**THE LORENZ ATTRACTOR**

A Mathcad 13 Document by Roger L. Mansfield
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The "strange attractor" of E. N. Lorenz is documented in many places in the literature of fractals and chaos. See, for example, [1], [2], and [3]. But the example below is based upon an image I saw in Mathsoft's sales brochure for Mathcad PLUS 6 for Power Macintosh, distributed circa 1996.

The Lorenz attractor is a smooth curve in 3-dimensional space. It is the solution to an initial value problem consisting of (a) a system of three non-linear, first-order ODEs and (b) a vector of initial conditions, defined as follows.

$\sigma := 10 \qquad r := 28 \qquad b := \frac{8}{3}$

$$Q_{dot}(Q) := \begin{bmatrix} \sigma\cdot\left(Q_1 - Q_0\right) \\ \left(-Q_1 - Q_0\cdot Q_2\right) + r\cdot Q_0 \\ Q_0\cdot Q_1 - b\cdot Q_2 \end{bmatrix} \qquad Q_o := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Here Q is a real-valued 3-vector function, $Q_{dot}$ is dQ/dt, and $Q_o$ is the 3-vector of initial values. We can define the derivative function formally to Mathcad with the following assignment.

$D\left(t, Q_o\right) := Q_{dot}\left(Q_o\right)$

"Npts" is the number of points to be computed on the space curve, i.e, Npts is the number of Runge-Kutta integration steps to take.

$Npts := 10000$

We use Mathcad's Rkadapt function to perform the integration from t = 0 to t = 100.

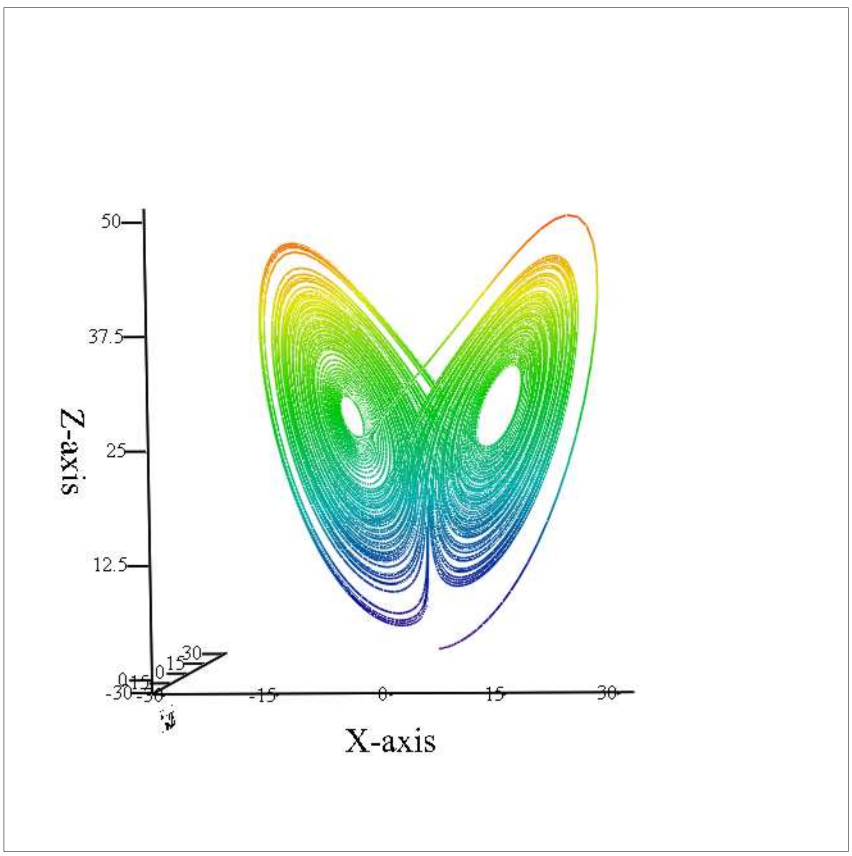
$L := \mathrm{Rkadapt}\left(Q_o, 0, 100, Npts, D\right)$

Below are the tabular results. Click in the integration table and scroll down to see all of the points.

$L =$

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0.00000000 | 1.00000000 | 1.00000000 | 1.00000000 |
| 1 | 0.01000000 | 1.01256573 | 1.25992003 | 0.98489104 |
| 2 | 0.02000000 | 1.04882146 | 1.52400085 | 0.97311434 |
| 3 | 0.03000000 | 1.10720628 | 1.79831461 | 0.96515909 |
| 4 | 0.04000000 | 1.18686546 | 2.08854557 | 0.96173737 |
| 5 | 0.05000000 | 1.28755477 | 2.40016045 | 0.96380619 |
| 6 | 0.06000000 | 1.40956882 | 2.73855211 | 0.97260826 |
| 7 | 0.07000000 | 1.55368880 | 3.10916101 | 0.98973118 |
| 8 | 0.08000000 | 1.72114581 | 3.51757715 | 1.01718654 |
| 9 | 0.09000000 | 1.91359641 | 3.96962352 | 1.05751184 |
| 10 | 0.10000000 | 2.13310762 | 4.47142018 | 1.11389889 |
| 11 | 0.11000000 | 2.38214828 | 5.02942575 | 1.19035370 |
| 12 | 0.12000000 | 2.66358398 | 5.65045026 | ... |

Here is the 3D scatter plot. Note that the solution points are joined by interpolating lines.




$\left(L^{\langle 1\rangle}, L^{\langle 2\rangle}, L^{\langle 3\rangle}\right)$

Click in the plot, then hold down the mouse and drag in varying directions to see different aspects of the strange attractor.

What is "strange" about the behavior of the solution curve is that the curve starts drawing at (1,1,1) and loops around a number of times to form an "eye" on the left (as seen in the original, undragged scatter plot), then jumps over and starts looping an eye on the right, and then back and forth from eye to eye, chaotically.

This well-known chaotic behavior is not easy to show in a Mathcad 3D scatter plot, since Mathcad plots all of the points in one complete "batch" after the Runge-Kutta integration is complete. See [3] for a Turbo Pascal program which plots the points "sequentially", as they are integrated, and which illustrates thereby the actual chaotic behavior.

REFERENCES

[1] Drazin, P. G., *Nonlinear Systems,* Cambridge University Press (1992), Chapter 8.

[2] Robbins, Judd, *Fun with Fractals,* Sybex, Inc., San Francisco (1993), Chapter 14.

[3] Stevens, Roger T., F*ractal Programming in Turbo Pascal,* M&T Publishing, Inc., Redwood City, California (1991), Chapter 4.