.

### 3.2. Complex Hermite polynomials $\tilde{H}_\lambda^{(mk)}(Z)$

The complex Hermite polynomials $\tilde{H}_\lambda^{(mk)}(Z)$, $\lambda \vdash l = 0, 1, \ldots$, constitute the complete system of unitary polynomials associated with the normal $\tilde{N}_{m,k}(0; I_m, I_k)$ distribution.

*Generating function*

The g.f. for the $\tilde{H}_\lambda^{(mk)}(Z)$ is

$$\sum_{l=0}^{\infty}\sum_{\lambda\vdash l}\frac{\tilde{H}_\lambda^{(mk)}(Z)\tilde{C}_\lambda(T^*T)}{4^l[k]_\lambda l!\tilde{C}_\lambda(I_m)}$$
$$= \operatorname{etr}\left(-\frac{1}{4}T^*T\right)\int_{\tilde{O}(m)}\int_{\tilde{O}(k)}\operatorname{etr}[\operatorname{Re}(HZQ^*T^*)][\mathrm{d}Q][\mathrm{d}H]. \tag{3.3}$$

As in Section 2.2, we can obtain from (3.3) the following results, where the discussions involving the ${}_0\tilde{F}_0$ functions carried out in Section 2.2 are replaced by those involving the ${}_1\tilde{F}_1$ functions in this section.

*Fourier transform*

$$\int_{\tilde{R}_{m,k}}\operatorname{etr}[\mathrm{i}\operatorname{Re}(Y^*Z)]\tilde{\varphi}^{(mk)}(Z)\tilde{H}_\lambda^{(mk)}(Z)(\mathrm{d}Z)$$
$$= \operatorname{etr}\left(-\frac{1}{4}Y^*Y\right)\tilde{C}_\lambda\left(-\frac{1}{4}Y^*Y\right). \tag{3.4}$$

*Rodrigues formulae*

$$\tilde{H}_\lambda^{(mk)}(Z)\tilde{\varphi}^{(mk)}(Z)$$
$$= \tilde{C}_\lambda\left(\frac{1}{4}\partial Z^*\partial Z\right)\tilde{\varphi}^{(mk)}(Z) \tag{3.5}$$
$$= \frac{1}{(2\pi)^{2km}}\int_{\tilde{R}_{m,k}}\operatorname{etr}\left[-\mathrm{i}\operatorname{Re}(Z^*T)-\frac{1}{4}T^*T\right]\tilde{C}_\lambda\left(-\frac{1}{4}T^*T\right)(\mathrm{d}T). \tag{3.6}$$

*Series expression for $\tilde{H}_\lambda^{(mk)}(Z)$*

$$\frac{\tilde{H}_\lambda^{(mk)}(Z)}{[k]_\lambda\tilde{C}_\lambda(I_m)} = \sum_{s=0}^{l}\sum_{\sigma\vdash s}(-1)^{l-s}\tilde{b}_{\lambda,\sigma}\frac{\tilde{C}_\sigma(Z^*Z)}{[k]_\sigma\tilde{C}_\sigma(I_m)}, \tag{3.7}$$

where the coefficients $\tilde{b}$ are defined by (1.8).

*Series expression for the differential*

$$\frac{\tilde{C}_\nu(\partial Z^*\partial Z)\tilde{H}_\lambda^{(mk)}(Z)}{4^l[k]_\lambda l!\tilde{C}_\lambda(I_m)} = \sum_{\substack{\sigma\vdash(l-n)\\(\lambda\in\nu\cdot\sigma)}} \frac{\tilde{g}_{\nu,\sigma}^\lambda \tilde{H}_\sigma^{(mk)}(Z)}{4^{l-n}[k]_\sigma(l-n)!\tilde{C}_\sigma(I_m)}. \tag{3.8}$$

## 4. Laguerre polynomials

### 4.1. Complex Wishart distributions

The following lemma is useful.

**Lemma 4.1** (Chikuse [8]). *Let the unique complex polar decomposition of $Z \in \tilde{R}_{m,k}$ be*

$$Z = \tilde{H}_Z\tilde{T}_Z^{1/2}, \quad \textit{with } \tilde{H}_Z = Z(Z^*Z)^{-1/2} \in \tilde{V}_{k,m} \textit{ and } \tilde{T}_Z = Z^*Z \in \tilde{S}_m^+. \tag{4.1}$$

*Then the Lebesgue measure* $(\mathrm{d}Z)$ *is decomposed as*

$$(\mathrm{d}Z) = [\pi^{km}/\tilde{\Gamma}_k(m)]|\tilde{T}_Z|^{m-k}(\mathrm{d}\tilde{T}_Z)[\mathrm{d}\tilde{H}_Z]. \tag{4.2}$$

Letting the $n \times m$ complex matrix $Z$ be distributed as normal $\tilde{N}_{n,m}(0; I_n, \Sigma)$ with $\Sigma \in \tilde{S}_m^+$ and using Lemma 4.1, $W = Z^*Z$ has the complex Wishart $\tilde{W}_m(n, \Sigma)$ distribution having the p.d.f.

$$\tilde{w}_m(W; n, \Sigma) = [\tilde{\Gamma}_m(n)|\Sigma|^n]^{-1}\operatorname{etr}(-\Sigma^{-1}W)|W|^{n-m}. \tag{4.3}$$

The m.g.f. is given by

$$E\operatorname{etr}(TW) = |I_m - \Sigma T|^{-n}, \quad \text{for } T \in \tilde{S}_m. \tag{4.4}$$

If $Z$ is distributed as normal $\tilde{N}_{n,m}(M; I_n, \Sigma)$, $W = Z^*Z$ has the noncentral complex Wishart $\tilde{W}(n, \Sigma; \Omega)$ distribution, with noncentrality matrix $\Omega = \Sigma^{-1}M^*M$, having the p.d.f.

$$\begin{aligned}&\tilde{w}_m(W; n, \Sigma; \Omega)\\&\quad = [\tilde{\Gamma}_m(n)|\Sigma|^n]^{-1}\operatorname{etr}(-\Omega)\operatorname{etr}(-\Sigma^{-1}W)|W|^{n-m}{}_0\tilde{F}_1(n; \Omega\Sigma^{-1}W).\end{aligned} \tag{4.5}$$

### 4.2. Complex Laguerre polynomials $\tilde{L}_\lambda^u(W)$

The complex Laguerre polynomials $\tilde{L}_\lambda^u(W)$, $\lambda \vdash l = 0, 1, \ldots$, constitute the complete system of unitary polynomials associated with the Wishart $\tilde{W}_m(2u + m, \frac{1}{2}I_m)$ distribution. We give some results on the polynomials $\tilde{L}_\lambda^u(W)$ in the following.

*Generating function*

$$\sum_{l=0}^{\infty}\sum_{\lambda\vdash l}\frac{\tilde{L}_\lambda^u(W)\tilde{C}_\lambda(T)}{l!\tilde{C}_\lambda(I_m)}$$
$$= \left|I_m - \frac{1}{2}T\right|^{-2u-m}\int_{\tilde{O}(m)} \mathrm{etr}\left[-HWH^*T\left(I_m - \frac{1}{2}T\right)^{-1}\right][\mathrm{d}H]. \qquad (4.6)$$

*Laplace transform*

$$\int_{\tilde{S}_m^+} \mathrm{etr}(-YW)|W|^{2u}\tilde{L}_\lambda^u(W)(\mathrm{d}W)$$
$$= \tilde{\Gamma}_m(2u+m)[2u+m]_\lambda |Y|^{-2u-m}\tilde{C}_\lambda(I_m - Y^{-1}). \qquad (4.7)$$

We may define the polynomials $\tilde{L}_\lambda^u(W)$ by

$$\tilde{L}_\lambda^u(W) = \mathrm{etr}(W)\int_{\tilde{S}_m^+} \mathrm{etr}(-R)|R|^{2u}\tilde{C}_\lambda(R)\tilde{A}_u(WR)(\mathrm{d}R), \qquad (4.8)$$

where the complex Bessel function $\tilde{A}_u$ is defined by

$$\tilde{A}_u(R) = {}_0\tilde{F}_1(2u+m; -R)/\tilde{\Gamma}_m(2u+m). \qquad (4.9)$$

*Rodrigues formulae*

The inverse Laplace transform (integral version of Rodrigues formulae) of $\tilde{L}_\lambda^u(W)$ is given by

$$\frac{\tilde{L}_\lambda^u(W)\tilde{w}_m(W; n, I_m)}{[n]_\lambda}$$
$$= \frac{2^{m(m-1)}}{(2\pi)^{m^2}}\int_{\tilde{S}_m} \mathrm{etr}(-\mathrm{i}WT)|I_m - \mathrm{i}T|^{-n}\tilde{C}_\lambda(I_m - (I_m - \mathrm{i}T)^{-1})(\mathrm{d}T),$$
$$\text{with } u = \frac{1}{2}(n-m), \qquad (4.10)$$

noting that $|I_m - \mathrm{i}T|^{-n}$ is the characteristic function of the Wishart $\tilde{W}_m(n, I_m)$ distribution.

From (4.10), the property (1.17), and the uniqueness of the Laplace transform, we obtain the *differential version* of Rodrigues formulae

$$g_\lambda(\partial W)\tilde{w}_m(W; 2u+2l+m, I_m) = \tilde{L}_\lambda^u(W)\tilde{w}_m(W; 2u+m, I_m), \qquad (4.11)$$

where we have the differential operator

$$g_\lambda(\partial W) = [2u+m]_\lambda |I + \partial W|^{2l}\tilde{C}_\lambda(I_m - (I_m + \partial W)^{-1}), \qquad (4.12)$$

and $\partial W$ is defined by (1.15).

*Limit normal property*

The p.d.f. (2.1) of the $\tilde{N}_{mm}(0, I_m)$ distribution and the g.f. (2.3) for the $\tilde{H}_\lambda^{(m)}(S)$ are obtained by letting

$$W \to u^{1/2}S + uI_m \quad \text{and} \quad T \to -u^{-1/2}T, \quad \text{and then } u \to \infty,$$

in the p.d.f. (see (4.3)) of the $\tilde{W}_m(2u + m, \frac{1}{2}I_m)$ distribution and the g.f. (4.6) for the $\tilde{L}_\lambda^u(W)$, respectively. So we have

$$\tilde{H}_\lambda^{(m)}(S) = \lim_{u\to\infty} u^{-l/2}\tilde{L}_\lambda^u(u^{1/2}S + uI_m). \tag{4.13}$$

*Series expression for $\tilde{L}_\lambda^u(W)$*

As for the case of the $\tilde{H}_\lambda^{(m)}(S)$ polynomials, we can obtain the series expression for $\tilde{L}_\lambda^u(W)$ based on the g.f. (4.6), and, in view of (3.7), we can establish the relationship

$$\tilde{H}_\lambda^{(mk)}(Z) = (-1)^l\tilde{L}_\lambda^u(Z^*Z), \quad \text{with } u = \frac{1}{2}[\max(m, k) - \min(m, k)]. \tag{4.14}$$

It is seen from (1.8) in conjunction with the property (1.10) that, if we write the series expression for $\tilde{L}_\lambda^u(W)$ as $\tilde{L}_\lambda^u(W) = \tilde{\Sigma}_\lambda^u[\tilde{C}_\sigma(W)]$, then we can express $\tilde{C}_\lambda(W)$ as $\tilde{\Sigma}_\lambda^u[\tilde{L}_\sigma^u(W)]$.

## 5. Multivariate complex Meixner classes of invariant distributions

We define the multivariate invariant (biinvariant) Meixner distributions of random complex matrices, extending the real ones discussed by Chikuse [2]; the Meixner distributions were first characterized by Meixner [16]. The distribution of an $m \times m$ Hermitian ($m \times k$ complex) random matrix $S$ is defined to be invariant (biinvariant) if the distribution of $S$ is invariant under the transformation $S \to HSH^*$, for $H \in \tilde{O}(m)$ ($S \to HSQ^*$, for $H \in \tilde{O}(m)$ and $Q \in \tilde{O}(k)$). Here we confine our discussion to continuous random complex matrices; a similar argument can be applied to the case of discrete random complex matrices, which is omitted in this paper though.

A random matrix $S \in \tilde{S}_m$ is said to belong to the class of invariant complex Meixner distributions defined by $U(T)$ if the g.f. for the associated unitary polynomials $\{\tilde{P}_\lambda^{(m)}(S)\}$ is of the from

$$\sum_{l=0}^{\infty}\sum_{\lambda\vdash l}\frac{\tilde{P}_\lambda^{(m)}(S)\tilde{C}_\lambda(T)}{l!C_\lambda(I_m)} = \int_{\tilde{O}(m)}\frac{\operatorname{etr}[HSH^*U(T)]}{\tilde{M}_S(U(T))}[\mathrm{d}H], \tag{5.1}$$

for $T \in \tilde{S}_m$ and the m.g.f. $\tilde{M}_S(\cdot)$ of $S$. Here $U(T)$ is an analytic $m \times m$ Hermitian matrix-valued function of $T$ with $U(0) = 0$.

A random matrix $Z \in \tilde{R}_{m,k}$ is said to belong to the class of biinvariant complex Meixner distributions defined by $U(T)$ if the g.f. for the associated unitary polynomials $\{\tilde{P}_{\lambda}^{(mk)}(Z)\}$ is of the form

$$\sum_{l=0}^{\infty}\sum_{\lambda\vdash l}\frac{\tilde{P}_{\lambda}^{(mk)}(Z)\tilde{C}_{\lambda}(T^*T)}{l!\tilde{C}_{\lambda}(I_m)}=\int_{\tilde{O}(m)}\int_{\tilde{O}(k)}\frac{\operatorname{etr}[\operatorname{Re}(HZQ^*U^*(T))]}{\tilde{M}_Z(U(T))}[\mathrm{d}Q][\mathrm{d}H], \tag{5.2}$$

for $T \in \tilde{R}_{m,k}$ and the m.g.f. $\tilde{M}_Z(\cdot)$ of $Z$. Here $U(T)$ is an analytic $m \times k$ complex matrix-valued function of $T$ with $U(0) = 0$.

Thus, we see from (2.3), (3.3), and (4.6) that our distributions $\tilde{N}_{mm}(0, I_m)$, $\tilde{N}_{m,k}(0; I_m, I_k)$, and $\tilde{W}_m(n, \frac{1}{2}I_m)$ are the invariant (biinvariant) complex Meixner distributions with the associated unitary polynomials $\tilde{H}_{\lambda}^{(m)}(S)$, $\tilde{H}_{\lambda}^{(mk)}(Z)$ (or $\tilde{H}_{\lambda}^{(mk)}(Z)/4^l[k]_{\lambda}$), and $\tilde{L}_{\lambda}^{u}(W)$ $(n = 2u + m)$, defined by $U(T) = T \in \tilde{S}_m$, $T \in \tilde{R}_{m,k}$, and $-T(I_m - \frac{1}{2}T)^{-1}$ for $T \in \tilde{S}_m$, respectively.

## 6. Applications

We define the *complex* Stiefel manifold $\tilde{V}_{k,m}$ $(m \geqslant k)$ as being represented by the space of all $m \times k$ complex matrices $Z$ satisfying $Z^*Z = I_k$, that is, $X'X + Y'Y = I_k$ and $X'Y - Y'X = 0$ for $Z = X + \mathrm{i}Y$. It is known that the most commonly used and tractable distribution defined on the (real) Stiefel manifold $V_{k,m} = \{X(m \times k)$ real; $X'X = I_m\}$ is the matrix Langevin distribution; see e.g., [3,7,12,14,15], for discussions and statistical analyses of the matrix Langevin distributions on $V_{k,m}$. We introduce the complex matrix Langevin distribution on $\tilde{V}_{k,m}$ whose p.d.f. with respect to the normalized invariant measure is given by

$$\operatorname{etr}[\operatorname{Re}(F^*Z)]/{}_0\tilde{F}_1\left(m; \frac{1}{4}F^*F\right), \tag{6.1}$$

where $F$ is an $m \times k$ complex matrix. The distribution (6.1) has properties similar to the (real) matrix Langevin distribution. Putting $F = 0$ gives the uniform distribution on $\tilde{V}_{k,m}$.

We are now interested in testing the null hypothesis $H_0$ of uniformity, against a sequence of local alternative hypotheses $H_1 : F = n^{-1/2}F_0$ for an $m \times k$ constant complex matrix $F_0$, indicating a slight departure from the uniformity as $n$ becomes large.

Given a random sample $Z_1, \ldots, Z_n$ of size $n$ from the complex matrix Langevin distribution (6.1), we define the standardized sample mean matrix as

$$U = (nm)^{1/2}\bar{Z}, \quad \text{with } \bar{Z} = \sum_{i=1}^{n} Z_i/n.$$

The statistic $U$ may play important roles in the above mentioned test of uniformity, but the exact distribution of $U$ is only given in an integral form which is difficult to evaluate. We will derive asymptotic expansions for the distributions of $U$ and $W = U^*U$, under the hypotheses $H_0$ and $H_1$, for large $n$.

The inversion of the characteristic function $E\,\mathrm{etr}[\mathrm{i}\,\mathrm{Re}(T^*U)]$ of $U$ leads to the p.d.f. of $U$ under the hypothesis $H_1$

$$\begin{aligned} f_U(U) = {} & \frac{\mathrm{etr}[m^{-1/2}\,\mathrm{Re}(F_0^*U)]}{\left[{}_0\tilde{F}_1\left(m; \frac{1}{4n}F_0^*F_0\right)\right]^n} \int_{\tilde{R}_{m,k}} \frac{1}{(2\pi)^{2km}}\, \mathrm{etr}[-\mathrm{i}\,\mathrm{Re}(U^*Z)] \\ & \times \left[{}_0\tilde{F}_1\left(m; -\frac{m}{4n}Z^*Z\right)\right]^n (\mathrm{d}Z). \end{aligned} \tag{6.2}$$

We expand the ${}_0\tilde{F}_1$ functions in (6.2) in view of (1.11) and (1.6), for large $n$. Then the integral with respect to $(\mathrm{d}Z)$ in (6.2) becomes

$$\begin{aligned} & \frac{1}{(2\pi)^{2km}} \int_{\tilde{R}_{m,k}} \mathrm{etr}\left[-\mathrm{i}\,\mathrm{Re}(U^*Z) - \frac{1}{4}Z^*Z\right] \\ & \quad \times \left[1 + \frac{1}{n}\sum_{\lambda\vdash 2} c_\lambda \tilde{C}\left(-\frac{1}{4}Z^*Z\right) + \mathrm{O}(n^{-2})\right] (\mathrm{d}Z), \\ & \text{with } c_\lambda = \frac{1}{2}\left(\frac{m^2}{[m]_\lambda} - g^\lambda_{(1),(1)}\right), \end{aligned} \tag{6.3}$$

and using the Rodrigues formula (3.6) in (6.3) yields an asymptotic expansion for the p.d.f. of $U$ in the form

$$\begin{aligned} f_U(U) = {} & \tilde{\varphi}^{(mk)}\left(U - \frac{F_0}{2m^{1/2}}\right) \\ & \times \left\{1 + \frac{1}{n}\sum_{\lambda\vdash 2} c_\lambda\left[\tilde{H}_\lambda^{(mk)}(U) - \tilde{C}_\lambda(\Omega)\right] + \mathrm{O}(n^{-2})\right\}, \\ & \text{with } \Omega = (4m)^{-1}F_0^*F_0, \end{aligned} \tag{6.4}$$

in terms of the normal $\tilde{N}_{m,k}((2m^{1/2})^{-1}F_0; I_m, I_k)$ distribution and the associated complex Hermite polynomials $\tilde{H}_\lambda^{(mk)}(U)$.

The p.d.f. of $W = U^*U$ is expanded, using Lemma 4.1 in (6.4), as

$$\begin{aligned} f_W(W) = {} & \tilde{w}_k(W; m, I_k; \Omega) \\ & \times \left\{1 + \frac{1}{n}\sum_{\lambda\vdash 2} c_\lambda\left[\tilde{L}_\lambda^{(m-k)/2}(W) - \tilde{C}_\lambda(\Omega)\right] + \mathrm{O}(n^{-2})\right\}, \end{aligned} \tag{6.5}$$

in terms of the noncentral Wishart $\tilde{W}_m(m, I_k; \Omega)$ distribution and the associated complex Laguerre polynomials $\tilde{L}_\lambda^{(m-k)/2}(W)$.

Putting $F_0 = 0$ in (6.4) and (6.6) gives the asymptotic expansions under the null hypothesis of uniformity.

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