Discrete Fourier Transform (DFT) Scaling

L. Poulo Apr. 7, 2006 Rev 0

Introduction

DFT's have some degrees of freedom in defining the scaling constants used in the transform definitions. Various analysis programs and software routines do not use a universal, commonly agreed upon choice of these constants. Thus, there is the need to understand which choices are used in any particular application, and how the results are to be interpreted and translated to physical parameters. This note is intended to address this issue. A tutorial derivation of the relations is given below. A summary of the methodology to translate results into physical spectra is given at the end of the document.

Transform Definitions

The (1-dimensional) DFT is a discrete transform relating two vectors, often interpreted in engineering work as relating time-frequency or space-frequency vector pairs. A sampled time sequence vector xt_k of N points, with k = (0,1, 2,...N -1) and its transform vector Xf_m, with m also in the range m = (0,1, 2,...N -1) are related by the forward transform

$$Xf_m = \frac{C_1}{N} \cdot \sum_{k=0}^{N-1} \left(xt_k \cdot e^{-j \cdot \frac{2 \cdot \pi \cdot k \cdot m}{N}} \right) \qquad (1)$$

and the inverse transform

$$xt_k = \frac{1}{C_1} \cdot \sum_{m=0}^{N-1} \left(Xf_m \cdot e^{j \cdot \frac{2 \cdot \pi \cdot k \cdot m}{N}} \right) \qquad (2)$$

The constants C_1 is arbitrary, and may be picked to satisfy other desired properties of the transform pair. The signs of the exponential terms can be, and sometimes are, reversed in the forward and inverse transform definitions. This has the effect of changing the phase properties of the transform, but not its magnitude characteristcs. The common choices for C_1 are 1, N, and $\sqrt{N}$. The latter choice gives mathermatical symmetry, but is generally not useful in engineering.

Parseval Relation

The DFT vector pair xt and Xf satisfy the Parseval energy/power relation

$$\frac{{C_1}^2}{N} \cdot \sum_k \left(\left| xt_k \right| \right)^2 = \sum_m \left(\left| Xf_m \right| \right)^2 \qquad (3)$$

Choice of Constant

Picking $C_1 = 1$ is a convenient engineering choice which give transforms with a number of desirable and familiar properties. The resulting forward and inverse transforms are:

Forward transform

$$Xf_m = \frac{1}{N}\cdot\sum_{k=0}^{N-1}\left(xt_k\cdot e^{-j\cdot\frac{2\cdot\pi\cdot k\cdot m}{N}}\right) \qquad (4)$$

and the inverse transform

$$xt_k = \sum_{m=0}^{N-1}\left(Xf_m\cdot e^{j\cdot\frac{2\cdot\pi\cdot k\cdot m}{N}}\right) \qquad (5)$$

Equation 5 gives the Fourier series expansion of the sampled sequence xt_k in terms of the Fourier coefficients Xf_m. The Xf_m need no further scaling, and represent the complex amplitude of the frequency bin m. The term Xf_0 gives the DC mean value of the time sequence, since it is computed from (4) as

$$Xf_0 = \frac{1}{N}\cdot\sum_{k=0}^{N-1} xt_k = \text{mean}(xt) \qquad (6)$$

The Parseval relation (3) becomes

$$Pavg = \frac{1}{N}\cdot\sum_{k}\left(\left|xt_k\right|\right)^2 = \sum_{m}\left(\left|Xf_m\right|\right)^2 \qquad (7)$$

In signal terms, $|p|^2$ is the power in a signal p. With this usual convention, the left side of this equation gives the time average power Pavg of the samples xt_k. The right side gives the sum of powers at the frequency components Xf_m of the signal xt_k. In words, this equation is

total average signal power = sum of powers at each frequency component (7a)

This choice of constant is implemented in Mathcad 11(& other) versions as the CFFT/ICFFT transform pair.

If $C_1 = N$, then the resulting transforms are:

forward transform

$$Xf_m = \sum_{k=0}^{N-1} \left(xt_k \cdot e^{-j \cdot \frac{2 \cdot \pi \cdot k \cdot m}{N}} \right) \tag{8}$$

and the inverse transform

$$xt_k = \frac{1}{N} \cdot \sum_{m=0}^{N-1} \left(Xf_m \cdot e^{j \cdot \frac{2 \cdot \pi \cdot k \cdot m}{N}} \right) \tag{9}$$

The corresponding Parseval relation is

$$N \cdot \sum_k \left(\left| xt_k \right| \right)^2 = \sum_m \left(\left| Xf_m \right| \right)^2 \tag{10}$$

This transform set is the one implemented in Matlab. Note that

$$Xf_0 = \sum_{k=0}^{N-1} xt_k = N \cdot \text{mean}(xt)$$

$$\text{mean}(xt) = \frac{1}{N} \cdot Xf_0 \tag{11}$$

showing the frequency components from (8) are scaled by a factor of N from the usual engineering interpretation. The Parseval relation (10), written to give the time average power, is

$$Pavg = \frac{1}{N} \cdot \sum_k \left(\left| xt_k \right| \right)^2 = \sum_m \left(\frac{\left| Xf_m \right|}{N} \right)^2 \tag{12}$$

Scaling and Physical Units

Typically, sampled data files don't have inherent units, they are just vector of numbers. The user must infer or add the appropriate units. In the following discussion and example, we will assume that the data file samples xt_k are the output of a sampled data system, with amplitude units in counts, with sampling rate at frequency fs, and N samples taken for the data record. A sample file will be analyzed as an example. Clearly, the signal could have any other units, e.g., uvolts, milliamps, pressure, etc.

The index has period N, corresponding to frequency fs. With Ts = 1/fs = sampling period, the fundamental frequency resolution is fs/N, corresponding to the full time window of the data sequence = N*Ts. The frequency f_m of bin m is given by $f_m = \frac{m}{N} \cdot fs$. The bin spacing corresponds to a frequency increment $df = \frac{fs}{N}$.

Single-sided vs.Double-sided Spectra

The complex exponential form DFT transforms defined above have components at both positive and negative frequencies. The DFT output vector Xf_m represents a single period of a periodic spectrum, corresponding to a spectral width of frequency fs. For continuous transforms and spectra, the normal frequency range of interest is from -fs/2 to +fs/2. With the indexing used in most software, DFT's use the range 0 to fs, with the upper part of the spectrum from fs/2 to fs being a periodic extension of the spectrum from -fs/2 to 0.

Regardless of the period starting point, the complex exponential DFT is inherently double-sided. For real time signals, the DFT spectrum Xf_m has conjugate symmetry about 0, or by period extension, about index N/2.

Assume the set of N indices for m are in the range 0, 1, 2, ... N-1. Xf_N (= Xf_0) is the beginning of the next spectral period. The symmetry gives the relations

$$Xf_1 = \overline{Xf_{N-1}}$$
$$Xf_2 = \overline{Xf_{N-2}} \qquad (13)$$
$$\text{etc}$$

If N is odd, then there is no component at m = N/2, and the last relation in this chain is

$$Xf_{\frac{N-1}{2}} = \overline{Xf_{\frac{N+1}{2}}} \qquad (14)$$

If N is even, then $Xf_{\frac{N}{2}}$ is a unique element. Xf_0, the DC component, is always a unique term, and has no conjugate counterpart in the spectrum.

Physical noise sources are often defined using single-sided real spectra, with each corresponding positive and negative complex frequency components combined into a single real frequency component. The power in a DFT component is its squared magnitude. If we denote the single sided PSD component for index m by $S1_m$, then we have

$$S1_0 = \left(\left|Xf_0\right|\right)^2 \qquad (15a)$$

$$S1_1 = \left(\left|Xf_1\right|\right)^2 + \left(\left|Xf_{N-1}\right|\right)^2 = 2\cdot\left(\left|Xf_1\right|\right)^2$$

$$\vdots \qquad (15b)$$

$$S1_m = \left(\left|Xf_m\right|\right)^2 + \left(\left|Xf_{N-m}\right|\right)^2 = 2\cdot\left(\left|Xf_m\right|\right)^2 \qquad 1 \le m < \frac{N}{2}$$

$$S1_{\frac{N}{2}} = \left(\left|Xf_{\frac{N}{2}}\right|\right)^2 \qquad m = \frac{N}{2} \text{ if N even} \qquad (15c)$$

Again, we note that the DC term and, for N even the Nyquist frequency term, are not doubled, as they have no conjugate counterparts in the spectrum.

If the transform scaling is selected according to (4) and (5) above with $C_1 = 1$, then the Parseval power relation (7) takes on the single-ended form

$$Pavg = \frac{1}{N}\cdot\sum_k \left(\left|xt_k\right|\right)^2 = \sum_{m=0}^{m \le \frac{N}{2}} S1_m \qquad (16)$$

Deterministic Signals with Discrete Frequency Components vs. Noise

When the underlying signal time signal, in counts, has power at discrete frequencies, then the DFT coefficients also have units of counts. For random signals with a continuous power spectral density (PSD), there is no finite power at any singe discrete frequency. The power is determined by integrating the PSD over the appropriate frequency band. In this case, for our sample noise signal, the PSD has physical units of $counts^2/Hz$.

When using a DFT on a sampled signal, there is no inherent calibration of the frequency axis, only an index or bin number. The DFT power components of a noise signal $\left|Xf_m\right|^2$ thus have the units $counts^2/bin$ in order that the total time average signal power be independent of the number of sample points N, as it must be. The unit "bin" is similar to radian, in that it defines a measure, but is inherently dimensionless.

For noise signals, the DFT power component $S1_m$ corresponds to the area under the PSD over the mth bin. Physically, each bin correspoonds to a frequency band $df = \frac{fs}{N}$. We can find the physical PSD, in units of counts2/Hz, since the total power defined over the physcial frequency band 0-fs is the same as the total power defined over the index range (0, N-1). With the physical PSD denoted by Sp, we then get

$$Pavg = \sum_{m=0}^{m \le \frac{N}{2}} S1_m = \sum_{m=0}^{m \le \frac{N}{2}} \left(Sp_m \cdot df \right)$$

where $Sp_m \cdot df$ is the power in the frequency interval df centered at $f_m = \frac{m}{N} \cdot fs$. Since the scaling does not affect the spectral shape, and using $df = \frac{fs}{N}$, we get the single-sided physical PSD as

$$Sp_m \cdot \left(\frac{counts^2}{Hz} \right) = \frac{N}{fs} \cdot S1_m \cdot \left(\frac{counts^2}{bin} \right) \qquad (17)$$

This formula gives the relation between the DFT of the sampled data file and the physical noise PSD Sp, which can be related to real devices. Various gain factors can be used to translate the results back to some other part of a design. If the gain from a source y(t), with samples y_k, to the captured output samples xt_k, is given by H(f), then $Sx_m = \left(\left| H\left(f_m\right) \right| \right)^2 \cdot Sy_m$, and Sy_m can be determined. The units of Sy will be determined by the units of H(f).

Note: this assumes that the DFT components $S1_m$ calculated from (15) above use the DFT with $C_1 = 1$. per (4) and (5) above. Otherwise the appropriate scalings need to be made.

A Note on Windowing

At times, it may be necessary to window the time data record to reduce edge effects of the limited time sample. This introduces some additional scaling requirements which are different for PSD's and for discrete components. The analysis presented here does not cover this more general case.

Summary - Interpreting DFT's of Acquired Noise Signals and Converting into Corresponding Physical PSD's

1. Data inputs:

data vector x, with known amplitude units
fs - physical sampling frequency
N - number of points in data record

2. Calculate X, the DFT of the input data record x. If the transform scaling is that given by (4) and (5) above, then no adjustment of the DFT is necessary. For example, the mean DC of the input is given directly by X_0.

If using the scaling of (8) and (9), as in Matlab, then the resulting DFT output should be divided by N before proceeding, to give a DFT vector with the Fourier series reconstruction defined in (5), as is typical in EE use.

3. Compute S1, the single-sided PSD in $\left(\frac{unit^2}{bin}\right)$, from the relations in (15). S1 is in $\frac{units^2}{bin}$.

$$S1_0 = \left(\left|Xf_0\right|\right)^2$$

$$S1_1 = 2\cdot\left(\left|Xf_1\right|\right)^2$$

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$$S1_m = 2\cdot\left(\left|Xf_m\right|\right)^2 \qquad 1 \le m < \frac{N}{2}$$

$$S1_{\frac{N}{2}} = \left(\left|Xf_{\frac{N}{2}}\right|\right)^2 \qquad \text{if N even}$$

4. Convert S1 to physical PSD from

$$Sp_m\cdot\left(\frac{units^2}{Hz}\right) = \frac{N}{fs}\cdot S1_m\cdot\left(\frac{units^2}{bin}\right)$$

Sp is the single-sided, physical PSD corresponding to the acquired data vector x, over the frequency band 0 to fs/2. The square root of the PSD may also be used, to give a signal domain result in $\frac{units}{\sqrt{Hz}}$.