INSTITUTO TECNOLÓGICO DE AERONÁUTICA

MP-288

OPTIMIZATION IN MECHANICAL ENGINEERING

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OPTIMALITY NECESSARY CONDITIONS

OPTIMALITY NECESSARY CONDITIONS

Reading material (see References at the last slide):

-Chapter 4 of ARORA (2012);

-Chapter 5 of ARORA (2012);

-Chapter 5 of HAFTKA and GÜRDAL (1992).

OPTIMALITY NECESSARY CONDITIONS

ÅPROBLEMS WITHOUT CONSTRAINTS

(FUNCTION MINIMIZATION)

ÅPROBLEMS WITH EQUALITY CONSTRAINTS

ÅPROBLEMS WITH INEQUALITY CONSTRAINTS

OPTIMALITY NECESSARY CONDITIONS

CAN BE USED TO CHECK IF A GIVEN POINT IS A LOCAL OPTIMUM FOR THE PROBLEM.

CAN BE SOLVED FOR OPTIMUM POINTS.

OPTIMALITY NECESSARY CONDITIONS

NECESSARY CONDITIONS: MUST BE RESPECTED BY AN OPTIMUM POINT.

IF NOT RESPECTED, THE POINT IS NOT AN OPTIMUM.

HOWEVER, MAY BE RESPECTED BY NON-OPTIMUM POINTS.

SUFFICIENT CONDITIONS: GUARANTEES THAT A POINT IS AN OPTIMUM.

POINTS THAT RESPECT THE SUFFICIENT CONDITIONS ALSO RESPECT THE NECESSARY.

A POINT THAT RESPECT BOTH IS INDEED AN OPTIMUM.

PROBLEMS WITHOUT CONSTRAINTS

ONE VARIABLE FUNCTION MINIMIZATION

Considering the following problem:

$$\min_{x} \; f(x)$$

Methodology: it will be assumed that *x** is a local minimum point. Its neighborhood will be investigated.

Defining: $$x - x^{*} = d$$ (*d* is a small increment)

$$\Delta f = f(x) - f(x^{*})$$

PROBLEMS WITHOUT CONSTRAINTS

ONE VARIABLE FUNCTION MINIMIZATION

x* **is a local minimum point of** ***f(x)*** **only if**

$$\Delta f = f(x) - f(x^*) \geq 0 \quad \forall d$$
$$d = x - x^*$$

Now, approximating *f(x)* around *x** using Taylorôs expansion:

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2} f''(x^*)(x - x^*)^2 + \ldots$$

or

$$f(x) \approx f(x^*) + f'(x^*)d + \frac{1}{2} f''(x^*)d^2 + \ldots$$

PROBLEMS WITHOUT CONSTRAINTS

ONE VARIABLE FUNCTION MINIMIZATION

Using the series developed, for *f(x*)* to be a minimum:

$$\Delta f = f(x) - f(x^*) \approx f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + \ldots \geq 0$$

If *d* is small the first term of the expansion dominates. For *x** to be a minimum point:

$$\Delta f \approx f'(x^*)d \geq 0$$

The increment *d* may have any sign (+/-). Due to this the inequality above only have chance to hold when:

$$f'(x^*) = 0 \quad \forall d \neq 0$$

This condition also permits to qualify *f(x*)* like a maximum point by analizying ***æfÒ****0*.

PROBLEMS WITHOUT CONSTRAINTS

ONE VARIABLE FUNCTION MINIMIZATION

Therefore:

$$f'(x^*) = 0$$

Is a **necessary condition** for *f(x*)* to be a minimum!

This is a first order necessary condition, since it is related to the first derivative of *f(x)*.

It is needed now a sufficient condition to qualify *f(x*)* like a minimum.

PROBLEMS WITHOUT CONSTRAINTS

ONE VARIABLE FUNCTION MINIMIZATION

Now, with $f'(x^*)=0$, the Taylor series for Δf is dominated by the term with the second derivative:

$$\Delta f = f(x) - f(x^*) \approx \cancelto{0}{f'(x^*)}d + \frac{1}{2}f''(x^*)d^2 + \ldots \geq 0$$

Then, to a minimum, it is needed:

$$\Delta f \approx f''(x^*)d^2 \geq 0$$

Due to the power 2 in *d*, it can be observed that:

$$if \quad f''(x^*) > 0, \quad \Delta f > 0 \quad \forall d \neq 0$$

Which strictly qualifies *f(x*)* like a minimum.

PROBLEMS WITHOUT CONSTRAINTS

ONE VARIABLE FUNCTION MINIMIZATION

Therefore:

$$f''(x^*) > 0$$

Is a **sufficient condition** for *f(x*)* to be a minimum!

This is a second order sufficient condition, since it is related to the second derivative of *f(x)*.

PROBLEMS WITHOUT CONSTRAINTS

ONE VARIABLE FUNCTION MINIMIZATION

Further analyzing f(x) and its expansion leads to:

$f'(x^*)=0$ stationarity first order necessary condition

$f''(x^*)>0$ local minimum second order sufficient condition

$f''(x^*)<0$ local maximum second order sufficient condition

If $f''(x^*)=0$ it is needed to analyze other terms in the series to be sure about the nature of x*.

In general, for local minimum points, the lowest nonzero derivative in the series must be positive as sufficient condition.

For stationary points, the lowest nonzero derivative in the series must even ordered as necessary condition.

PROBLEMS WITHOUT CONSTRAINTS

MULTI VARIABLE FUNCTION MINIMIZATION

Considering now the following problem:

$$\min_{\mathbf{x}}: \ f(\mathbf{x})$$

$$\mathbf{x}^{T} = \{x_1, x_2, \ldots, x_n\}$$

Methodology: the same adopted before but *f(***x***)* depends now on several variables x_i.

Defining: $$\mathbf{x} - \mathbf{x}^{*} = \mathbf{d}$$ (**d** is a small increment)

$$\Delta f = f(\mathbf{x}) - f(\mathbf{x}^{*})$$

PROBLEMS WITHOUT CONSTRAINTS

MULTI VARIABLE FUNCTION MINIMIZATION

The Taylor series expansion in this case becomes:

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T \mathbf{d} + \frac{1}{2}\mathbf{d}^T \mathbf{H}(\mathbf{x}^*)\mathbf{d} + \dots$$

This considers the gradient and Hessian matrix of f at x* like follows:

$$\nabla f(\mathbf{x}^*) = \left\{ \begin{array}{c} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{array} \right\}_{\mathbf{x}^*} \qquad \mathbf{H}(\mathbf{x}^*) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial \partial x_n x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{\mathbf{x}^*}$$

PROBLEMS WITHOUT CONSTRAINTS

MULTI VARIABLE FUNCTION MINIMIZATION

By the same arguments used before, it is possible to show:

$$\nabla f(\mathbf{x}^*) = 0$$

Is a first order **necessary condition** for *f(***x****)* to be a minimum!

$\mathbf{H}(\mathbf{x}^*)$ is positive definite

Is a second order **sufficient condition** for *f(***x****)* to be a minimum!

By definition, **H** being positive definite guarantees:

$$\mathbf{d}^T\mathbf{H}(\mathbf{x}^*)\mathbf{d} > 0 \quad \forall d \neq 0$$

(Check: the eigenvalues of a positive definite matrix are all positive.)

PROBLEMS WITHOUT CONSTRAINTS

MULTI VARIABLE FUNCTION MINIMIZATION

Further analyzing $f(\mathbf{x})$ and its expansion leads to:

$\nabla f(\mathbf{x}^*) = 0$ stationarity first order necessary condition

$\mathbf{H}(\mathbf{x}^*)$ positive definite local minimum second order sufficient condition

$\mathbf{H}(\mathbf{x}^*)$ negative definite local maximum second order sufficient condition

If $\mathbf{H}(\mathbf{x}^*)$ is not positive or negative definite it is needed to analyze other terms in the series to be sure about the nature of $\mathbf{x}^*$.

PROBLEMS WITH CONSTRAINTS

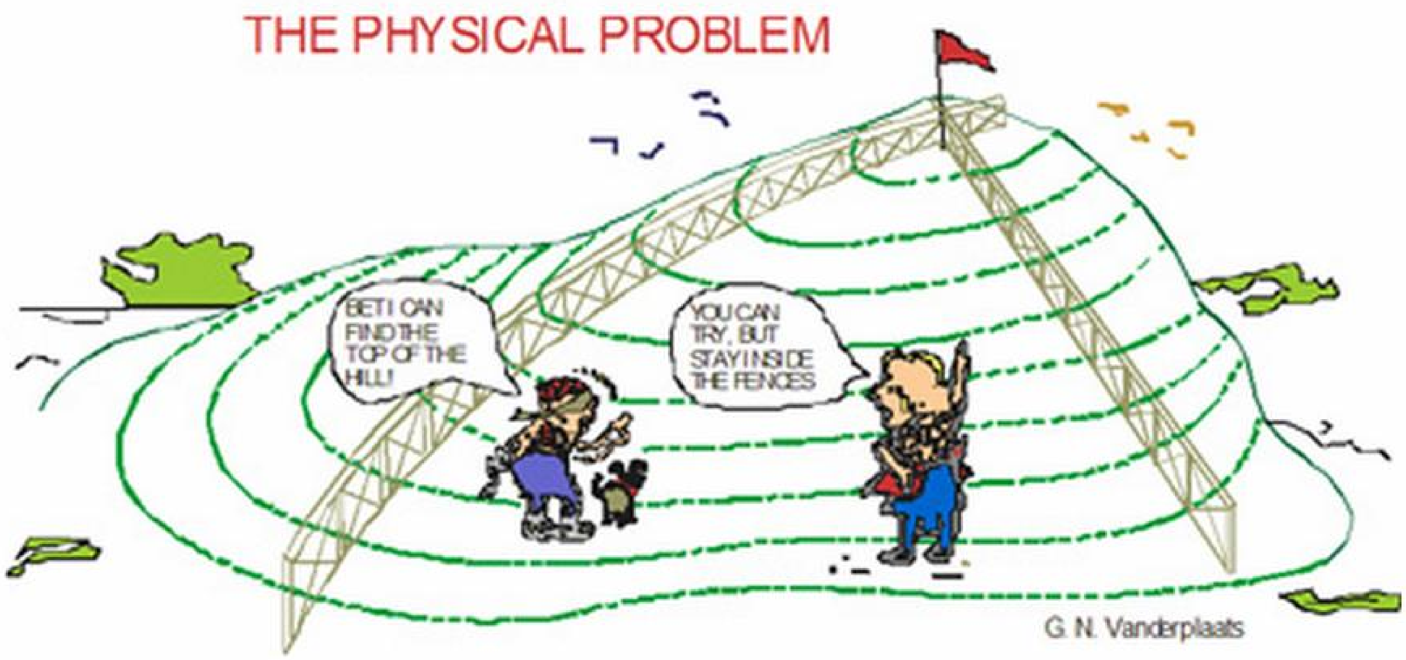


Figure 1. Locate the top of the hill while blindfolded.

(VANDERPLAATS, 2005)

PROBLEMS WITH CONSTRAINTS

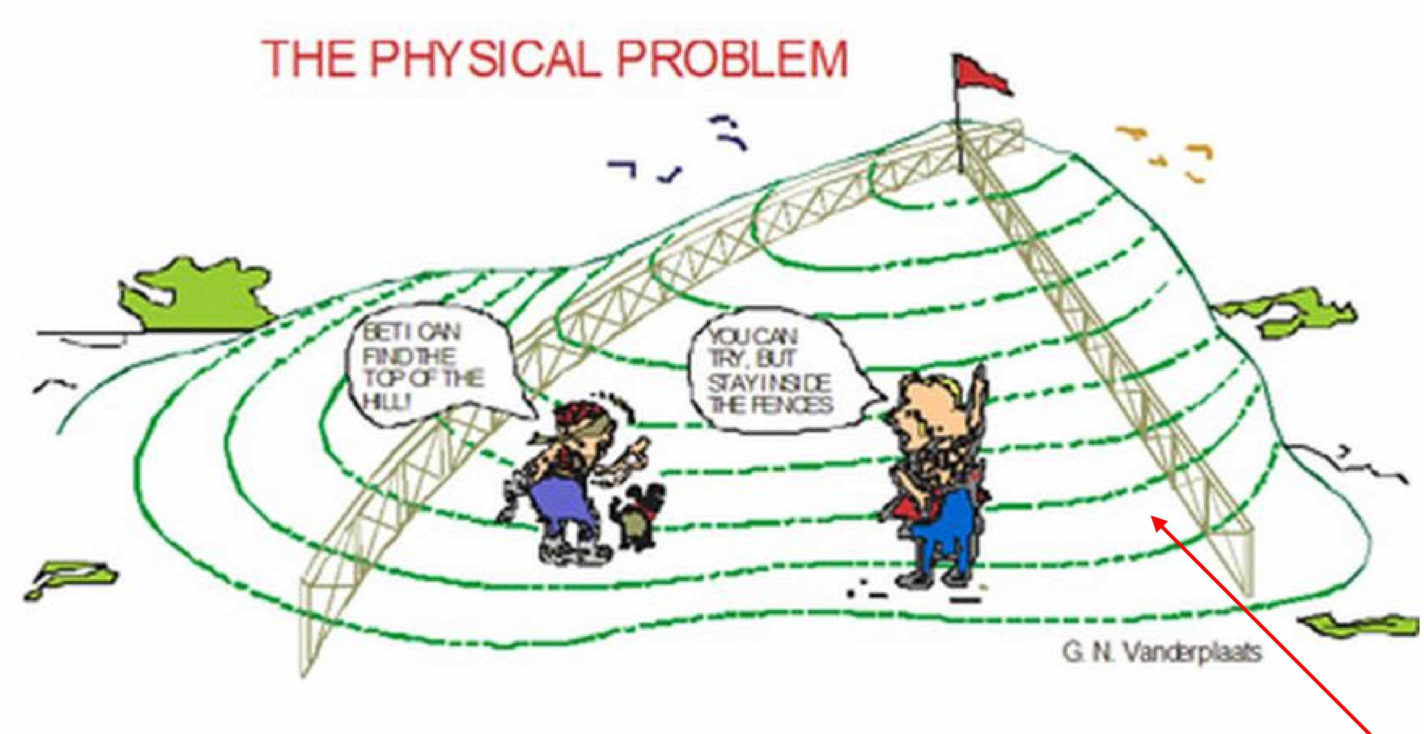


Figure 1. Locate the top of the hill while blindfolded.

(VANDERPLAATS, 2005)

FEASIBLE REGION (CONSTRAINTS RESPECTED)

PROBLEMS WITH EQUALITY CONSTRAINTS

Now, considering the following problem, with equality constraints:

$$\underset{\mathbf{x}}{\text{minimize:}} \; f(\mathbf{x})$$

subject to:

$$h_k(\mathbf{x}) = 0 \quad k = 1, \ldots, l$$

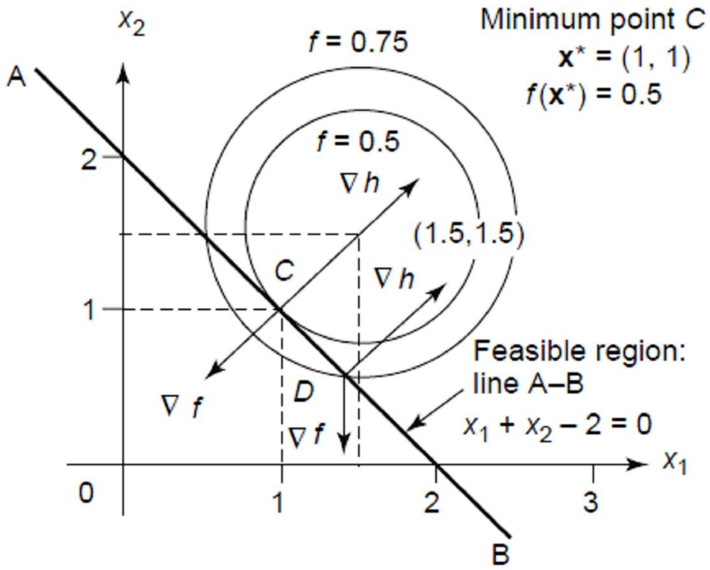
An equality constraint need always to be **active** (å0, very close to zero) in a feasible solution to the problem above.

If the design variables x_i are in number n, usually $n>l$.

The feasible region of the problem is reduced.

PROBLEMS WITH EQUALITY CONSTRAINTS

Example:



(ARORA, 2012)

minimize: $f(x_1,x_2)=(x_1-1.5)^2+(x_2-1.5)^2$
x_1,x_2

subject to: $x_1+x_2-2=0$

PROBLEMS WITH EQUALITY CONSTRAINTS

The Lagrange multiplier theorem permits to find optimum points for a problem of function minimization with equality constraints.

LAGRANGE MULTIPLIER THEOREM

minimize: $\underset{\mathbf{x}}{} f(\mathbf{x})$

subject to: $h_k(\mathbf{x}) = 0 \quad k = 1, \ldots, l$

A regular point ***x**** is a local minima for the problem above.

Then, there are unique Lagrange Multipliers v_j^* such that:

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^{p} v_j^* \frac{\partial h_j(\mathbf{x}^*)}{\partial x_i} = 0 \quad i = 1, \ldots, n$$

$$h_j(\mathbf{x}^*) = 0 \quad j = 1, \ldots, p$$

PROBLEMS WITH EQUALITY CONSTRAINTS

LAGRANGE MULTIPLIER THEOREM

minimize: $\underset{\mathbf{x}}{} f(\mathbf{x})$

subject to: $h_k(\mathbf{x}) = 0 \quad k = 1, \ldots, l$

A regular point ***x**** is a local minima for the problem above.

Then, there are unique Lagrange Multipliers v_j* such that:

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^{p} v_j^* \frac{\partial h_j(\mathbf{x}^*)}{\partial x_i} = 0 \quad i = 1, \ldots, n$$

$$h_j(\mathbf{x}^*) = 0 \quad j = 1, \ldots, p$$

Regular point x*:

- the constraints are satisfied;

- f(x) is differentiable;

- the constraintsô gradients are linearly independent.

PROBLEMS WITH EQUALITY CONSTRAINTS

LAGRANGE MULTIPLIER THEOREM

minimize: $\underset{\mathbf{x}}{} f(\mathbf{x})$

subject to: $h_k(\mathbf{x}) = 0 \quad k = 1, \ldots, l$

A regular point ***x**** is a local minima for the problem above.

Then, there are unique Lagrange Multipliers v_j^* such that:

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^{p} v_j^* \frac{\partial h_j(\mathbf{x}^*)}{\partial x_i} = 0 \quad i = 1, \ldots, n$$

$$h_j(\mathbf{x}^*) = 0 \quad j = 1, \ldots, p$$

Lagrange Multiplier:* *is a scalar factor associated with each problem constraint.

***Varies with the shape of** f(**x**) **and** h_j(**x**).*

It is unrestricted in sign for equality constraints.

PROBLEMS WITH EQUALITY CONSTRAINTS

LAGRANGE MULTIPLIER THEOREM

minimize: $\underset{\mathbf{x}}{} f(\mathbf{x})$

subject to: $h_k(\mathbf{x}) = 0 \quad k = 1, ..., l$

A regular point ***x**** is a local minima for the problem above.

Then, there are unique Lagrange Multipliers v_j* such that:

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^{p} v_j^* \frac{\partial h_j(\mathbf{x}^*)}{\partial x_i} = 0 \quad i = 1, ..., n$$

$$h_j(\mathbf{x}^*) = 0 \quad j = 1, ..., p$$

For the first condition, it is possible to write:

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_i} = -\sum_{j=1}^{p} v_j^* \frac{\partial h_j(\mathbf{x}^*)}{\partial x_i} \quad i = 1, ..., n$$

This shows that, at a candidate minimum point,...

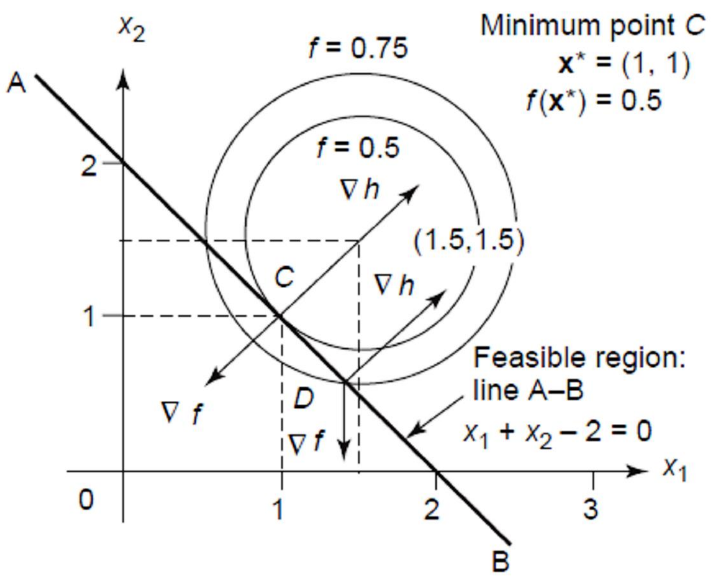
...the gradient of the objective function is a linear combination of the gradients of the constraints,...

... whose weights are the v_j*.

PROBLEMS WITH EQUALITY CONSTRAINTS

LAGRANGE MULTIPLIER THEOREM

Example:



(ARORA, 2012)

$$\underset{x_1,x_2}{\text{minimize:}}\ f(x_1,x_2)=(x_1-1.5)^2+(x_2-1.5)^2$$

subject to: $x_1+x_2-2=0$

At C, the $\mathbf{x}^*=(1,1)$:

$$\nabla f(\mathbf{x}^*)=\begin{Bmatrix}-1\\-1\end{Bmatrix}\quad \nabla h(\mathbf{x}^*)=\begin{Bmatrix}1\\1\end{Bmatrix}\quad v^*=-1$$

At D the condition is not observed.

PROBLEMS WITH EQUALITY CONSTRAINTS

LAGRANGE MULTIPLIER THEOREM

It is convenient to write the discussed conditions in respect to a Lagrangian function, defined by:

$$L(\mathbf{x},\mathbf{v}) = f(\mathbf{x}) + \sum_{j=1}^{p} v_j^* h(\mathbf{x}) = f(\mathbf{x}) + \mathbf{v}^T\mathbf{h}(\mathbf{x})$$

Therefore, the conditions become:

$$\nabla L(\mathbf{x}^*,\mathbf{v}^*) = \mathbf{0}$$

Because this zero gradient implies:

$$\frac{\partial L}{\partial x_i}(\mathbf{x}^*,\mathbf{v}^*) = 0 \implies \frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^{p} v_j^* \frac{\partial h_j(\mathbf{x}^*)}{\partial x_i} = 0 \quad i = 1,\ldots,n$$

$$\frac{\partial L}{\partial v_j}(\mathbf{x}^*,\mathbf{v}^*) = 0 \implies h_j(\mathbf{x}^*) = 0 \quad j = 1,\ldots,p$$

PROBLEMS WITH EQUALITY CONSTRAINTS

LAGRANGE MULTIPLIER THEOREM

These conditions show that the Lagrangian function $L(\mathbf{x},\mathbf{v})$ is stationary with respect to $\mathbf{x},\mathbf{v}$ at $\mathbf{x}^*,\mathbf{v}^*$.

Therefore, $L(\mathbf{x},\mathbf{v})$ can be treated as an unconstrained function in $\mathbf{x},\mathbf{v}$ in order to find stationary points.

Nevertheless, to qualify such points as minimum or maximum, second order conditions are needed.

PROBLEMS WITH INEQUALITY CONSTRAINTS

(STANDARD OPTIMIZATION PROBLEM)

Now, considering the problem with equality and inequality constraints:

$$\underset{\mathbf{x}}{\text{minimize:}}\ f(\mathbf{x})$$

subject to:

$$g_j(\mathbf{x}) \leq 0 \quad j = 1, ..., m$$

$$h_k(\mathbf{x}) = 0 \quad k = 1, ..., l$$

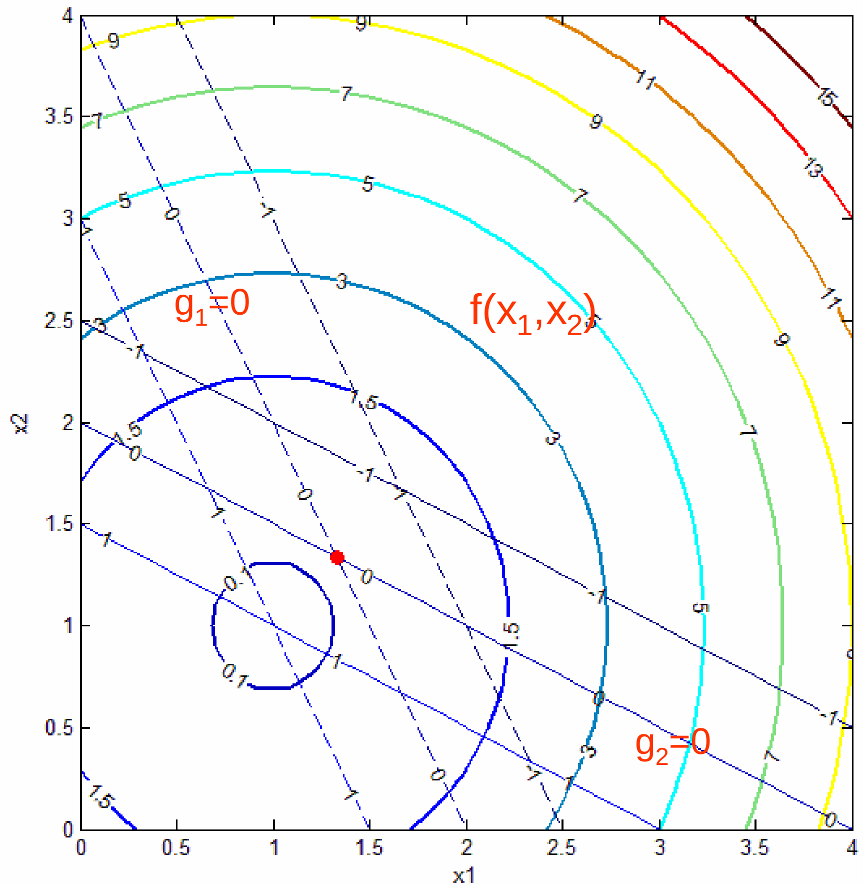
$$\mathbf{x} = \{x_1, x_2, ..., x_n\}$$

This is the standard optimization problem (the most general).

Inequality constraints increase the feasible region of a problem (where constraints are respected) in a direct comparison with equality constraints.

PROBLEMS WITH INEQUALITY CONSTRAINTS

Example:



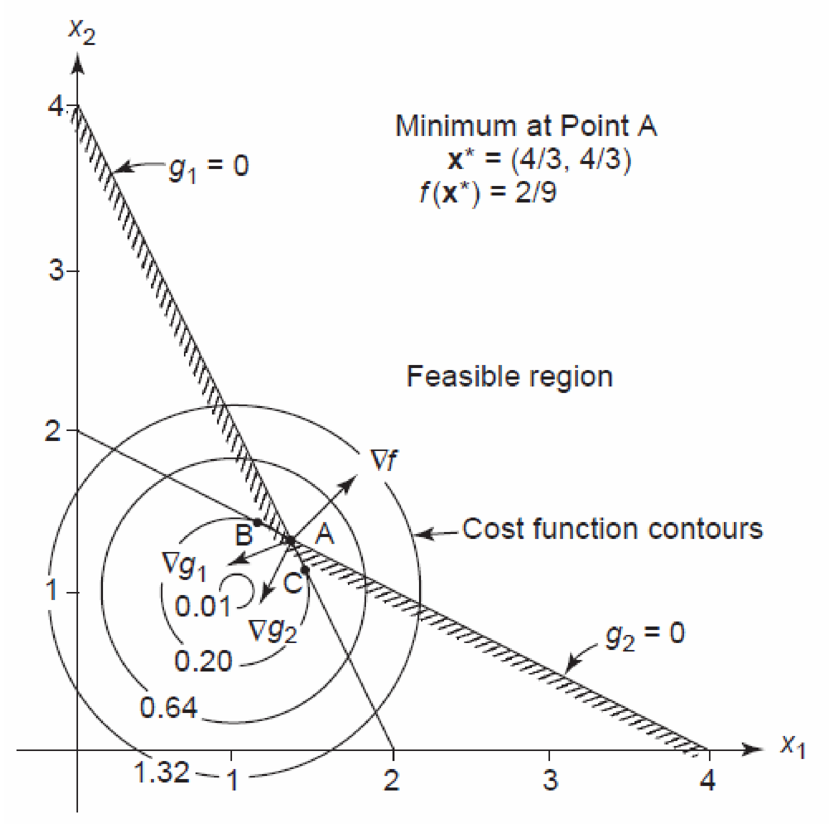
$$\underset{x_1,x_2}{\text{minimize:}} \quad f(x_1,x_2)= x_1^2+x_2^2-2x_1-2x_2+2$$

$$\text{subject to:} \quad g_1(x_1,x_2)= -2x_1-x_2+4 \leq 0$$

$$g_2(x_1,x_2)= -x_1-2x_2+4 \leq 0$$

PROBLEMS WITH INEQUALITY CONSTRAINTS

Example:



(ARORA, 2012)

minimize: $f(x_1,x_2)= x_1^2+x_2^2-2x_1-2x_2+2$
x_1,x_2

subject to: $g_1(x_1,x_2)= -2x_1-x_2+4$ Ò0

$g_2(x_1,x_2)= -x_1-2x_2+4$ Ò0

The feasible region is
delimited by g_1=0 and g_2=0.

PROBLEMS WITH INEQUALITY CONSTRAINTS

In terms of optimality conditions, the problem with inequality constraints $g_j(\mathbf{x})\leq 0$ is a bit more complicated.

To interpret the optimality conditions easier, these constraints are transformed in equality constraints by the use of slack variables s_j^2.

$$g_j(\mathbf{x}) \leq 0 \quad \longrightarrow \quad g_j(\mathbf{x}) + s_j^2 = 0 \mid s_j^2 \geq 0 \quad j = 1, ..., m$$

PROBLEMS WITH INEQUALITY CONSTRAINTS

$$g_j(\mathbf{x}) \leq 0 \quad \longrightarrow \quad g_j(\mathbf{x}) + s_j^2 = 0 \mid s_j^2 \geq 0 \quad j = 1, ..., m$$

The **slack variables** s_j^2 indicate the status of the constraints $g_j(\mathbf{x})$:

If $s_j^2=0$, $g_j(\mathbf{x})=0$ the constraint is active (on the feasibility bound)

If $s_j^2>0$, $g_j(\mathbf{x})<0$ the constraint is feasible with a slack

If $s_j^2<0$, $g_j(\mathbf{x})>0$ the constraint is not feasible (not acceptable!)

The s_j^2 are obtained from the optimality conditions considering $g_j(\mathbf{x})$.

PROBLEMS WITH INEQUALITY CONSTRAINTS

KARUSH-KUHN-TUCKER (KKT) NECESSARY CONDITIONS

A regular point ***x****** is a local minima for the standard optimization problem. The point ***x****** is feasible (respects $h_i(\mathbf{x}^*)=0$, $g_j(\mathbf{x}^*)$Ò0).

Then, there are unique Lagrange Multipliers v_i^*, u_j^* such that the Lagrangian function $L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s})$ is stationary at ***x******.

$$L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) = f(\mathbf{x})+\sum_{i=1}^{p} v_i^* h_i(\mathbf{x})+\sum_{j=1}^{m} u_j^* \left(g_j(\mathbf{x}) + s_j^2\right) = f(\mathbf{x})+\mathbf{v}^T\mathbf{h}(\mathbf{x})+\mathbf{u}^T\mathbf{g}(\mathbf{x})$$

$$\nabla L(\mathbf{x}^*,\mathbf{v}^*,\mathbf{u}^*,\mathbf{s}^*) = \mathbf{0}$$

PROBLEMS WITH INEQUALITY CONSTRAINTS

KARUSH-KUHN-TUCKER (KKT) NECESSARY CONDITIONS

Analizing the gradient of *L*:

$$L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i^* h_i(\mathbf{x}) + \sum_{j=1}^{m} u_j^* \left(g_j(\mathbf{x}) + s_j^2\right) = f(\mathbf{x}) + \mathbf{v}^T\mathbf{h}(\mathbf{x}) + \mathbf{u}^T\mathbf{g}(\mathbf{x})$$

$$\left.\frac{\partial L}{\partial x_k}\right|_{\mathbf{x}^*} = 0 \implies \frac{\partial f(\mathbf{x}^*)}{\partial x_k} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i(\mathbf{x}^*)}{\partial x_k} + \sum_{j=1}^{m} u_j^* \frac{\partial g_j(\mathbf{x}^*)}{\partial x_k} = 0 \quad k = 1, \dots, n$$

$$\left.\frac{\partial L}{\partial v_i}\right|_{\mathbf{x}^*} = 0 \implies h_i(\mathbf{x}^*) = 0 \quad i = 1, \dots, p$$

$$\left.\frac{\partial L}{\partial u_j}\right|_{\mathbf{x}^*} = 0 \implies g_j(\mathbf{x}^*) + s_j^2 = 0 \quad j = 1, \dots, m$$

$$s_j^2 \geq 0 \quad j = 1, \dots, m$$

$$u_j^* \geq 0 \quad j = 1, \dots, m$$

$$\left.\frac{\partial L}{\partial s_j}\right|_{\mathbf{x}^*} = 0 \implies 2u_j^* s_j = 0 \quad j = 1, \dots, m$$ (switching conditions)

PROBLEMS WITH INEQUALITY CONSTRAINTS

KARUSH-KUHN-TUCKER (KKT) NECESSARY CONDITIONS

Remarks:

The KKT necessary conditions give equations to be solved for candidate minimum points.

It is needed to check the feasibility of the points found through slack variables and/or constraints values.

$$s_j^2 \geq 0 \quad j = 1, ..., m$$

The Lagrange multipliers associated to the inequality constraints need to be checked: they are non-negative at a constrained minimum.

$$u_j^* \geq 0 \quad j = 1, ..., m$$

When a Lagrange multiplier is zero at a candidate point, its associated constraint is not active.

PROBLEMS WITH INEQUALITY CONSTRAINTS

KARUSH-KUHN-TUCKER (KKT) NECESSARY CONDITIONS

Remarks:

The first KKT condition has a physical interpretation. It is possible to write:

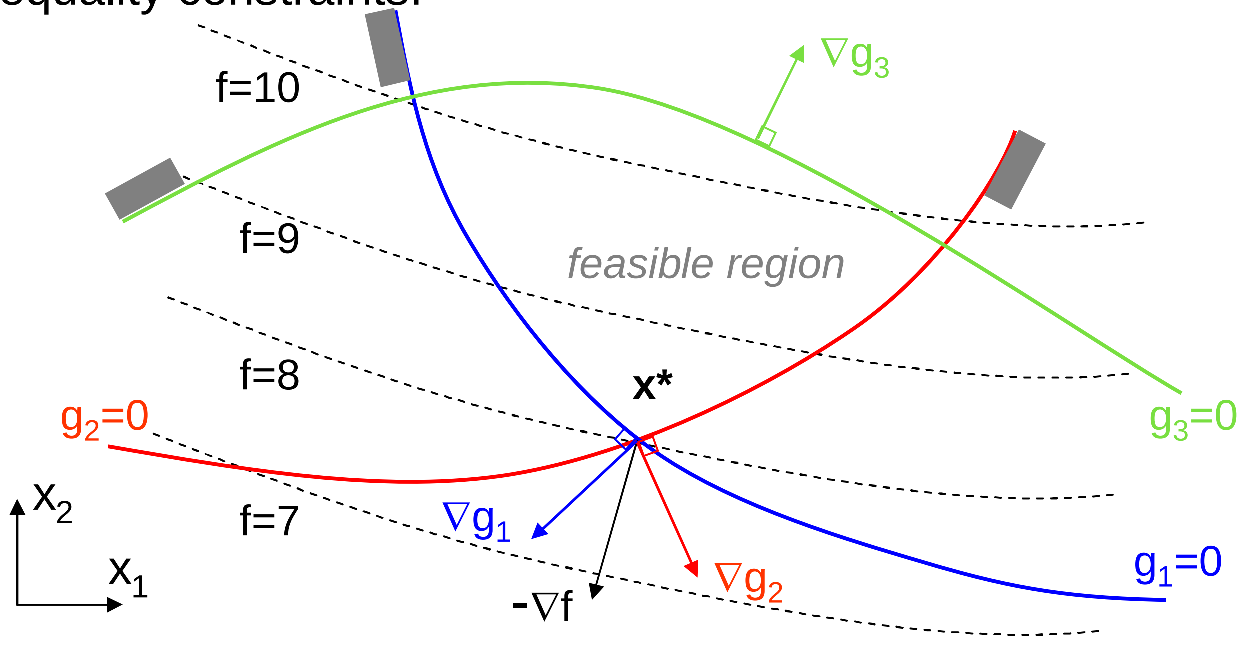
$$-\frac{\partial f(\mathbf{x}^*)}{\partial x_k} = \sum_{i=1}^{p} v_i^* \frac{\partial h_i(\mathbf{x}^*)}{\partial x_k} + \sum_{j=1}^{m} u_j^* \frac{\partial g_j(\mathbf{x}^*)}{\partial x_k} \quad k = 1, \ldots, n$$

This shows that, at the stationary point ***x****, the negative of the gradient (steepest descent direction) of the objective function *f(**x**)* is a linear combination of the gradients of the active constraints.

The Lagrange multipliers are the scalars of this combination.

PROBLEMS WITH INEQUALITY CONSTRAINTS

1st KKT condition graphical interpretation in a minimum point of a problem with inequality constraints:



$$\min_{\mathbf{x}} \; f(x_1,x_2)$$

s.t.: $g_1(x_1,x_2) \le 0$

$g_2(x_1,x_2) \le 0$

$g_3(x_1,x_2) \le 0$

$$-\nabla f = v^*_1 \nabla g_1 + v^*_2 \nabla g_2$$

At $\boldsymbol{x}^*$, there is no “usable direction” (decreases $f(\mathbf{x})$) which is “feasible” (respect the constraints).

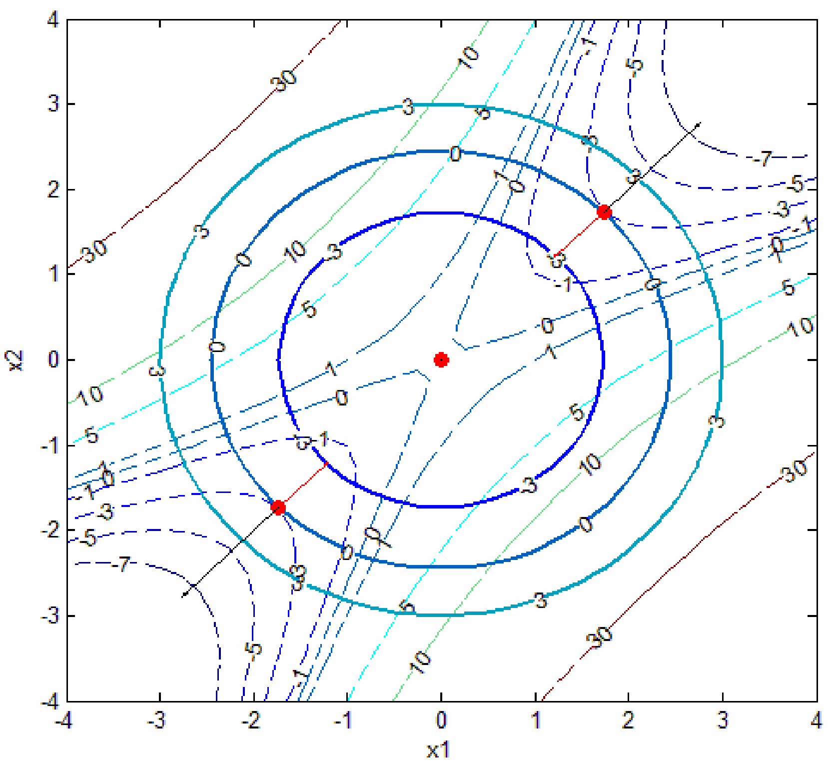
The Lagrange multipliers u^*_1, u^*_2 must be positive. (The u_3=0 since g_3 is not active).

PROBLEMS WITH INEQUALITY CONSTRAINTS

Example to solve:

$$\underset{x_1,x_2}{\text{minimize:}}\quad f(x_1,x_2)= x_1^2+x_2^2-3x_1x_2$$

$$\text{subject to:}\quad g(x_1,x_2)= x_1^2+x_2^2-6 \le 0$$



Methodology:

- Write the KKT conditions;
- Evaluate the possibilities for the switching conditions;
- Solve the resulting equations;
- Evaluate results observing $u_j \ge 0$ and $s_j^2 \ge 0$.

PROBLEMS WITH INEQUALITY CONSTRAINTS

ALTERNATE FORM OF THE KKT NECESSARY CONDITIONS

Slack variables can be eliminated:

$$L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) = f(\mathbf{x})+\sum_{i=1}^{p} v_i^* h_i(\mathbf{x})+\sum_{j=1}^{m} u_j^*\left(g_j(\mathbf{x})+s_j^2\right) = f(\mathbf{x})+\mathbf{v}^T\mathbf{h}(\mathbf{x})+\mathbf{u}^T\mathbf{g}(\mathbf{x})$$

$$\left.\frac{\partial L}{\partial x_k}\right|_{\mathbf{x}^*} = 0 \implies \frac{\partial f(\mathbf{x}^*)}{\partial x_k} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i(\mathbf{x}^*)}{\partial x_k} + \sum_{j=1}^{m} u_j^* \frac{\partial g_j(\mathbf{x}^*)}{\partial x_k} = 0 \quad k = 1, \ldots, n$$

$$\left.\frac{\partial L}{\partial v_i}\right|_{\mathbf{x}^*} = 0 \implies h_i(\mathbf{x}^*) = 0 \quad i = 1, \ldots, p$$

$$\left.\frac{\partial L}{\partial u_j}\right|_{\mathbf{x}^*} = 0 \implies g_j(\mathbf{x}^*) + s_j^2 = 0 \quad j = 1, \ldots, m$$

$$\left.\frac{\partial L}{\partial s_j}\right|_{\mathbf{x}^*} = 0 \implies 2u_j^* s_j = 0 \quad j = 1, \ldots, m$$

Since s_j^2 Ó0 guarantees feasibility, this condition can be changed to $g_j(x^*)$ Ò0.

$$s_j^2 \geq 0 \quad j = 1, \ldots, m$$

$$u_j^* \geq 0 \quad j = 1, \ldots, m$$

Multiplying this one by s_j and using the relation between $g_j(x^*)$ and s_j^2, it is changed to $u_j^* g(x^*)=0$.

PROBLEMS WITH INEQUALITY CONSTRAINTS

ALTERNATE FORM OF THE KKT NECESSARY CONDITIONS

$$L(\mathbf{x},\mathbf{v},\mathbf{u}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i^* h_i(\mathbf{x}) + \sum_{j=1}^{m} u_j^* g_j(\mathbf{x}) = f(\mathbf{x}) + \mathbf{v}^T\mathbf{h}(\mathbf{x}) + \mathbf{u}^T\mathbf{g}(\mathbf{x})$$

$$\left.\frac{\partial L}{\partial x_k}\right|_{\mathbf{x}^*} = \frac{\partial f(\mathbf{x}^*)}{\partial x_k} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i(\mathbf{x}^*)}{\partial x_k} + \sum_{j=1}^{m} u_j^* \frac{\partial g_j(\mathbf{x}^*)}{\partial x_k} = 0 \quad k = 1, \ldots, n$$

$$\left.\frac{\partial L}{\partial v_i}\right|_{\mathbf{x}^*} = h_i(\mathbf{x}^*) = 0 \quad i = 1, \ldots, p$$

$$\left.\frac{\partial L}{\partial u_j}\right|_{\mathbf{x}^*} = g_j(\mathbf{x}^*) \begin{cases} < 0 \text{ if } u_j^* = 0 \\ = u_j^* g_j(\mathbf{x}^*) = 0 \text{ if } u_j^* > 0 \end{cases} \quad j = 1, \ldots, m$$

$$u_j^* \geq 0 \quad j = 1, \ldots, m$$

PROBLEMS WITH INEQUALITY CONSTRAINTS

ALTERNATE FORM OF THE KKT NECESSARY CONDITIONS

Remarks:

This form is completely analogous to the version with slack variables.

It is needed to check the feasibility of the points found directly through constraints values.

$$g_j(\mathbf{x}^*) \leq 0 \quad j = 1, ..., m$$

The non-negativity of the Lagrange multipliers associated to the inequality constraints need to be checked.

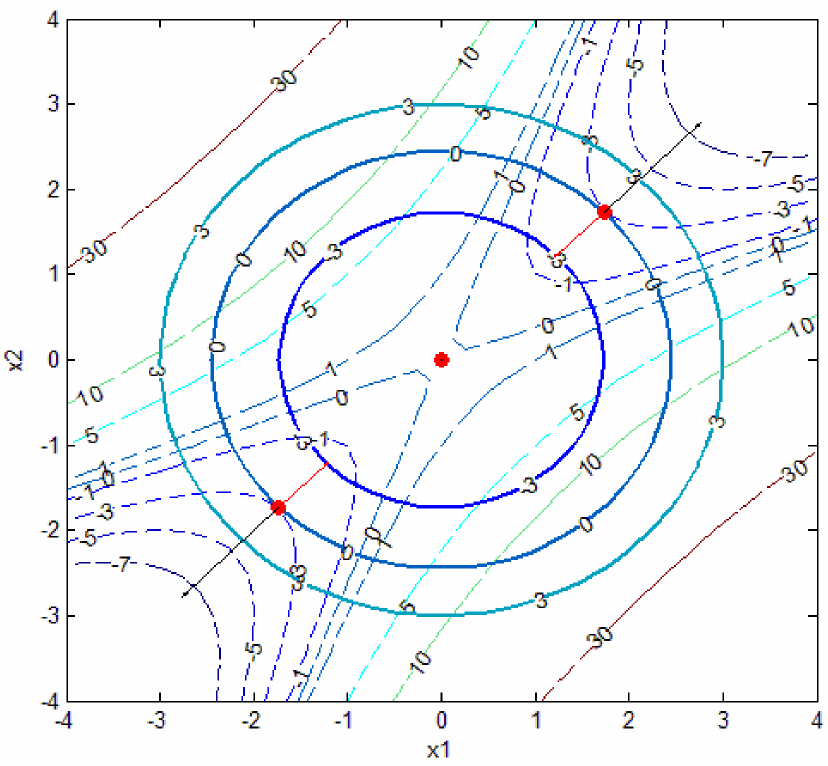
$$u_j^* \geq 0 \quad j = 1, ..., m$$

PROBLEMS WITH INEQUALITY CONSTRAINTS

Example to solve:

minimize: $f(x_1,x_2)= x_1^2+x_2^2-3x_1x_2$
x_1,x_2

subject to: $g(x_1,x_2)= x_1^2+x_2^2-6$ Ò0



Methodology:
- Write the KKT conditions;
- Evaluate the possibilities for the switching conditions;
- Solve the resulting equations;
- Evaluate results observing v_jÓ0 and g_j^2Ò0.

PROBLEMS WITH INEQUALITY CONSTRAINTS

SECOND ORDER CONDITIONS

An analysis of the Hessian matrix of the Lagrangian function L is used to develop second order sufficient conditions for the general constrained problem, as done in the unconstrained problem.

However, now only perturbing directions $\boldsymbol{d}$ wich are feasible to the problem are taken into account.

PROBLEMS WITH INEQUALITY CONSTRAINTS

SECOND ORDER SUFFICIENT CONDITIONS

Consider a point ***x**** wich satisfy the first order KKT necessary conditions for a general constrained minimization problem.

Define the Hessian of the Lagrangian as:

$$\nabla^2 L = \nabla^2 f(\mathbf{x}) + \sum_{i=1}^{p} v_i^* \nabla^2 h_i(\mathbf{x}) + \sum_{j=1}^{m} u_j^* \nabla^2 g_j(\mathbf{x})$$

Define nonzero feasible directions ***dÍ 0*** solution of:

$$\nabla h_i^T \mathbf{d} = 0 \quad i = 1, ..., p \qquad \nabla g_j^T \mathbf{d} = 0 \text{ for all active } g_j \text{ with } u_j^* > 0$$

Also let:

$$\nabla g_j^T \mathbf{d} \leq 0 \text{ for all active } g_j \text{ with } u_j^* = 0$$

If it is verifyied:

$$\mathbf{d}^T \nabla^2 L(\mathbf{x}^*) \mathbf{d} > 0$$

Then ***x**** is an isolated minimum point (there are no other minimum points at the neighborhood of ***x****).

PROBLEMS WITH INEQUALITY CONSTRAINTS

SECOND ORDER STRONG SUFFICIENT CONDITION

Consider a point ***x**** wich satisfy the first order KKT necessary conditions for a general constrained minimization problem.

Define the Hessian of the Lagrangian as:

$$\nabla^2 L = \nabla^2 f(\mathbf{x}) + \sum_{i=1}^{p} v_i^* \nabla^2 h_i(\mathbf{x}) + \sum_{j=1}^{m} u_j^* \nabla^2 g_j(\mathbf{x})$$

If:

$\nabla^2 L(\mathbf{x}^*)$ is positive definite

Then ***x**** is an isolated minimum point .

PROBLEMS WITH INEQUALITY CONSTRAINTS

SECOND ORDER CONDITIONS

Remarks:

If the strong sufficient conditions of are met (the Hessian of the Lagrangian is positive definite), a minimum point in ***x**** is guaranteed.

If not (the Hessian of the Lagrangian is not positive definite), there may occur directions ***dÍ 0*** that ensure the sufficient condition:

$$\mathbf{d}^T \nabla^2 L(\mathbf{x}^*)\mathbf{d} > 0$$

Therefore, if the strong sufficient condition is not met it is necessary to check the sufficient condition in order to conclude something about ***x****.

Check example 5.5 from ARORA (2012).

REFERENCES

ARORA, J.S. *Introduction to optimum design*, 3rd ed., Elsevier, Amsterdan, 2012.

HAFTKA, R.T. and GÜRDAL, Z. *Elements of structural optimization*, 3rd. ed., Kluwer, Dordrecht, 1992.