Creating Gears and Splines: 3 Methods for Generating True Involute Curves

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Background and Objectives

- Many methods have been presented by PTC and others to help users develop involute curves for the generation of spline and gear teeth. However, these methods are sometimes confusing, and not all are completely accurate.

- The methods in this presentation expound and improve on current formulae commonly used.

- After developing the equations, I will suggest some “good modeling” practices for given situations.

- At the end, I will perform several live demos of involute creation involving the methods presented within.
What is an Involute Curve?

- An Involute is described as the path of a point on a straight line, called the generatrix, as it rolls along a convex base curve (the evolute).
- The Involute Curve is most often used as the basis for the profile of a spline or gear tooth.
Generating the Involute Curve

Imagine a cylinder and a piece of string.

Wrap the string tightly around the cylinder.

Pull the string tight while unwinding it from the cylinder.

Trace the end of the string as it is unwrapped – the result is the involute curve.
Involute Tooth Profile Terminology

Figure courtesy of ANSI B-92.1-1996, pg. 9, © Society of Automotive Engineers, 1996.
## Involute Tooth Profile Specifications

### Drawing Data

<table>
<thead>
<tr>
<th>Internal Involute Spline Data</th>
<th>External Involute Spline Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat Root Side Fit</td>
<td>Flat Root Side Fit</td>
</tr>
<tr>
<td>Number of Teeth</td>
<td>xx</td>
</tr>
<tr>
<td>Spline Pitch</td>
<td>xx/xx</td>
</tr>
<tr>
<td>Pressure Angle</td>
<td>30°</td>
</tr>
<tr>
<td>Base Diameter</td>
<td>x.xxxxxx REF</td>
</tr>
<tr>
<td>Pitch Diameter</td>
<td>x.xxxxxx REF</td>
</tr>
<tr>
<td>Major Diameter</td>
<td>x.xxx MAX</td>
</tr>
<tr>
<td>Form Diameter</td>
<td>x.xxxx</td>
</tr>
<tr>
<td>Minor Diameter</td>
<td>x.xxxx/x.xxxx</td>
</tr>
<tr>
<td>Circular Space Width</td>
<td></td>
</tr>
<tr>
<td>Max Actual</td>
<td>x.xxxxx</td>
</tr>
<tr>
<td>Min Effective</td>
<td>x.xxxxx</td>
</tr>
</tbody>
</table>

The following information may be added as required:

- Max Measurement Between Pins: x.xxxxx REF
- Pin Diameter: x.xxxx

<table>
<thead>
<tr>
<th>Circular Tooth Thickness</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Max Effective</td>
<td>x.xxxxx</td>
</tr>
<tr>
<td>Min Actual</td>
<td>x.xxxxx</td>
</tr>
</tbody>
</table>

The following information may be added as required:

- Min Measurement Over Pins: x.xxxxx REF
- Pin Diameter: x.xxxx

More Involute Profile Terminology

• Pin Diameter and Diameter over/between pins: Place a circular measurement object tangent to the teeth at the pitch diameter. Measure over the pins for an external tooth, and between pins for an internal tooth.

• Pressure Angle: the angle between a line tangent to an involute at the pitch diameter and a radial line through the point of tangency.
Reasons to Use an Involute Curve

Reasons of Importance:

1. Conjugate action is independent of changes in the center distance.

2. The form of the basic rack tooth is straight-sided, and therefore is relatively simple. Thus, it can be accurately made. As a cutting tool, it imparts high accuracy to the cut gear or spline tooth.

3. One cutter can generate all gear or spline tooth numbers of the same pitch.

Image courtesy of Roy Beardmore, http://www.roymech.co.uk/Useful_Tables/Drive/Gears.html
How Do I Model the Involute Curve in Pro/Engineer?

- The easiest way is to create the profile of the tooth (the Involute Curve) by using a datum curve by equation. This can be done in both Cartesian (X, Y, Z) and Cylindrical (R, ?, Z) coordinates.

- Another more difficult method involves a Variable Section Sweep surface feature created by equation. This is done in Cylindrical coordinates only.
Overall Procedure for the Involute Tooth Profile Geometry Creation

Steps to completion

- Set up parameters for key variables!
- Create basic geometry in support of the spline or gear tooth
- Define the Tooth Profile with the Involute Datum Curve or Variable Section Sweep (VSS) Surface
- Create the Tooth solid feature with a cut or protrusion
  - Design vs. Manufacturing intent
  - May need helical curves for VSS generation of helical gear teeth
- Pattern the Tooth around the centerline axis
Choosing the Appropriate Method

What to use, and when ...

- Use the Cylindrical coordinate method if you desire the easiest and most versatile method of involute creation, or if you have to use polar coordinates.

- Use Cartesian Coordinates if you have to have the equations in terms of X,Y,Z only.

- Use the Variable Section Sweep method only if you don’t know/care about the major dia., and you are creating a straight (non-helical) tooth surface.
Deriving the Involute Curve Equations -- Terms

Cartesian Coordinates

- $R_i = \text{Base dia.}/2$
- $R_o = \text{Major dia.}/2$
- $S_{\alpha} = \text{arc length}$
- $S_R = \text{tangent line length at any point } X,Y \text{ on the involute}$
- $S_{R_0} = \text{tangent line length at major diameter on involute}$
- $\beta = \text{angle from start of involute to tangent point on base circle}$
- $X_c,Y_c = \text{tangent point on base circle corresponding to tangent line } S_R$
Deriving the Involute Curve Equations

Cartesian Coordinates

- \( X_c = R_i \times \cos(Beta) \)
- \( Y_c = R_i \times \sin(Beta) \)
- \( S_{\alpha} = S_R = R_i \times Beta \)
- \( X_R = X_c + S_R \times \sin(Beta) \)
- \( Y_R = Y_c - S_R \times \cos(Beta) \)
- \( R_o = \sqrt{(X_R^2 + Y_R^2)} \) for Beta = \( t \)

- For now, we will assume a start angle of 0° for simplicity, and remove it from the formulae.
Deriving the Involute Curve Equations

Cartesian Coordinates

- Substituting $R_i \cdot \cos(Beta)$ for $X_c$ and simplifying:
  
  $X_R = X_c + S_R \cdot \sin(Beta)$
  
  $X_R = R_i \cdot \cos(Beta) + \beta \cdot R_i \cdot \sin(Beta)$
  
  $X_R = R_i \cdot [\cos(Beta) + \beta \cdot \sin(Beta)]$

- Substituting $R_i \cdot \sin(Beta)$ for $Y_c$ and simplifying:
  
  $Y_R = Y_c - S_R \cdot \cos(Beta)$
  
  $Y_R = R_i \cdot \sin(Beta) - \beta \cdot R_i \cdot \cos(Beta)$
  
  $Y_R = R_i \cdot [\sin(Beta) - \beta \cdot \cos(Beta)]$
Deriving the Involute Curve Equations

Cartesian Coordinates

- Substituting for $X_R$, $Y_R$, and simplifying:

$$R_o = v(X_R^2 + Y_R^2)$$

$$R_o = v[(R_i*(\cos(Beta)+\beta*\sin(Beta))^2 + (R_i*(\sin(Beta) - \beta*\cos(Beta))^2]$$

$$R_o = v[(R_i*\cos(Beta)+R_i*\beta*\sin(Beta))^2 + (R_i*\sin(Beta) - R_i*\beta*\cos(Beta))^2]$$

$$R_o = v[R_i^2*\cos^2(Beta) + 2*R_i^2*\beta*\cos(Beta)*\sin(Beta) + R_i^2*\beta^2*\sin^2(Beta) + R_i^2*\sin^2(Beta) - 2*R_i^2*\sin(Beta)*\cos(Beta) + R_i^2*\beta^2*\cos^2(Beta)]$$

$$R_o = v[R_i^2*(\sin^2(Beta) + \cos^2(Beta)) + R_i^2*\beta^2*(\sin^2(Beta) + \cos^2(Beta)) + R_i^2(2*\beta*\cos(Beta)*\sin(Beta) - 2*\beta*(\cos(Beta)*\sin(Beta)))]$$
Deriving the Involute Curve Equations

Cartesian Coordinates

- Substituting for \( X_R, Y_R \), and simplifying, cont.:

\[
R_o = v[R_i^2(\sin^2(Beta) + \cos^2(Beta)) + \\
R_i^2\beta^2(\sin^2(Beta) + \cos^2(Beta)) + \\
R_i^2(2\beta^2\cos(Beta)\sin(Beta) - \\
2\beta^2\cos(Beta)\sin(Beta))] 
\]

\[
R_o = v[R_i^2(1) + R_i^2\beta^2(1) + 0] 
\]

\[
R_o = v[R_i^2(1 + \beta^2)] 
\]
Deriving the Involute Curve Equations

Cartesian Coordinates

- Squaring $R_o$ gives us:
  
  $R_o^2 = \{v[R_i^2*(1+Beta^2)]\}^2$

  $R_o^2 = R_i^2*(1+Beta^2)$

- Solving the above equation for Beta gives:
  
  $R_o^2 / R_i^2 = 1+Beta^2$

  $(R_o^2 / R_i^2) -1 = Beta^2$

  $v[(R_o^2 / R_i^2) -1] = Beta$

  or, $Beta = v[(R_o^2 / R_i^2) -1]$
Deriving the Involute Curve Equations

Cartesian Coordinates

- We need to define a term, alpha, in terms of $R_i$ and $R_o$, so that we can solve the parametric equation for the creation of the datum curve.

- We need to evaluate Beta over its full range (from $R_i$ to $R_o$) to derive the involute curve, so we multiply by $t$ in the equation ($t$ varies linearly from 0 to 1):

$$\alpha = t \times \beta$$

$$\alpha = t \times \sqrt{\left(\frac{R_o^2}{R_i^2}\right) - 1}$$
Deriving the Involute Curve Equations

Cartesian Coordinates

- We need the parametric equations for X and Y in terms of $R_i$ and alpha. We will use $X_c$ and $Y_c$ as the basis, substituting alpha for Beta:

\[
X = R_i \times \left[ \cos(\alpha \times (360/2\times p)) + (\alpha \times \sin(\alpha \times (360/2\times p))) \right]
\]

\[
Y = R_i \times \left[ \sin(\alpha \times (360/2\times p)) - (\alpha \times \cos(\alpha \times (360/2\times p))) \right]
\]

(Note that we have converted the angles from radians to degrees by multiplying them by $360/2\times p$)

Finally, $Z = 0$ (since we wish to create a 2-D planar curve!)
Deriving the Involute Curve Equations

Cartesian Coordinates

- So, the relation equations used in the creation of the involute profile datum curve will be:

solve

\[ \alpha = t \times v \left( \frac{R_o^2}{R_i^2} - 1 \right) \]

for \( \alpha \)

\[ X = R_i \times [\cos(\alpha \times (360/2p)) + (\alpha \times \sin(\alpha \times (360/2p)))] \]

\[ Y = R_i \times [\sin(\alpha \times (360/2p)) - (\alpha \times \cos(\alpha \times (360/2p)))] \]

\[ Z = 0 \]

Remember that all variables (\( \alpha, R_o, R_i \)) must be predefined. Since we don’t know \( \alpha \) yet, just preset it initially to a value of 1.
Deriving the Involute Curve Equations

Cartesian Coordinates

- If we include a start angle of some value other than 0°, the equations become:

\[
\text{solve} \\
\alpha = t*v\left(\frac{R_o^2}{R_i^2} - 1\right) \\
\text{for } \alpha \\
X = R_i\left[\cos(\text{Start Angle} + \alpha*(360/2*p)) + (\alpha\sin(\text{Start Angle} + \alpha*(360/2*p)))\right] \\
Y = R_i\left[\sin(\text{Start Angle} + \alpha*(360/2*p)) - (\alpha\cos(\text{Start Angle} + \alpha*(360/2*p)))\right] \\
Z = 0
\]
Deriving the Involute Curve Equations -- Terms

**Cylindrical Coordinates**

- $R_i = \text{Base dia.}/2$
- $R_o = \text{Major dia.}/2$
- $R = \text{Radius to any point on the involute curve}$
- $S_{\alpha} = \text{arc length from start of the involute to the tangent point}$
- $S_R = \text{tangent line length at any point X,Y on the involute}$
- $S_{Ro} = \text{tangent line length at major diameter on involute}$
- $\beta = \text{angle from start of involute to tangent point on base circle}$
- $\theta = \text{angle from start of involute to any point on the involute between Ro and Ri}$
- $\alpha = \text{angle from a point on the involute to the tangent point on base circle}$
Deriving the Involute Curve Equations

Cylindrical Coordinates

- \( S_{\text{alpha}} = S_R = 2\pi R_i \frac{\text{Beta}}{360} \)
- \( \text{Theta} = \text{Beta} - \text{Alpha} \)
- \( \text{Alpha} = \tan^{-1}\left(\frac{S_{\text{alpha}}}{R_i}\right) = \tan^{-1}\left(\frac{S_R}{R_i}\right) \)
- \( S_R = S_{R_0} \times t \)
- By the Pythagorean Theorem – \( R = \sqrt{S_R^2 + R_i^2} \)
- By observation and Pythagorean Theorem – \( S_{R_0} = \sqrt{(R_0^2 - R_i^2)} \)
- Again, we are setting the Start Angle to 0° for simplicity.

Cylindrical Coordinates
Deriving the Involute Curve Equations

Cylindrical Coordinates

- Substituting $S_R$ into the equation for $R$:
  \[ R = v \left[ (v(R_o^2 - R_i^2) t)^2 + R_i^2 \right] \]

- Solving the equations for Alpha and Beta, substituting $(S_{R_0} t)$ for $S_R$:
  \[ S_R = 2 \times p \times R_i \times \text{Beta} / 360 \]
  \[ \text{Beta} = (S_{R_0} t \times 360) / (2 \times p \times R_i) \]
  and:
  \[ \text{Alpha} = \text{atan}(S_R / R_i) \]
  \[ \text{Alpha} = \text{atan} \left( S_{R_0} t / R_i \right) \]
Deriving the Involute Curve Equations

Cylindrical Coordinates

- Substituting for $S_{R_0}$ in the equations for Alpha and Beta:

$$\text{Beta} = \frac{(v[(R_o^2-R_i^2)]\times t \times 360)}{(R_i \times 2 \times p)}$$

$$\text{Alpha} = \text{atan}((v(R_o^2-R_i^2)\times t)/R_i)$$
Deriving the Involute Curve Equations

Cylindrical Coordinates

- Substituting for Beta and Alpha in the equation for Theta:
  \[ \text{Theta} = \left( \frac{v[(R_o^2 - R_i^2)]t}{R_i^2} \right) \cdot \frac{360}{R_i^2} - \arctan \left( \frac{v(R_o^2 - R_i^2) \cdot t}{R_i} \right) \]

- As in the case for the equations for Cartesian involute curves, we still want the curve to be 2-D and planar, so:
  \[ Z = 0 \]
Deriving the Involute Curve Equations

Cylindrical Coordinates

- We need to make the equations parametric based on $R_0$ and $R_i$ and $t$ (which varies linearly from 0 to 1), so we create a variable “Gamma”, similar to the alpha term in the Cartesian Coordinate equations:

  \[
  \text{Gamma} = v(R_0^2 - R_i^2) \times t
  \]

- Substituting Gamma into the equations for R and Theta gives us:

  \[
  R = v(Gamma^2 + R_i^2)
  \]

  \[
  \text{Theta} = Gamma \times 360 / (R_i \times 2 \times p) - \text{atan}(\text{Gamma} / R_i)
  \]
Deriving the Involute Curve Equations

Cylindrical Coordinates

- So, the relation equations used in the creation of the involute profile datum curve will be:

```
solve
Gamma = [v(R_o^2-R_i^2)]*t
for Gamma
R = v(Gamma^2 + R_i^2)
Theta = Gamma*360/(R_i*2* p) – atan(Gamma/R_i)
Z = 0
```
Deriving the Involute Curve Equations

Cylindrical Coordinates

- Note: to account for a start angle $\theta < 0$, use:

$$\Theta = \text{start angle} + \left[ \frac{\Gamma \cdot 360}{R_i \cdot 2 \cdot p} - \arctan \left( \frac{\Gamma}{R_i} \right) \right]$$

- Remember to predefine $\Gamma$ (preset = 1), $R_o$, and $R_i$ before solving the relations !!! Setting them up as parameters makes your life easier in the long run!
Another Method for Involute Curve Creation

Using the Variable Section Sweep Feature

- Create a cylindrical protrusion with OD = major dia. or minor dia. (depending on whether we are extruding or cutting the feature)

- Create a datum curve at the pitch dia. with CL’s at angles Alpha (X-axis to 1st endpt. of curve) and Beta (angle between curve endpoints)

- Create a projected datum curve on the back surface of the protrusion

- Begin the process of creating a Variable Section Sweep
Another Method for Involute Curve Creation

Using the Variable Section Sweep Feature

- In the dashboard under references, select the spine trajectory by picking the first datum curve.
- Select the x-dir trajectory by picking the projected curve.
- Go to the sketcher, choose the CL Axis of the cylinder and the front and rear surfaces of the cylinder as your sketching references.
- Sketch a line between the front and rear surfaces, parallel to the CL Axis.
- Dimension the line to the CL Axis.
- Make sure you choose the option to be a variable section. This allows you to use relations (and input the involute equations!)
Another Method for Involute Curve Creation

Using the Variable Section Sweep Feature

- Similar to the Cylindrical coordinate equations for involute curve creation, the equations for the creation of a Variable Section Sweep uses the trajpar function (instead of the variable $t$) that varies from 0 to 1. Also, since the VSS runs from the front datum curve to a projected curve (in the $Z$-dir.), there is no need for an equation to define $Z$.

Solve

\[ \Gamma \times \left( \frac{360}{2 \times \pi} \right) - \text{atan}(\Gamma) = \text{trajpar} \times \beta \]

for $\Gamma$

\[ R = R_i \times [v_1 + (\Gamma^2)] \]
Another Method for Involute Curve Creation

Using the Variable Section Sweep Feature

- Note: Because we are varying the angle (Beta) rather than the radius, we end up solving for Gamma. This equation is similar to the equation for cylindrical coordinate involute curves where we solve for Theta. The radius, $R$, is dependent on the angle rather than the outside diameter/radius, hence the use of Gamma instead of $R_o$. 
Works Consulted


- CADQuest: Involute Gear Design Tutorial (www.cadquest.com)

- PTC Knowledgebase, Suggested Techniques:
  - Sugg. Tech. for the Creation of an Involute Gear Cutting (3 Methods)
  - Sugg. Tech. for Creating a Cylindrical Gear with Helical Teeth
  - Sugg. Tech. for Creating an Involute Curve


Live Demos!!!

(Time Permitting!!)

1. Cylindrical Coordinate Method for a Standard External Spline Tooth
2. Variable Section Sweep Method for a Standard External Spline Tooth
3. Helical and Worm Gear Creation Suggestions