13.3 Digital Elliptic Filter Design

This document carries out design of a discrete-time elliptic lowpass filter. The user specifies the following parameters: passband edge, passband and stopband ripple, and filter order.

- **T**, the sampling rate
- **fp**, the passband edge frequency
- **fs**, the stopband edge frequency
- δ, the maximum ripple

Mathcad then calculates the remaining parameters and finds the zeros, poles, and coefficients for the filter transfer function. This procedure begins by computing the transfer function for a corresponding analog filter and then transforms the poles and roots of this function to produce the digital filter transfer function required.

For more information on the bilinear transformation used in this application, see **Section 13.1: Analog/Digital Lowpass Butterworth Filter**.

Background

Elliptic filters, so called because they employ elliptic functions to generate the filter function, are useful because they have equiripple characteristics in the pass and stopband regions. This ensures that the lowest order filter can be used to meet design constraints.

Elliptic filters, like all other lowpass filters, are feedback systems. All of the usual stability rules for feedback systems in the **z**-domain apply. Elliptic filters have all their zeros on the unit circle, and all their poles within the unit circle.

For information on how to transform a continuous-time IIR filter to a discrete-time IIR filter, see **Section 13.1: Analog/Digital Lowpass Butterworth Filter**.

For information on analog elliptical filter design, see **Section 13.2: Analog Elliptic Filter Design**.

Mathcad Implementation

This document implements a digital elliptic lowpass filter design. The definitions below of elliptic functions are used throughout the document to calculate filter characteristics.

Elliptic Function Definitions

$$\mathbf{TOL} \equiv 10^{-4}$$

$$U(\phi,k) \equiv 1\mathrm{i} \cdot \int_0^{\mathrm{Im}(\phi)} \frac{1}{\sqrt{1-k^2 \cdot \sin(1\mathrm{j}\cdot y)^2}}\,dy + \int_0^{\mathrm{Re}(\phi)} \frac{1}{\sqrt{1-k^2 \cdot \sin(1\mathrm{j}\cdot \mathrm{Im}(\phi)+y)^2}}\,dy$$

$$\phi \equiv 1 + 1\mathrm{i} \qquad sn(u,k) \equiv \sin(\mathbf{root}(U(\phi,k)-u,\phi))$$

$$L(k) \equiv \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2 \cdot \sin(y)^2}}\,dy$$

$$V(k) \equiv \frac{2}{1+\sqrt{1-k^2}} \cdot L\left(\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)\right)$$

$$K(k) \equiv \mathbf{if}(k < .9999, L(k), V(k))$$

Design Specifications

Sampling interval in seconds: $T := .001$

implies a bandlimit (Hz) of: $f_{max} := \dfrac{1}{2\cdot T} = 500$

Pass band edge in Hz: $f_p := 100$ Stop band edge in Hz: $f_s := 105$

Pass band ripple: $\delta_1 := .05$ Stop band ripple: $\delta_2 := .05$

Pass band edge is normalized to one: $\omega p_p := 1$

For digital design, prewarp the stopband edge to the corresponding analog value.

$$\omega s_s := \frac{\dfrac{2}{T}\cdot\tan(\pi\cdot f_s \cdot T)}{\dfrac{2}{T}\cdot\tan(\pi\cdot f_p \cdot T)} = 1.0537$$

Calculation of Filter Order

Definitions of moduli:

$$\varepsilon := \sqrt{\frac{2 \cdot \delta_1 - \delta_1{}^2}{1 - 2 \cdot \delta_1 + \delta_1{}^2}}$$

$$k_1 := \frac{\varepsilon}{\sqrt{\frac{1}{\delta_2{}^2} - 1}} \qquad\qquad k'_1 := \sqrt{1 - k_1{}^2}$$

$$k := \frac{\omega p_p}{\omega s_s} \qquad\qquad k' := \sqrt{1 - k^2}$$

For correct calculations, check to see that **k** and **k'** are not too close to 1. For accuracy, they should be less than 1 - 10-9.

$$check := k < 1 - 10^{-9} = 1 \qquad\qquad check_1 := k'_1 < 1 - 10^{-9} = 1$$

$$N := \text{ceil}\left(\frac{K(k) \cdot K(k'_1)}{K(k') \cdot K(k_1)}\right)$$

The minimum filter order is:

$$N = 6$$

Finding the Zeros and Poles of the Transfer Function

Zeros and poles of the transfer function are calculated using the elliptic functions. If the filter order is odd, there will be one real pole of order 1, in addition to several complex conjugate poles.

$$KK := K(k) = 2.581$$

$$odd := \text{mod}(N, 2) \qquad\qquad I := \textbf{if}(odd, 2, 1)$$

$$m := 0 .. N - I \qquad\qquad c_m := I + 2 \cdot \text{floor}\left(\frac{m}{2}\right)$$

$$sr_m := \frac{(-1)^m \cdot 1\text{j}}{k \cdot \text{sn}\left(c_m \cdot \frac{KK}{N}, k\right)}$$

$$l := 0 .. N - 1 \qquad\qquad conj(z, l) := \textbf{if}\left(\text{mod}(l, 2) = 0, z, \overline{z}\right)$$

$$d_l := 1 - \text{mod}(N, 2) + 2 \cdot \text{floor}\left(\frac{l + \text{mod}(N, 2)}{2}\right) \qquad v := \frac{KK}{N \cdot K(k_1)} \cdot U\left(\text{atan}\left(\frac{1}{\varepsilon}\right), k'_1\right) = 0.5015$$

$$sp_l := conj\left(1\text{j} \cdot \text{sn}\left(\frac{d_l \cdot KK}{N} + 1\text{j} \cdot v, k\right), l\right)$$

Analog Transfer Function

Use the poles and zeros of the system to construct the transfer function. It will be written as a factored expression of each pole and zero in the form **(s - p1)(s - p2)(...)**, etc..

$$A := \mathbf{if}\left(\mathrm{mod}\left(N,2\right) = 0, 1 - \delta_1, 1\right) \cdot \frac{\prod_l sp_l}{\prod_m sr_m}$$

The transfer function, **H**, normalized so that max **H(jω)** = 1:

$$H_s(s) := A \cdot \frac{\prod_m \left(s - sr_m\right)}{\prod_l \left(s - sp_l\right)}$$

Digital Transfer Function

When you have calculated the roots and poles of the analog transfer function, the corresponding digital transfer function is given by

$$q := \frac{1}{\tan\left(\pi \cdot f_p \cdot T\right)} \qquad\qquad H(z) := H_s\left(q \cdot \frac{z-1}{z+1}\right)$$

$$f := 0 .. \frac{1}{2 \cdot T}$$

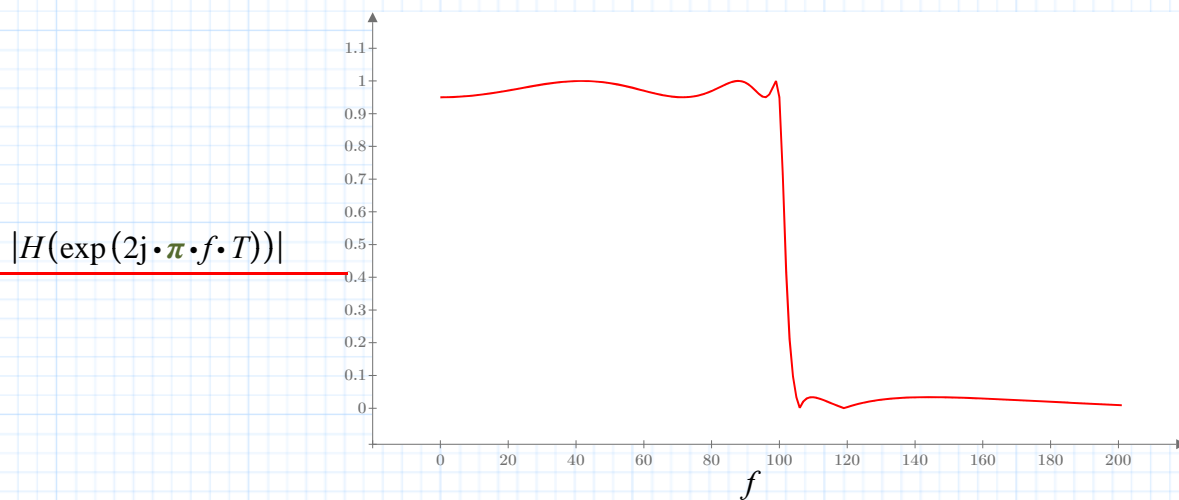$$\left|H\left(\exp\left(2\mathrm{j} \cdot \pi \cdot f \cdot T\right)\right)\right|$$

Fig. 13.3.1 Frequency response of the digital transfer function

Check magnitude at passband and stopband edges:

$$\left| H\left(\exp\left(2\mathrm{j}\cdot\pi\cdot f_p\cdot T\right)\right)\right|=0.95$$

$$\left| H\left(\exp\left(2\mathrm{j}\cdot\pi\cdot f_s\cdot T\right)\right)\right|=0.0337$$

The zeros are

$$a_m:=\frac{q+sr_m}{q-sr_m} \qquad a_{N-1}:=\mathbf{if}\left(odd,-1,\frac{q+sr_{N-1}}{q-sr_{N-1}}\right)$$

The poles are

$$b_l:=\frac{q+sp_l}{q-sp_l}$$
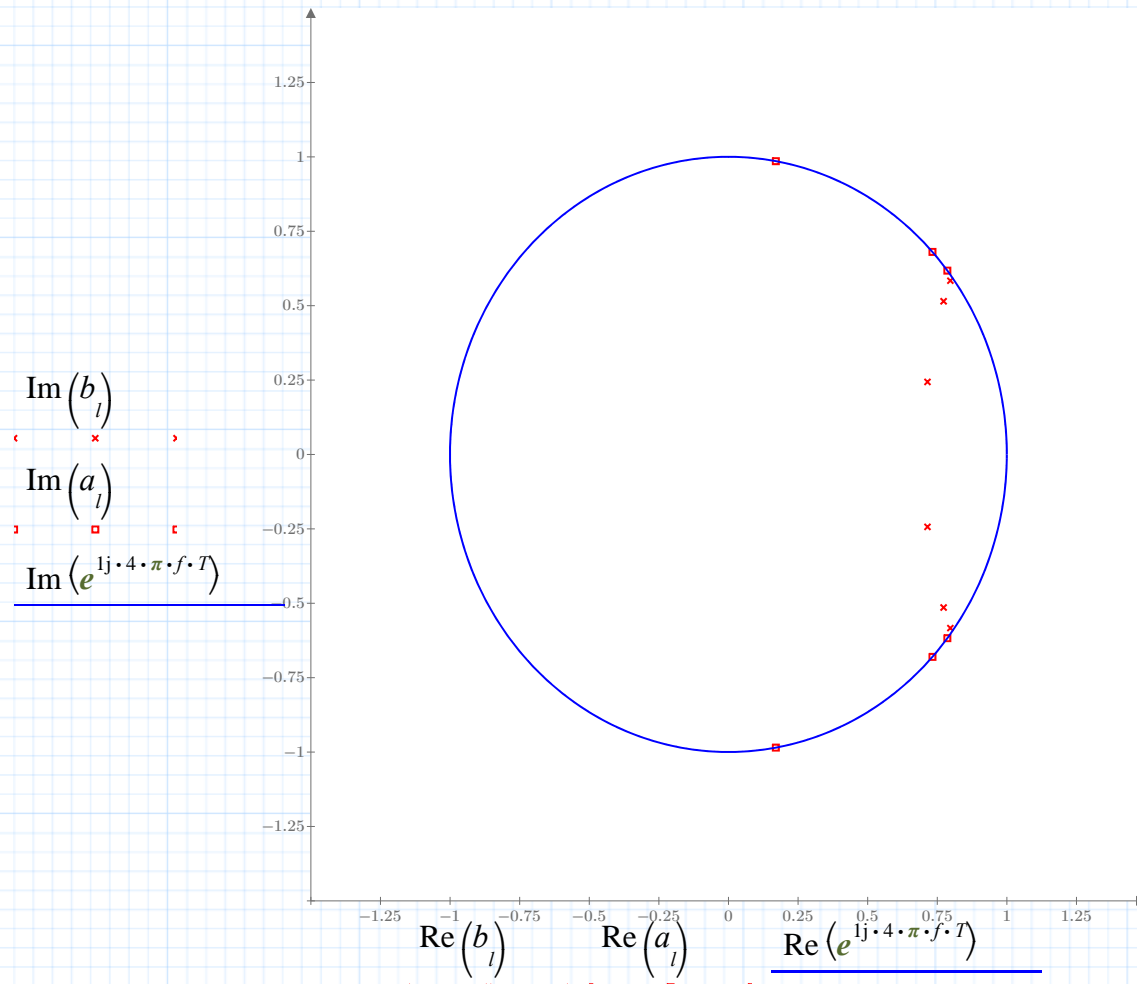
Fig. 13.3.2 Poles and zeros of the filter in the z-plane

Polynomial Expansion

To expand the transfer function into an **N**th order polynomial, find the transfer function coefficients for numerator and denominator, and integrate according to Cauchy's theorem:

$$HNUM(s) := \prod_l \left(s - a_l\right)$$

$$HDEN(s) := \prod_l \left(s - b_l\right)$$

Multiplying numerator coefficients by **B** will normalize **H** so that at frequency zero the response is 1 for **N** odd or 1 - δ for **N** even.

$$B := \frac{HDEN(1)}{HNUM(1)} \cdot \mathbf{if}\,(odd, 1, 1 - \delta_l) \qquad\qquad w := 0..N$$

$$hnum_w := B \cdot \frac{\displaystyle\int_0^1 HNUM\left((\exp(2j \cdot \pi \cdot x))\right) \cdot \cos(w \cdot 2 \cdot \pi \cdot x)\, \mathrm{d}x}{\mathbf{if}\,(w = 0, 1, .5)}$$

$$hden_w := \frac{\displaystyle\int_0^1 HDEN\left((\exp(2j \cdot \pi \cdot x))\right) \cdot \cos(w \cdot 2 \cdot \pi \cdot x)\, \mathrm{d}x}{\mathbf{if}\,(w = 0, 1, .5)}$$

Numerator coefficients:

Denominator coefficients:

$$w = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

$$hnum_w = \begin{bmatrix} 0.043 + 9.775i \cdot 10^{-12} \\ -0.144 - 4.98i \cdot 10^{-11} \\ 0.27 + 1.132i \cdot 10^{-10} \\ -0.321 - 1.464i \cdot 10^{-10} \\ 0.27 + 1.132i \cdot 10^{-10} \\ -0.144 - 4.98i \cdot 10^{-11} \\ 0.043 + 9.775i \cdot 10^{-12} \end{bmatrix}$$

$$hden_w = \begin{bmatrix} 0.48 + 8.882i \cdot 10^{-16} \\ -2.849 - 8.882i \cdot 10^{-16} \\ 7.421 - 8.882i \cdot 10^{-16} \\ -10.83 \\ 9.365 - 4.441i \cdot 10^{-16} \\ -4.571 - 3.851i \cdot 10^{-15} \\ 1 + 1.069i \cdot 10^{-15} \end{bmatrix}$$