Structural Analysis Example Problems

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Structural Analysis
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List of Examples

2.1 Stress Vectors

For the following stress tensor at point P

$$\sigma := \begin{pmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} := \sigma = \begin{pmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Determine the traction (or stress vector) **t** passing through P and parallel to the plane ABC where A(4,0,0), B(0,2,0) and C(0,0,6)

Solution:

The vector normal to the plane can be found by taking the cross products of vectors AB and AC

$$N = AB \times AC = \begin{pmatrix} e_1 & e_2 & e_3 \\ -4 & 2 & 0 \\ -4 & 0 & 6 \end{pmatrix} = 12\,e_1 + 24\,e_2 + 8\,e_3$$

The unit normal of N is given by

$$\mathbf{n} = \frac{3}{7}e_1 + \frac{6}{7}e_2 + \frac{2}{7}e_3$$

$$n := \begin{pmatrix} \frac{3}{7} & \frac{6}{7} & \frac{2}{7} \end{pmatrix}$$

Hence the stress vector traction will be

$$t := n\cdot\sigma = \begin{pmatrix} -\frac{9}{7} & \frac{5}{7} & \frac{10}{7} \end{pmatrix}$$

and thus $\mathbf{t} = -\frac{9}{7}e_1 + \frac{5}{7}e_2 + \frac{10}{7}e_3$

2.2 Stress Vectors 2

The state of stress through a continuum is given with respect to the cartesian axes by

$$\sigma(x) := \begin{bmatrix} 3\,x_1\,x_2 & 5\left(x_2\right)^2 & 0 \\ 5\left(x_2\right)^2 & 0 & 2\,x_3 \\ 0 & 2\,x_3 & 0 \end{bmatrix} \cdot \text{MPa}$$

Determine the stress vector at $P\left(1, 1, \sqrt{3}\right)$ of the plane that is normal to the tangent to the cylindrical surface $\left(x_2\right)^2 + \left(x_3\right)^2 = 4$ at P

Solution:

Vector Equation

$$v(x) := \left(x_2\right)^2 + \left(x_3\right)^2 - 4$$

Gradient

$$\text{Vect1}(x) := \nabla_x v(x) \rightarrow \begin{pmatrix} 0 \\ 2 \cdot x_2 \\ 2 \cdot x_3 \end{pmatrix}$$

For $x_2 := 1$

$x_1 = 0$

$x_2 = 1$

$$x_3 := \sqrt{4 - \left(x_2\right)^2} = 1.732$$

Normalized Vector

$$n := \frac{x}{|x|} = \begin{pmatrix} 0 \\ 0.5 \\ 0.866 \end{pmatrix}$$

$$\sigma(x) \cdot n = \begin{pmatrix} 2.5 \\ 3 \\ 1.732 \end{pmatrix} \cdot \text{MPa}$$

2.3 Principal Stresses

The stress tensor is given at a point by point P

$$\sigma := \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

Determine the principal stress values and the corresponding directions

Solution:

$$\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 0-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{pmatrix} = 0$$

Simplifying the determinant

$$\left|\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 0-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{pmatrix}\right| = 0 \text{ simplify } \rightarrow -(\lambda-1)\cdot(\lambda+2)\cdot(\lambda-4) = 0$$

thus the roots are

$$\lambda := -(\lambda-1)\cdot(\lambda+2)\cdot(\lambda-4) = 0 \text{ solve } \rightarrow \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$$\lambda = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

Note that those are the three eigenvalues of the stress vector. If we let the x_1 axis be the one corresponding to the direction of λ_2 and n_i^2 be the direction cosines of this axis, then we have

$$(3-\lambda_2)n_1^{\,2} + n_2^{\,2} + n_3^{\,2} = 0 \rightarrow 5\cdot n_1^{\,2} + n_2^{\,2} + n_3^{\,2} = 0$$

$$n_1^{\,2} + (0-\lambda_2)\cdot n_2^{\,2} + 2n_3^{\,2} = 0 \rightarrow n_1^{\,2} + 2\cdot n_2^{\,2} + 2\cdot n_3^{\,2} = 0$$

$$n_1^{\,2} + 2\cdot n_2^{\,2} + (0-\lambda_2)n_3^{\,2} = 0 \rightarrow n_1^{\,2} + 2\cdot n_2^{\,2} + 2\cdot n_3^{\,2} = 0$$

$$n_1^{\,2} = 0 \qquad n_2^{\,2} = \frac{1}{\sqrt{2}} \qquad n_3^{\,2} = -\frac{1}{\sqrt{2}}$$

Similarly, if we let x_2 be the one corresponding to the direction of λ_1 and n_i^1 be the direction cosines of this axis,

$$(3-\lambda_1)n_1^{\,2} + n_2^{\,2} + n_3^{\,2} = 0 \rightarrow 2\cdot n_1^{\,2} + n_2^{\,2} + n_3^{\,2} = 0$$

$$n_1^{\,2} + (0-\lambda_1)\cdot n_2^{\,2} + 2n_3^{\,2} = 0 \rightarrow n_1^{\,2} - n_2^{\,2} + 2\cdot n_3^{\,2} = 0$$

$$n_1^{\,2} + 2\cdot n_2^{\,2} + (0-\lambda_1)n_3^{\,2} = 0 \rightarrow n_1^{\,2} + 2\cdot n_2^{\,2} - n_3^{\,2} = 0$$

$$n_1^{\,1} = \frac{1}{\sqrt{3}} \qquad n_2^{\,1} = -\frac{1}{\sqrt{3}} \qquad n_3^{\,1} = -\frac{1}{\sqrt{3}}$$

Finally, if we let x_3 be the one corresponding to the direction of λ_3 and n_i^3 be the direction cosines of this axis,

$$(3-\lambda_3)n_1^2 + n_2^2 + n_3^2 = 0 \rightarrow -n_1^2 + n_2^2 + n_3^2 = 0$$

$$n_1^2 + (0-\lambda_3)\cdot n_2^2 + 2n_3^2 = 0 \rightarrow n_1^2 - 4\cdot n_2^2 + 2\cdot n_3^2 = 0$$

$$n_1^2 + 2\cdot n_2^2 + (0-\lambda_3)n_3^2 = 0 \rightarrow n_1^2 + 2\cdot n_2^2 - 4\cdot n_3^2 = 0$$

$$n_1^3 = -\frac{2}{\sqrt{6}} \quad n_2^3 = -\frac{1}{\sqrt{6}} \quad n_3^3 = -\frac{1}{\sqrt{6}}$$

$$n_{pd} := \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

2.4 Stress Tensor Operations

For the following stress tensor

$$\sigma := \begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

a) Determine directly the three Invariants I_σ, II_σ and III_σ
b) Determine the principal stresses and principal stress directions
c) Show that the transformation tensor of direction cosines transforms the original stress tensor into the diaganol axes stress tensor
d) Recompute the three invariants from the principal stresses
e) Split the stress tensor into its spherical and deviator parts
f) Show that the first invariant of the deviator is zero

Solution:

a) $I_\sigma := \sigma_{1,1} + \sigma_{2,2} + \sigma_{3,3} = 20$

$$II_\sigma := \sigma_{1,1}\cdot\sigma_{2,2} + \sigma_{2,2}\cdot\sigma_{3,3} + \sigma_{3,3}\cdot\sigma_{1,1} - \sigma_{1,2}\cdot\sigma_{2,1} = 123$$

$$III_\sigma := |\sigma| = 216$$

b) $\sigma_p := \text{eigenvals}(\sigma) = \begin{pmatrix} 3 \\ 8 \\ 9 \end{pmatrix}$

$$n_p := \text{eigenvecs}(\sigma) = \begin{pmatrix} -0.707 & 0 & -0.707 \\ -0.707 & 0 & 0.707 \\ 0 & 1 & 0 \end{pmatrix}$$

c) $n_p^T\cdot\sigma\cdot n_p = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

d) $I_{\sigma pd} := \sigma_{p_1} + \sigma_{p_2} + \sigma_{p_3} = 20$

$$II_{\sigma pd} := \sigma_{p_1}\cdot\sigma_{p_2} + \sigma_{p_2}\cdot\sigma_{p_3} + \sigma_{p_3}\cdot\sigma_{p_1} = 123$$

$$III_{\sigma pd} := \sigma_{p_1}\cdot\sigma_{p_2}\cdot\sigma_{p_3} = 216$$

e) $\sigma_{mean} := \dfrac{I_{\sigma pd}}{3} = 6.667$

$$\sigma_{sperical} := \begin{pmatrix} \sigma_{mean} & 0 & 0 \\ 0 & \sigma_{mean} & 0 \\ 0 & 0 & \sigma_{mean} \end{pmatrix} = \begin{pmatrix} 6.667 & 0 & 0 \\ 0 & 6.667 & 0 \\ 0 & 0 & 6.667 \end{pmatrix}$$

f) $\sigma_{deviator} := \sigma - \sigma_{sperical} = \begin{pmatrix} -0.667 & -3 & 0 \\ -3 & -0.667 & 0 \\ 0 & 0 & 1.333 \end{pmatrix}$

2.5 Stress Transformation

Show that the transformation tensor of direction cosines for the stress tensor

$$\sigma := \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

transforms the original stress tensor into the diaganol principal axes stress tensor

Solution:

$$n_{pd} := \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\bar{\sigma} := n_{pd} \cdot \sigma \cdot n_{pd}^{T} = ▮$$

2.6 Stress Transformation 2

The octahedral plane is the plane which makes equal angles with the principal stress directions. Show that the shear stress on this plane, the so-called octahedral shear stress, is given by

$$\sigma_{oct}(\sigma_I,\sigma_{II},\sigma_{III}) := \frac{1}{3}\cdot\sqrt{(\sigma_I-\sigma_{II})^2+(\sigma_{II}-\sigma_{III})^2+(\sigma_{III}-\sigma_I)^2}$$

Solution:

$$v := \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$n := \left(\frac{v}{|v|}\right)^T = (0.577 \;\; 0.577 \;\; 0.577)$$

$$\sigma(\sigma_I,\sigma_{II},\sigma_{III}) := \begin{pmatrix}\sigma_I & 0 & 0\\ 0 & \sigma_{II} & 0\\ 0 & 0 & \sigma_{III}\end{pmatrix}$$

$$t(\sigma_I,\sigma_{II},\sigma_{III}) := n\cdot\sigma(\sigma_I,\sigma_{II},\sigma_{III}) \rightarrow (0.577\cdot\sigma_I \;\; 0.577\cdot\sigma_{II} \;\; 0.577\cdot\sigma_{III})$$

$$tn(\sigma_I,\sigma_{II},\sigma_{III}) := n\cdot t(\sigma_I,\sigma_{II},\sigma_{III})^T \rightarrow 0.333\cdot\sigma_I + 0.333\cdot\sigma_{II} + 0.333\cdot\sigma_{III}$$

$$t_{shear}(\sigma_I,\sigma_{II},\sigma_{III}) := \sqrt{t(\sigma_I,\sigma_{II},\sigma_{III})\cdot t(\sigma_I,\sigma_{II},\sigma_{III})^T - tn(\sigma_I,\sigma_{II},\sigma_{III})\cdot tn(\sigma_I,\sigma_{II},\sigma_{III})}$$

$$t_{shear}(\sigma_I,\sigma_{II},\sigma_{III}) \text{ simplify } \rightarrow \sqrt{0.222\cdot\sigma_I^2 - 0.222\cdot\sigma_I\cdot\sigma_{II} - 0.222\cdot\sigma_I\cdot\sigma_{III} + 0.222\cdot\sigma_{II}^2 - 0.222\cdot\sigma_{II}\cdot\sigma_{III} + 0.222\cdot\sigma_{III}^2}$$

$$t_{shear}(\sigma_I,\sigma_{II},\sigma_{III}) - \sigma_{oct}(\sigma_I,\sigma_{II},\sigma_{III}) = 0$$

2.7 Strain Invariants & Principal Strains

Determine the planes of principal strains for the following strain tensor

$$E := \begin{pmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution:

The strain invariants are given by

$$I_E = E_{ii} = 2$$

$$II_E = \frac{1}{2}\cdot\left(E_{ij}\cdot E_{ij} - E_{ii}\cdot E_{jj}\right) = -1 + 3 = 2$$

$$III_E = \left|E_{ij}\right| = -3$$

The principal strains by

$$E_{ij} - \lambda\delta_{ij} = \begin{pmatrix} 1-\lambda & \sqrt{3} & 0 \\ \sqrt{3} & 0-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)\cdot\left(\lambda - \frac{1+\sqrt{13}}{2}\right)\cdot\left(\lambda - \frac{1-\sqrt{13}}{2}\right)$$

$$E_1 = \lambda_1 = \frac{1+\sqrt{13}}{2} = 2.3$$

$$E_2 = \lambda_2 = 1$$

$$E_3 = \lambda_3 = \frac{1-\sqrt{13}}{2} = -1.3$$

The eigenvectors for $E_1 = \frac{1+\sqrt{13}}{2}$ give the principal directions n^1

$$\begin{pmatrix} 1-\frac{1+\sqrt{13}}{2} & \sqrt{3} & 0 \\ \sqrt{3} & 0-\frac{1+\sqrt{13}}{2} & 0 \\ 0 & 0 & 1-\frac{1+\sqrt{13}}{2} \end{pmatrix}\cdot\begin{pmatrix} n_{1,1} \\ n_{2,1} \\ n_{3,1} \end{pmatrix} \rightarrow \begin{bmatrix} -n_{1,1}\cdot\left(\frac{\sqrt{13}}{2}-\frac{1}{2}\right)+\sqrt{3}\cdot n_{2,1} \\ -n_{2,1}\cdot\left(\frac{\sqrt{13}}{2}+\frac{1}{2}\right)+\sqrt{3}\cdot n_{1,1} \\ -n_{3,1}\cdot\left(\frac{\sqrt{13}}{2}-\frac{1}{2}\right) \end{bmatrix}$$

$$\begin{bmatrix} -n_{1,1}\cdot\left(\frac{\sqrt{13}}{2}-\frac{1}{2}\right)+\sqrt{3}\cdot n_{2,1} \\ -n_{2,1}\cdot\left(\frac{\sqrt{13}}{2}+\frac{1}{2}\right)+\sqrt{3}\cdot n_{1,1} \\ -n_{3,1}\cdot\left(\frac{\sqrt{13}}{2}-\frac{1}{2}\right) \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$n_{1,1} = \frac{1+\sqrt{13}}{2\sqrt{3}} \cdot n_{2,1}$$

$$n_{3,1} = 0$$

$$n^1 \cdot n^1 = 1$$

$$\left(\frac{1+2\sqrt{13}+13}{12}+1\right)\cdot\left(n_{2,1}\right)^2 = 1$$

$$n_{2,1} = 0.6$$

$$n_{1,1} = \frac{1+\sqrt{13}}{2\sqrt{3}} \cdot n_{2,1} = \frac{1+\sqrt{13}}{2\sqrt{3}} \cdot 0.6 = 0.8$$

$$n^1 = (0.8 \;\; 0.6 \;\; 0)$$

For the second eigenvector λ_2=1

$$\begin{pmatrix} 1-1 & \sqrt{3} & 0 \\ \sqrt{3} & 0-1 & 0 \\ 0 & 0 & 1-1 \end{pmatrix} \cdot \begin{pmatrix} n_{1,2} \\ n_{2,2} \\ n_{3,2} \end{pmatrix} \rightarrow \begin{bmatrix} \sqrt{3}\cdot n_{2,2} \\ -\left(n_{2,2}\right)+\sqrt{3}\cdot n_{1,2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}\cdot n_{2,2} \\ -\left(n_{2,2}\right)+\sqrt{3}\cdot n_{1,2} \\ 0 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$n^2 = (0 \;\; 0 \;\; 1)$$

Finally, the third eigenvector can be obtained by the same manner, but more easily from

$$n^3 = n^1 \times n^2 = \det\begin{pmatrix} e_1 & e_2 & e_3 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0.6e_1 - 0.8e_2$$

Therefore

$$a_i^j = \begin{pmatrix} n^1 \\ n^2 \\ n^3 \end{pmatrix} \qquad a := \begin{pmatrix} 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.8 & 0 \end{pmatrix}$$

$$a\cdot E\cdot a^T = \begin{pmatrix} 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.8 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.8 & 0 & 0.6 \\ 0.6 & 0 & -0.8 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2.3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1.3 \end{pmatrix}$$

2.8 Equilibrium Equations

In the absence of body forces, does the following stress distribution

$$T(x,\nu) = \begin{bmatrix} (x_2)^2 + \nu\left[(x_1)^2 - (x_2)^2\right] & -2\cdot\nu\cdot x_1\cdot x_2 & 0 \\ -2\nu\cdot x_1\cdot x_2 & (x_1)^2 + \nu\left[(x_2)^2 - (x_1)^2\right] & 0 \\ 0 & 0 & \nu\left[(x_1)^2 + (x_2)^2\right] \end{bmatrix}$$

where v is a constant, satisfy equilibrium?

Solution:

$$\frac{\partial}{\partial x_j}T_{1j} = \frac{\partial}{\partial x_1}T_{11} + \frac{\partial}{\partial x_2}T_{12} + \frac{\partial}{\partial x_3}T_{13} = 2\cdot\nu\cdot x_1 - 2\cdot\nu\cdot x_1 = 0$$

$$\frac{\partial}{\partial x_j}T_{2j} = \frac{\partial}{\partial x_1}T_{21} + \frac{\partial}{\partial x_2}T_{22} + \frac{\partial}{\partial x_3}T_{23} = -2\cdot\nu\cdot x_2 + 2\cdot\nu\cdot x_2 = 0$$

$$\frac{\partial}{\partial x_j}T_{3j} = \frac{\partial}{\partial x_1}T_{31} + \frac{\partial}{\partial x_2}T_{32} + \frac{\partial}{\partial x_3}T_{33} = 0$$

Therefore, equilibrium is satisfied.

2.9 Stress-Strain

Determine the stress tensor at a point where the Lagrangian strain tensor is given by

$$E := \begin{pmatrix} 30 & 50 & 20 \\ 50 & 40 & 0 \\ 20 & 0 & 30 \end{pmatrix} \cdot 10^{-6}$$

and the material is steel with $\lambda := 119.2\text{GPa}$ and $\mu := 79.2\text{GPa}$

Solution:

$$C := \begin{pmatrix} \lambda + 2\cdot\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\cdot\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\cdot\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\cdot\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\cdot\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\cdot\mu \end{pmatrix} \text{simplify} \rightarrow \begin{pmatrix} 277.6\cdot\text{GPa} & 119.2\cdot\text{GPa} & 119.2\cdot\text{GPa} & 0 & 0 & 0 \\ 119.2\cdot\text{GPa} & 277.6\cdot\text{GPa} & 119.2\cdot\text{GPa} & 0 & 0 & 0 \\ 119.2\cdot\text{GPa} & 119.2\cdot\text{GPa} & 277.6\cdot\text{GPa} & 0 & 0 & 0 \\ 0 & 0 & 0 & 158.4\cdot\text{GPa} & 0 & 0 \\ 0 & 0 & 0 & 0 & 158.4\cdot\text{GPa} & 0 \\ 0 & 0 & 0 & 0 & 0 & 158.4\cdot\text{GPa} \end{pmatrix}$$

$$\gamma := \begin{pmatrix} E_{1,1} \\ E_{2,2} \\ E_{3,3} \\ E_{1,2} \\ E_{1,3} \\ E_{2,3} \end{pmatrix} = \begin{pmatrix} 3 \times 10^{-5} \\ 4 \times 10^{-5} \\ 3 \times 10^{-5} \\ 5 \times 10^{-5} \\ 2 \times 10^{-5} \\ 0 \times 10^{0} \end{pmatrix}$$

$$\sigma := C\cdot\gamma = \begin{pmatrix} 0.01667 \\ 0.01826 \\ 0.01667 \\ 0.00792 \\ 0.00317 \\ 0 \end{pmatrix} \cdot\text{GPa}$$

2.10 Stress-Strain

Determine the stress tensor at a point where the Cauchy stress tensor is given by

$$\sigma := \begin{pmatrix} 100 & 42 & 6 \\ 42 & -2 & 0 \\ 6 & 0 & 15 \end{pmatrix} \cdot \text{MPa}$$

with $E := 207\text{GPa}$, $\mu := 79.2\text{GPa}$, and $\nu := 0.3$

Solution:

$$S := \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix} \text{simplify} \rightarrow \begin{pmatrix} \frac{1}{207 \cdot \text{GPa}} & -\frac{0.001}{\text{GPa}} & -\frac{0.001}{\text{GPa}} & 0 & 0 & 0 \\ -\frac{0.001}{\text{GPa}} & \frac{1}{207 \cdot \text{GPa}} & -\frac{0.001}{\text{GPa}} & 0 & 0 & 0 \\ -\frac{0.001}{\text{GPa}} & -\frac{0.001}{\text{GPa}} & \frac{1}{207 \cdot \text{GPa}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{0.013}{\text{GPa}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{0.013}{\text{GPa}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{0.013}{\text{GPa}} \end{pmatrix}$$

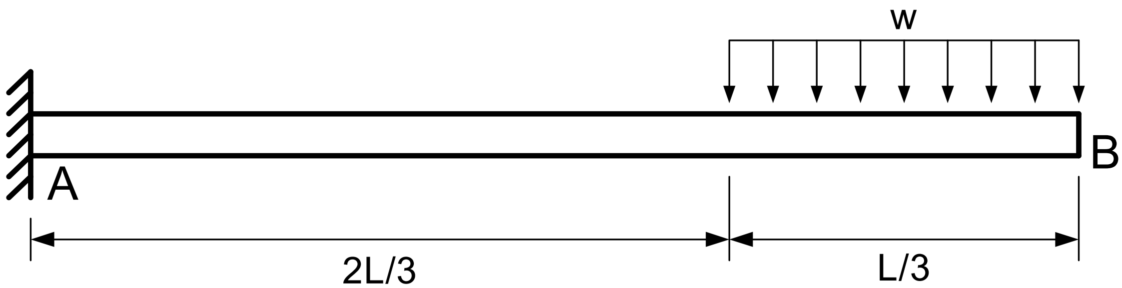
$$s := \begin{pmatrix} \sigma_{1,1} \\ \sigma_{2,2} \\ \sigma_{3,3} \\ \sigma_{1,2} \\ \sigma_{1,3} \\ \sigma_{2,3} \end{pmatrix} = \begin{pmatrix} 100 \\ -2 \\ 15 \\ 42 \\ 6 \\ 0 \end{pmatrix} \cdot \text{MPa}$$

$$\gamma := S \cdot s = \begin{pmatrix} 4.643 \times 10^{-4} \\ -1.763 \times 10^{-4} \\ -6.957 \times 10^{-5} \\ 5.303 \times 10^{-4} \\ 7.576 \times 10^{-5} \\ 0 \times 10^{0} \end{pmatrix}$$

$$E := \begin{pmatrix} \gamma_1 & \frac{\gamma_4}{2} & \frac{\gamma_5}{2} \\ \frac{\gamma_4}{2} & \gamma_2 & \frac{\gamma_6}{2} \\ \frac{\gamma_5}{2} & \frac{\gamma_6}{2} & \gamma_3 \end{pmatrix} = \begin{pmatrix} 4.643 \times 10^{-4} & 2.652 \times 10^{-4} & 3.788 \times 10^{-5} \\ 2.652 \times 10^{-4} & -1.763 \times 10^{-4} & 0 \times 10^{0} \\ 3.788 \times 10^{-5} & 0 \times 10^{0} & -6.957 \times 10^{-5} \end{pmatrix}$$

3.1 Displacement by Double Integration

Determine the deflection at B for the following cantilevered beam.



Solution:

At $0 \le x \le \frac{2L}{3}$

1. Moment Equation

$$E \cdot I \cdot \frac{\partial^2}{\partial x^2} y = M_x = \frac{w \cdot L}{3} \cdot x - \frac{5}{18} \cdot w \cdot L^2$$

2. Integrate Once

$$E \cdot I \cdot \frac{\partial}{\partial x} y = \frac{w \cdot L}{6} \cdot x^2 - \frac{5}{18} \cdot w \cdot L^2 \cdot x + C_1$$

However we have at x=0, dy/dx=0, C_1=0

3. Integrate Twice

$$E \cdot I \cdot y = \frac{w \cdot L}{18} \cdot x^3 - \frac{5 \cdot w \cdot L^2}{36} \cdot x^2 + C_2$$

Again we have at x=0, y=0, C_2=0

At $\frac{2L}{3} \le x \le L$

1. Moment Equation

$$E \cdot I \cdot \frac{\partial^2}{\partial x^2} y = M_x = \frac{w \cdot L}{3} \cdot x - \frac{5}{18} \cdot w \cdot L^2 - w \cdot \left(x - \frac{2L}{3}\right) \cdot \left(\frac{x - \frac{2L}{3}}{2}\right)$$

2. Integrate Once

$$E \cdot I \cdot \frac{\partial}{\partial x} y = \frac{w \cdot L}{6} \cdot x^2 - \frac{5}{18} \cdot w \cdot L^2 \cdot x - \frac{w}{6} \cdot \left(x - \frac{2L}{3}\right)^3 + C_3$$

Applying the boundary condition at $x = \frac{2L}{3}$, we must have $\frac{dy}{dx}$ equal to the value coming frmo the left C_3=0

3. Integrate Twice

$$E \cdot I \cdot y = \frac{w \cdot L}{18} \cdot x^3 - \frac{5 \cdot w \cdot L^2}{36} \cdot x^2 - \frac{w}{24} \cdot \left(x - \frac{2L}{3}\right)^4 + C_4$$

Again following the same argument as above, C_4=0

Substituting for x=L we obtain

$$y = \frac{163}{1944} \cdot \frac{wL^4}{E \cdot I}$$

4.1 Live Load Reduction

A four storey office building has interior columns spaced 30ft apart in the two directions. If the flat roof loading is 50 lb/ft^2, determine the reduced live load supported by a typical interior column located on the ground level

Solution:

$L_0 := 50psf$

$A_T := 30ft \cdot 30ft = 900 \cdot ft^2$

$K_{LL} := 4$

$L_{floor}(L_0, A_T, K_{LL}) := L_0 \cdot \left(0.25 + \frac{15}{\sqrt{K_{LL} \cdot A_T}}\right) \cdot psf$

$L_{floor} := L_{floor}\left(\frac{L_0}{psf}, \frac{A_T}{ft^2}, K_{LL}\right) = 25 \cdot psf$

$Reduction := \frac{L_{floor}}{L_0} = 50 \cdot \%$

$L_{roof} := 0.6 \cdot L_0 = 30 \cdot psf$

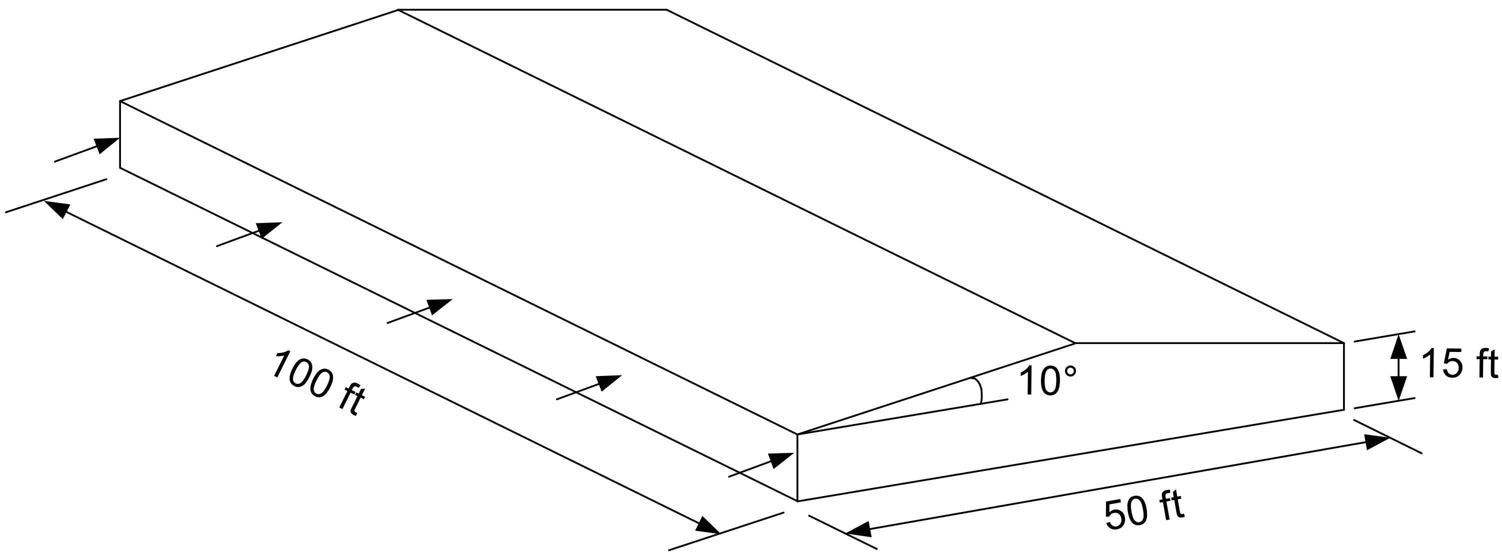
$F_1 := \left(3 \cdot L_{floor} \cdot A_T + L_{roof} \cdot A_T\right) = 94.5 \cdot kip$

Note that without reduction the total load woul dhave been

$F_2 := 4 \cdot L_0 \cdot A_T = 180 \cdot kip$

4.2 Wind Load

Wind blows on the side of the fully enclosed agricultural building located on open flat terrain in Oklahoma. Determine the external pressure acting on the roof. Also, what is the internal pressure in the building which acts on the roof? Use linear interpolation to determine q_h and C_p.



Solution:

$$q_z(K_z, K_{Zt}, K_d, V, I) := 0.00256 \cdot K_z \cdot K_{Zt} \cdot K_d \cdot V^2 \cdot I \cdot psf$$

$$K_{Zt} := 1$$

$$K_d := 1$$

$$V := 90mph$$

$$I := 0.87$$

$$q_z\left(K_z, K_{Zt}, K_d, \frac{V}{mph}, I\right) \rightarrow 18.04 \cdot K_z \cdot psf$$

$$K_{z15} := 0.85$$

$$q_{z15} := q_z\left(K_{z15}, K_{Zt}, K_d, \frac{V}{mph}, I\right) = 15.334 \cdot psf$$

$$K_{z20} := 0.9$$

$$q_{z20} := q_z\left(K_{z20}, K_{Zt}, K_d, \frac{V}{mph}, I\right) = 16.236 \cdot psf$$

$$h := 15ft + \frac{1}{2} \cdot 25ft \cdot \tan(10deg) = 17.204 \cdot ft$$

$$\frac{q_h - q_{z15}}{h - 15ft} = \frac{q_{z20} - q_{z15}}{20ft - 15ft}$$

$$q_h := \left(\frac{q_{z20} - q_{z15}}{20ft - 15ft}\right)\cdot(h - 15ft) + q_{z15}$$

$$q_h = 15.732\cdot psf$$

External pressure on windward side of roof

$$p\left(q_h, G, C_p\right) := q_h\cdot G\cdot C_p\cdot psf$$

$$G := 0.85$$

$$L := 50ft$$

$$\frac{h}{L} = 0.344$$

$$\frac{-0.9 - -0.7}{0.5 - 0.25} = \frac{-0.9 - C_p}{0.5 - \frac{h}{L}}$$

$$C_p := -0.9 - \frac{-0.9 - -0.7}{0.5 - 0.25}\cdot\left(0.5 - \frac{h}{L}\right)$$

$$C_p = -0.775$$

$$p_{Windward} := p\left(\frac{q_h}{psf}, G, C_p\right) = -10.367\cdot psf$$

External pressure on leeward side of roof

$$\frac{-0.5 - -0.3}{0.5 - 0.25} = \frac{-0.5 - C_p}{0.5 - \frac{h}{L}}$$

$$C_p := -0.5 - \frac{-0.5 - -0.3}{0.5 - 0.25}\cdot\left(0.5 - \frac{h}{L}\right)$$

$$C_p = -0.375$$

$$p_{Leeward} := p\left(\frac{q_h}{psf}, G, C_p\right) = -5.018\cdot psf$$

Internal pressure

$$p_{int}\left(q_h, G, C_{pi}\right) := -q_h\cdot G\cdot C_{pi}$$

$$C_{pi1} := 0.18$$

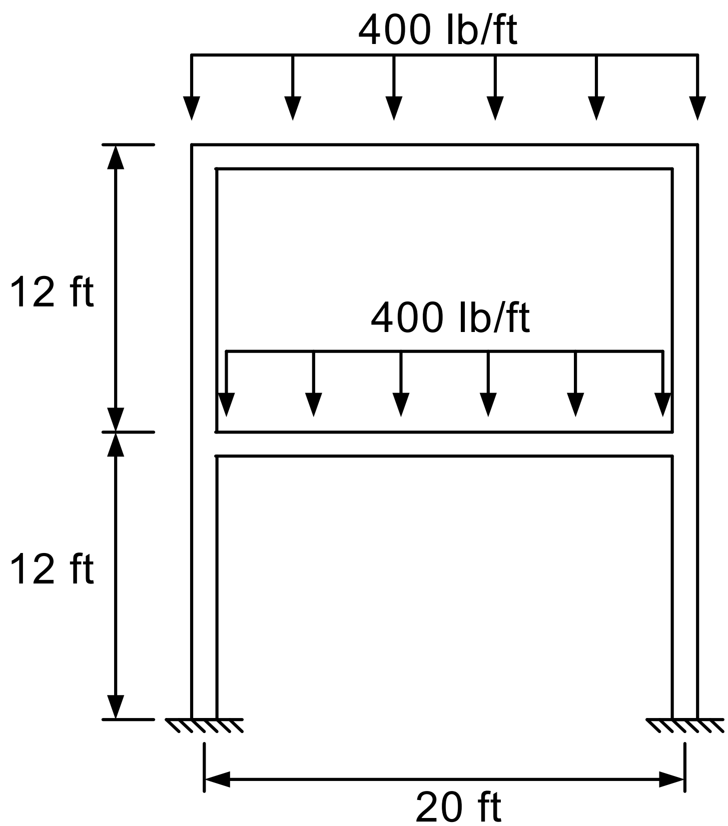
$$C_{pi2} := -0.18$$

$$p_{int1} := p_{int}\left(q_h, G, C_{pi1}\right) = -2.407\cdot psf$$

$$p_{int2} := p_{int}\left(q_h, G, C_{pi2}\right) = 2.407\cdot psf$$

4.3 Earthquake Load on a Frame

Determine the approximate earthquake forces for the ductile hospital frame structure shown below. The dead load for each floor is $DL := 200\frac{lbf}{ft}$ and the live load is $LL := 400\frac{lbf}{ft}$. The structure is built on soft clay. Use DL+0.5LL as the weight of each floor. The building is in seismic zone 3.



Solution:

1. The fundamental period of vibration is

$$T\left(C_t, h_n\right) := C_t \cdot h_n^{\frac{3}{4}} \cdot sec$$

$$C_t := 0.03$$

$$h := 24ft$$

$$T := T\left(C_t, \frac{h}{ft}\right) = 0.325s$$

2. The C coefficient is

$$C(S,T) := \frac{1.25 \cdot S}{T^{\frac{2}{3}}}$$

$$S := 2.0$$

$$C := C\left(S, \frac{T}{sec}\right) = 5.286 > 2.75$$

Use $C := 2.75$

3. The other coefficients are $Z := 0.3$, $I := 1.25$, $R_W := 12$

4. Check

$$\frac{C}{R_W} = \frac{2.75}{12} = 0.23 > 0.075 \text{ OK!}$$

5. The total vertical load is

$L := 20\text{ft}$ frame width

$$W := 2\cdot(DL + 0.5\cdot LL)\cdot L = 16000\cdot\text{lbf}$$

6. The total seismic base shear is

$$V := \frac{Z\cdot I\cdot C}{R_W}\cdot W = 1375\cdot\text{lbf}$$

7. Since T < 0.7sec, there is no whiplash

8. The toal load on each floor is given by

$$h_1 := 12\text{ft}$$

$$h_2 := 24\text{ft}$$

$$F_2 := \frac{V\cdot h_2}{h_1 + h_2} = 916.7\cdot\text{lbf}$$

$$F_1 := \frac{V\cdot h_1}{h_1 + h_2} = 458.3\cdot\text{lbf}$$

4.4 Earthquake Load on a Tall Building

Determine the approximate critical lateral loading for a 25 storey, ductile, rigid space frame concrete structure in the short direction. The rigid frames are spaced 25 ft apart in the cross section and 20 ft in the longitudinal direction. The plan dimension of the building is 175x100 ft, and the structure is 25(12ft)=300 ft high. This office building is located in an urban environment with a wind velocity of $V := 70\text{mph}$ and in seismic zone 4. For this investigation, an average building total dead load of $DL := 192\text{psf}$ is used. Soil conditions are unknown.

Solution:

1. The total building weight is

$$L := 100\text{ft}$$

$$B := 175\text{ft}$$

$$W := DL \cdot L \cdot B \cdot 25 = 84000 \cdot \text{kip}$$

2. The fundamental period of vibration for a rigid frame is

$$T(C_t, h_n) := C_t \cdot h_n^{\frac{3}{4}} \cdot \text{sec}$$

$$C_t := 0.03$$

$$h := 25 \cdot 12\text{ft} = 300 \cdot \text{ft}$$

$$T := T\left(C_t, \frac{h}{\text{ft}}\right) = 2.16\,\text{s} \;\; > 0.7\text{s} \;\; \text{OK!}$$

3. The C coefficient is

$$C(S,T) := \frac{1.25 \cdot S}{T^{\frac{2}{3}}}$$

$$S := 1.5$$

$$C := C\left(S, \frac{T}{\text{sec}}\right) = 1.12 \;\; < 2.75$$

4. The other coefficients are $Z := 0.4,\ I := 1,\ R_W := 12$

5. Check

$$\frac{C}{R_W} = \frac{1.12}{12} = 0.093 > 0.075 \;\; \text{OK!}$$

6. The total seismic base shear along the critical short direction is

$$V := \frac{Z \cdot I \cdot C}{R_W} \cdot W = 3139 \cdot \text{kip}$$

7. Since T > 0.7sec, the whiplash effect must be considered

$$F_t := 0.07 \cdot \frac{T}{\text{sec}} \cdot V = 475 \cdot \text{kip}$$

Hence the total triangular load is

$$F_{tot} := V - F_t = 2664 \cdot \text{kip}$$

8. Let us check if wind load governs. From table xx we conservatively assume a uniform wind pressure of 29 psf resulting in a total lateral force of

$$p := 29\text{psf}$$

$$P_w := p \cdot h \cdot B = 1522.5 \cdot \text{kip} < 3108 \text{ kip}$$

The magnitude of the total seismic load is clearly larger than the totoal wind force

4.5 Hydrostatic Load

The basement of a building is 12 ft below grade. Ground water is located 9 ft below grade. What thickness concrete slab is required to exactly balance the hydrostatic uplift?

Solution:

The hydrostatic pressure must be countered by the pressure caused by the weight of the concrete. Since p=γh we equate the two pressures and solve for h, the height of the concrete slab.

$h := 12ft$

$h_{water} := 9ft$

$d := h - h_{water} = 3 \cdot ft$

$\gamma_{water} := 62.4 \frac{lbf}{ft^2}$

$\gamma_{concrete} := 150 \frac{lbf}{ft^2}$

$\gamma_{water} \cdot d = \gamma_{concrete} \cdot h$

$h := \frac{\gamma_{water} \cdot d}{\gamma_{concrete}} = 15 \cdot in$

4.6 Thermal Expansion/Stress

A low-rise building is enclosed along one side by a 100 ft-long clay masonry ($\alpha := 3.6\cdot10^{-6}\frac{in}{in\cdot\Delta°F}$, $E := 2400000psi$)bearing wall. The structure was built at a temperature of $T := 60\Delta°F$ and is located in the northern part of the United States where the temperature range is between $T_{low} := -20\cdot\Delta°F$ and $T_{high} := 120\cdot\Delta°F$.

Solution:

1. Assume that the wall can move freely with no restraint from cross-walls and foundations. The wall expansion and contraction (summer and winter) are given by

$$\Delta L = \alpha\cdot\Delta T\cdot L$$

$$\Delta L_{Summer} := \alpha\cdot\left(T_{high} - T\right)\cdot 100ft = 0.26\cdot in$$

$$\Delta L_{Winter} := \alpha\cdot\left(T_{low} - T\right)\cdot 100ft = -0.35\cdot in$$

2. We now assume (conservatively) that the free movement cannot occur (ΔL=0) hence the resulting stress would be equal to

$$\sigma = E\cdot\varepsilon = E\cdot\frac{\Delta L}{L} = E\cdot\frac{\alpha\cdot\Delta T\cdot L}{L} = E\cdot\alpha\cdot\Delta T$$

$$\sigma = E\cdot\alpha\cdot\Delta T$$

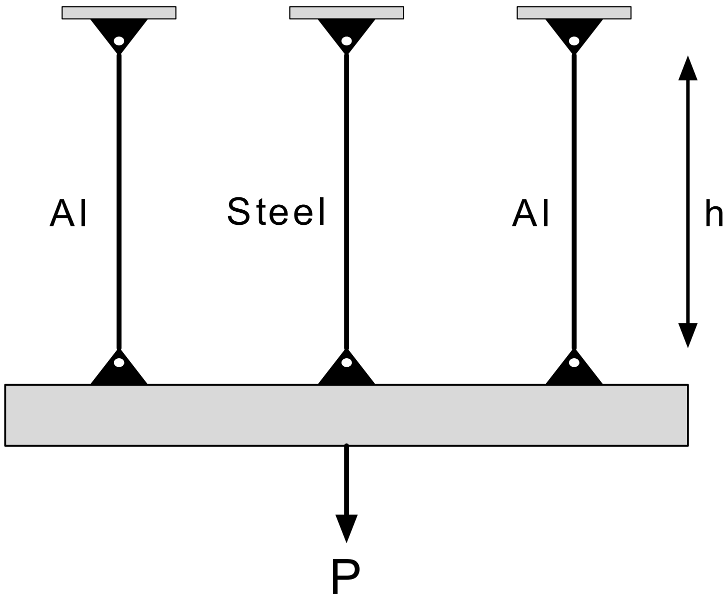
$$\sigma_{Summer} := E\cdot\alpha\cdot\left(T_{high} - T\right) = 518\cdot psi$$

$$\sigma_{Winter} := E\cdot\alpha\cdot\left(T_{low} - T\right) = -691\cdot psi$$

Note the tensile stress being beyond the masonry capacity. Cracking will occur.

5.1 Statically Indeterminate Cable Structures

A rigid plate is supported by two aluminum cables and a steel one. Determine the force in each cable.



If the rigid plate supports a load P, determine the stress in each of the three cables.

Solution:

1. We have three unknowns and only two independent equations of equilibrium. Hence the problem is statically indeterminate to the first degree.

$$\sum M_z = 0 \quad P_{Al}^{left} = P_{Al}^{right}$$

$$\sum F_y = 0 \quad 2P_{Al} + P_{St} = P$$

Thus we effectively have two unknowns and one equation

2. We need to have a third equation to solve for the three unknowns. This will be derived from the compatability of the displacements in all three cables i.e. all three displacements must be equal:

$$\sigma = \frac{P}{A}$$

$$\varepsilon = \frac{\Delta L}{L}$$

$$\varepsilon = \frac{\sigma}{E}$$

Combine to obtain

$$\Delta L = \frac{PL}{AE}$$

$$\frac{P_{Al}\cdot L}{E_{Al}\cdot A_{Al}} = \frac{P_{St}\cdot L}{E_{St}\cdot A_{St}} \quad \frac{P_{Al}}{P_{St}} = \frac{(EA)_{Al}}{(EA)_{St}}$$

or $-(EA)_{St}\cdot P_{Al} + (EA)_{Al}\cdot P_{St} = 0$

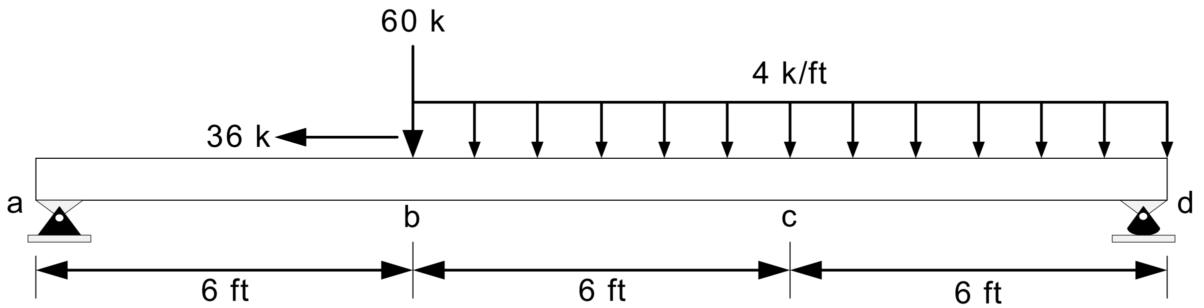
3. Solution of this system of two equations with two unknowns yield:

$$\begin{bmatrix} 2 & 1 \\ -(EA)_{St} & (EA)_{Al} \end{bmatrix} \cdot \begin{pmatrix} P_{Al} \\ P_{St} \end{pmatrix} = \begin{pmatrix} P \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} P_{Al} \\ P_{St} \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ -(EA)_{St} & (EA)_{Al} \end{bmatrix}^{-1} \cdot \begin{pmatrix} P \\ 0 \end{pmatrix} = \frac{1}{2 \cdot (EA)_{Al} + (EA)_{St}} \cdot \begin{bmatrix} (EA)_{Al} & -1 \\ (EA)_{St} & 2 \end{bmatrix} \cdot \begin{pmatrix} P \\ 0 \end{pmatrix}$$

5.2 Simply Supported Beam

Determine the reactions of the simply supported beam shown below.



The beam has 3 reactions. We have 3 equations of static equilibrium. Hence it is statically determinate.

Solution:

$$\sum F_x = 0 \quad R_{ax} - 36\text{kip} = 0$$

$$\sum F_y = 0 \quad R_{ay} + R_{dy} - 60\text{kip} - 4\frac{\text{kip}}{\text{ft}}\cdot 12\text{ft} = 0$$

$$\sum M_z{}^c = 0 \quad 12\cdot R_{ay} - 6\cdot R_{dy} - 60\text{kip}\cdot 6\text{ft} = 0$$

or

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 12 & -6 \end{pmatrix}\cdot\begin{pmatrix} R_{ax} \\ R_{ay} \\ R_{dy} \end{pmatrix} = \begin{pmatrix} 36\text{kip} \\ 108\text{kip} \\ 360\text{kip} \end{pmatrix}$$

$$\begin{pmatrix} R_{ax} \\ R_{ay} \\ R_{dy} \end{pmatrix} = \begin{pmatrix} 36\text{kip} \\ 56\text{kip} \\ 52\text{kip} \end{pmatrix}$$

Alternatively we could have used another set of equations:

$$\sum M_z{}^a = 0 \quad 60\text{kip}\cdot 6\text{ft} + 4\frac{\text{kip}}{\text{ft}}\cdot 12\text{ft}\cdot 12\text{ft} - R_{dy}\cdot 18\text{ft} = 0 \quad R_{dy} = 52\text{kip}$$

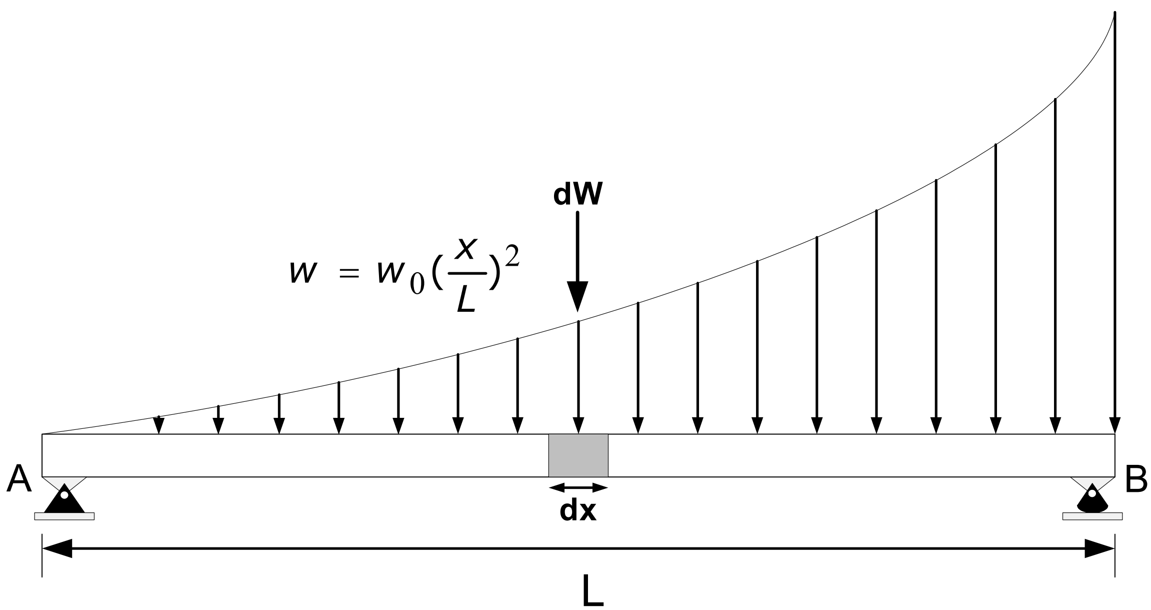
$$\sum M_z{}^d = 0 \quad R_{ay}\cdot 18\text{ft} - 60\text{kip}\cdot 12\text{ft} - 4\frac{\text{kip}}{\text{ft}}\cdot 12\text{ft}\cdot 12\text{ft} = 0 \quad R_{ay} = 56\text{kip}$$

Check:

$$\sum F_y = 0 \quad 56\text{kip} + 52\text{kip} - 60\text{kip} - 48\text{kip} = -0\cdot\text{kip}$$

5.3 Parabolic Load

Determine the reactions of the simply supported beam of length L subjected to a parabolic load $w = w_0 \cdot \left(\frac{x}{L}\right)^2$



Solution:

Since there are no axial forces, there are two unknowns and two equations of equilibrium. We have two equations of equilibrium ($\sum F_y$ and $\sum M$), we judiciously start with the second one, as it would directly give us the reaction at B. Considering an infinitesimal element of

$$\sum M_z{}^A = 0 \qquad \int_{x=0}^{x=L} w_0 \cdot \left(\frac{x}{L}\right)^2 dx \times x - R_B \cdot L = 0$$

$$R_B = \frac{1}{L} \cdot w_0 \cdot \left(\frac{L^4}{4L^2}\right) = \frac{1}{4} \cdot w_0 \cdot L$$

With R_B determined, we solve for R_A from

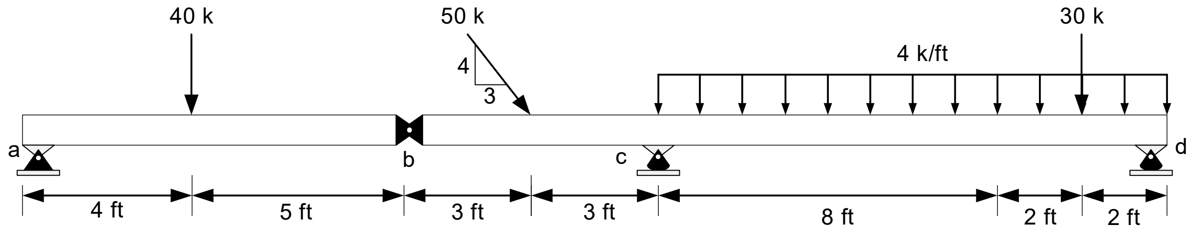
$$\sum F_y = 0$$

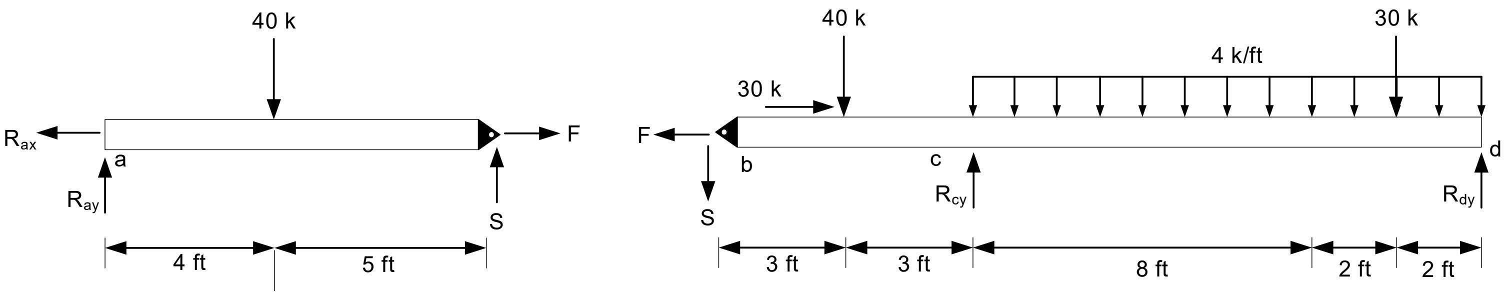
$$R_A + \frac{1}{4} \cdot w_0 \cdot L - \int_{x=0}^{x=L} w_0 \cdot \left(\frac{x}{L}\right)^2 dx = 0$$

$$R_A = \frac{w_0}{L^2} \cdot \frac{L^3}{3} - \frac{1}{4} \cdot w_0 \cdot L = \frac{1}{12} \cdot w_0 \cdot L$$

5.4 Three Span Beam

Determine the reactions of the following three span beam





Solution:

We have 4 unknowns (R_{ax},R_{ay},R_{cy}, and R_{dy}), three equations of equilibrium and one equation of condition ($\sum M_b = 0$), thus the structure is statically determinate. Though there are many approaches to solve for those four unknowns (all of them correct), a few are simpler to pursue. In this case, it is easiest to "break" the structure into substructures and examine the free body diagram of each one of them separately.

1. Isolating ab:

$$\sum M_z^{\,b} = 0 \qquad 9ft \cdot R_{ay} - 40kip \cdot 5ft = 0 \qquad R_{ay} := \frac{40kip \cdot 5ft}{9ft} = 22.2 \cdot kip$$

$$\sum M_z^{\,a} = 0 \qquad 40kip \cdot 4ft - S \cdot 9ft = 0 \qquad S := \frac{40kip \cdot 4ft}{9ft} = 17.8 \cdot kip$$

$$\sum F_x = 0 \qquad R_{ax} := 30kip$$

2. Isolating bd:

$$\sum M_z^{\,d} = 0 \qquad -S \cdot 18ft - 40kip \cdot 15ft - 4\frac{kip}{ft} \cdot 12ft \cdot 6ft - 30kip \cdot 2ft + R_{cy} \cdot 12ft = 0$$

$$R_{cy} := \frac{S \cdot 18ft + 40kip \cdot 15ft + 4\frac{kip}{ft} \cdot 12ft \cdot 6ft + 30kip \cdot 2ft}{12ft} = 105.7 \cdot kip$$

$$\sum M_z^{\,c} = 0 \qquad -S \cdot 6ft - 40kip \cdot 3ft + 4\frac{kip}{ft} \cdot 12ft \cdot 6ft + 30kip \cdot 10ft - R_{dy} \cdot 12ft = 0$$

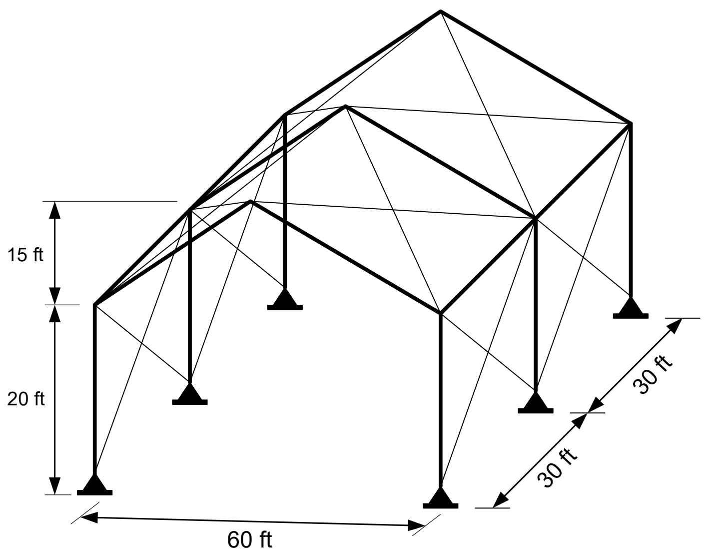
$$R_{dy} := \frac{-S \cdot 6ft - 40kip \cdot 3ft + 4\frac{kip}{ft} \cdot 12ft \cdot 6ft + 30kip \cdot 10ft}{12ft} = 30.1 \cdot kip$$

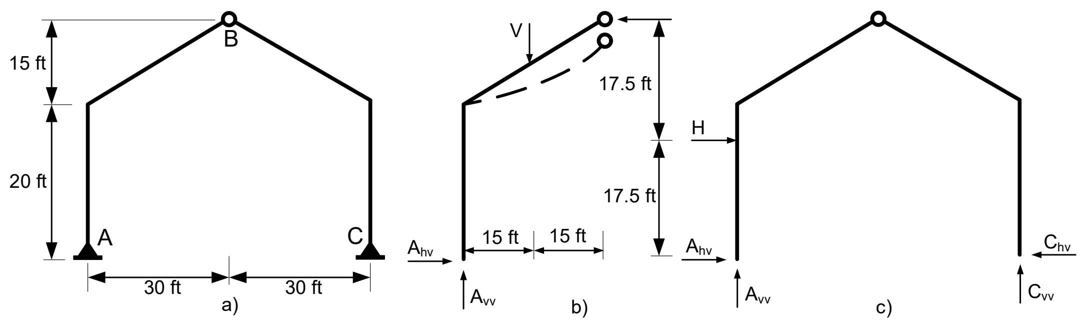
3. Check

$\sum F_y = 0$ $\qquad R_{ay} - 40kip - 40kip + R_{cy} - 4\frac{kip}{ft}\cdot 12ft - 30kip + R_{dy} = -0\cdot kip$ OK!

5.5 Three Hinged Gable Frame

The three-hinged gable frames are spaced at 30 ft on center. Determine the reactions components on the frame due to: 1) Roof dead load of 20 psf of roof area; 2) Snow load of 30 psf of horizontal projection; 3) Wind load of 15 psf of vertical projection. Determine the critical design values for the horizontal and vertical reactions.





Solution:

1. Due to smmetry, there is no vertical force transmitted by the hinge for snow and dead load, and thus we can consider only the left (or right) side of the frame.
2. Point equivalent loads:
   a) Roof dead load per one side of frame is

$$DL := 20psf\cdot 30ft\cdot\sqrt{(30ft)^2+(15ft)^2} = 20.12\cdot kip$$

   b) Snow load per one side of frame is

$$SL := 30psf\cdot 30ft\cdot 30ft = 27\cdot kip$$

   c) Wind load per per frame (ignoring the suction) is

$$WL := 15psf\cdot 30ft\cdot 35ft = 15.75\cdot kip$$

3. There are 4 reactions, 3 equations of equilibrium and one equation of condition; therefore, statically determinate. Alternatively, by symmetry there is no shear at the hinge, and we would have for the substructure two reactions at the support and one (horizontal) at the hinge.
4. The relationship between the horizontal and vertical reactions at A due to a centered vertical load, A_{hv} and A_{vv} respectively is determined by taking the moment with respect to the hinge (b):

$$\sum M_z^{\,B} = 0 \qquad 15\cdot V - 30\cdot A_{vv} + 35\cdot A_{hv} = 0$$

$$\sum F_y = 0 \qquad A_{vv} - V = 0$$

$$A_{hv} = \frac{15A_{vv}}{35}$$

Substituting for roof dead and snow load we obtain:

$A_{vvDL} := DL = 20.12 \cdot kip$ $\quad B_{vvDL} := A_{vvDL} = 20.12 \cdot kip$

$A_{hvDL} := \frac{15A_{vvDL}}{35} = 8.62 \cdot kip$ $\quad B_{hvDL} := A_{hvDL} = 8.62 \cdot kip$

$A_{vvSL} := SL = 27 \cdot kip$ $\quad B_{vvSL} := A_{vvSL} = 27 \cdot kip$

$A_{hvSL} := \frac{15A_{vvSL}}{35} = 11.57 \cdot kip$ $\quad B_{hvSL} := A_{hvSL} = 11.57 \cdot kip$

5. The reactions due to wind load (blowing from left) are determined as follows:
   a) Vertical reaction at A is determined by considering the entire structure and taking the moment with respect to C (c)

$$\sum M_z{}^C = 0 \qquad 15.75kip \cdot \left(\frac{20ft + 15ft}{2}\right) - 60ft \cdot A_{vh} = 0$$

$$A_{vh} := \frac{WL \cdot \left(\frac{20ft + 15ft}{2}\right)}{60ft} = 4.59 \cdot kip$$

A_{vh} is the vertical reaction at A due to the horizontal load, and from equilibrium of forces in the y-direction, we have

$$C_{vh} := -A_{vh} = -4.59 \cdot kip$$

(note that wind load does not have any vertical component)

   b) The horizontal reaction at B is determined by considering the right substructure and taking the moment with respect to the internal hinge at B

$$\sum M_z{}^B = 0 \qquad 35ft \cdot C_{hh} - C_{vh} \cdot 30ft = 0$$

$$C_{hh} := \frac{C_{vh} \cdot 30ft}{35ft} = -3.94 \cdot kip$$

   c) Horizontal reactino at A is taken by considering the entire structure and summing forces in the x-direction

$$\sum F_x = 0 \qquad WL + C_{hh} - A_{hh} = 0$$

$$A_{hh} := WL + C_{hh} = 11.81 \cdot kip$$

and note that A carries most of the horizontal load.

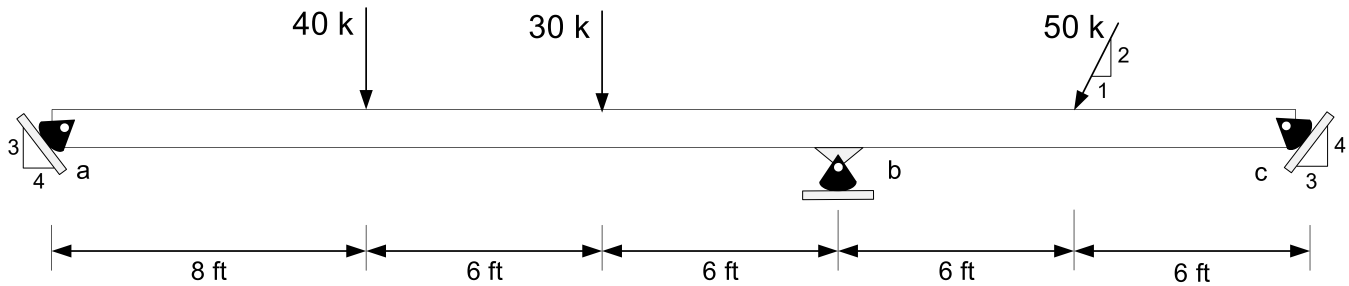
6. Finally, the supports should be designed for the most critical (plausible) combination of reactions

$$H := A_{hvDL} + A_{hvSL} + A_{hh} = 32.01 \cdot kip$$

$$V := A_{vvDL} + A_{vvSL} + A_{vh} = 51.72 \cdot kip$$

5.6 Inclined Supports

Determine the reactions of the following two span beam resting on inclined supports.



Solution:

A priori we would identify 5 reactions; however, we do have 2 equations of condition (one at each inclined support), thus with three equations of equilibrium, we have a statically determinate system.

$$\sum M_z{}^b = 0 \qquad R_{ay}\cdot 20ft - 40kip\cdot 12ft - 30kip\cdot 6ft + 50kip\cdot \sin\left(\operatorname{atan}\left(\frac{2}{1}\right)\right)\cdot 6ft - R_{cy}\cdot 12ft = 0$$

$$R_{ay}\cdot 20ft - R_{cy}\cdot 12ft = 40kip\cdot 12ft + 30kip\cdot 6ft - 50kip\cdot \sin\left(\operatorname{atan}\left(\frac{2}{1}\right)\right)\cdot 6ft$$

$$\sum F_x = 0 \qquad \frac{3}{4}R_{ay} - 50kip\cdot\cos\left(\operatorname{atan}\left(\frac{2}{1}\right)\right) - \frac{4}{3}\cdot R_{cy} = 0$$

$$R_{cy} = \frac{3}{4}\cdot\left(\frac{3}{4}R_{ay} - 50kip\cdot\cos\left(\operatorname{atan}\left(\frac{2}{1}\right)\right)\right) = \frac{9}{16}\cdot R_{ay} - 16.77kip$$

$$\frac{9}{16}\cdot R_{ay} - R_{cy} = 16.77kip$$

Solving for those two equations:

$$\begin{pmatrix} 20 & -12 \\ \frac{9}{16} & -1 \end{pmatrix}\cdot\begin{pmatrix} R_{ay} \\ R_{cy} \end{pmatrix} = \begin{pmatrix} 391.672\ kip \\ 16.77kip \end{pmatrix}$$

$$\begin{pmatrix} R_{ay} \\ R_{cy} \end{pmatrix} := \begin{pmatrix} 14.37kip \\ -8.69kip \end{pmatrix}$$

The horizontal components of the reactions at a and c are

$$R_{ax} := \frac{3}{4}\cdot R_{ay} = 10.78\cdot kip$$

$$R_{cx} := -\frac{4}{3}\cdot R_{cy} = 11.59\cdot kip$$

Finally, we solve for R_{by}

$$\sum M_z{}^a = 0 \qquad 40kip\cdot 8ft + 30kip\cdot 14ft - R_{by}\cdot 20ft + 50kip\cdot\sin\left(\operatorname{atan}\left(\frac{2}{1}\right)\right)\cdot 26ft - R_{cy}\cdot 32ft = 0$$

$$R_{by} := \frac{40kip\cdot 8ft + 30kip\cdot 14ft + 50kip\cdot\sin\left(\operatorname{atan}\left(\frac{2}{1}\right)\right)\cdot 26ft - R_{cy}\cdot 32ft}{20ft} = 109.04\cdot kip$$

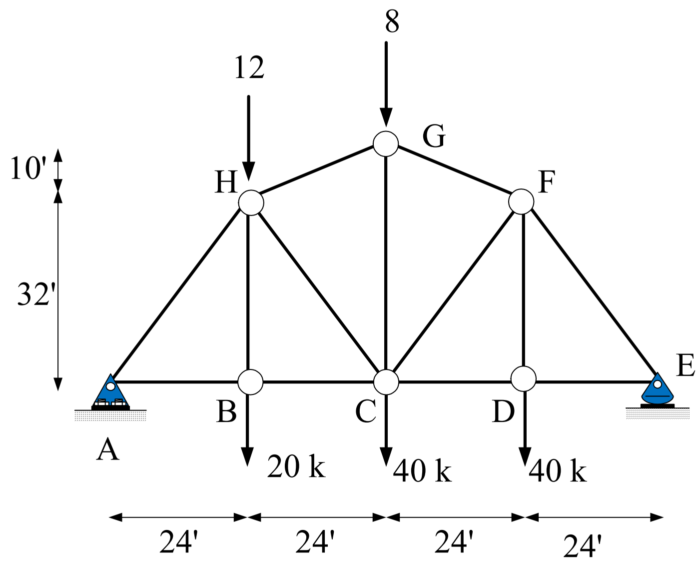
We check our results

$$\sum F_y = 0 \qquad R_{ay} - 40kip - 30kip - 50kip\cdot\sin\left(\operatorname{atan}\left(\frac{2}{1}\right)\right) + R_{by} + R_{cy} = 0\cdot kip \qquad \text{OK!}$$

$$\sum F_x = 0 \qquad R_{ax} - 50kip\cdot\cos\left(\operatorname{atan}\left(\frac{2}{1}\right)\right) + R_{cx} = 0\cdot kip \qquad \text{OK!}$$

7.1 Truss Method of Joints

Using the method of joints, analyze the following truss



Solution:

1. R=3, m=13, 2j=16, and m+R=2j
2. We compute the reactions

$$\sum M_z^E = 0 \qquad (20kip + 12kip)\cdot 72ft + (40kip + 8kip)\cdot 48ft + 40kip\cdot 24ft - R_{Ay}\cdot 96ft = 0$$

$$R_{Ay} := \frac{(20kip + 12kip)\cdot 72ft + (40kip + 8kip)\cdot 48ft + 40kip\cdot 24ft}{96ft}$$

$$\boxed{R_{Ay} = 58\cdot kip}$$

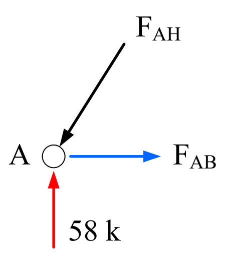
$$\sum F_y = 0 \qquad 20kip + 12kip + 40kip + 8kip + 40kip - R_{Ay} - R_{Ey} = 0$$

$$R_{Ey} := 20kip + 12kip + 40kip + 8kip + 40kip - R_{Ay}$$

$$\boxed{R_{Ey} = 62\cdot kip}$$

3. Consider each joint separately:

Node A: Clearly AH is under compression and AB is under tension



$$\sum F_y = 0 \qquad -F_{AHy} + R_{Ay} = 0$$

$$F_{AHy} := -R_{Ay} = -58\cdot kip$$

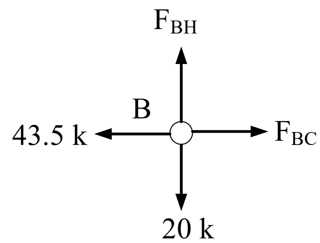
$$L_y := 32ft \qquad L_x := 24ft \qquad L := \sqrt{L_y^2 + L_x^2} = 40\cdot ft$$

$$F_{AH} := \frac{L}{L_y}\cdot F_{AHy} = -72.5\cdot kip$$ Compression

$\sum F_x = 0 \qquad -F_{AHx} + F_{AB} = 0$

$F_{AB} := \frac{L_x}{L_y} \cdot R_{Ay} = 43.5 \cdot kip$ Tension

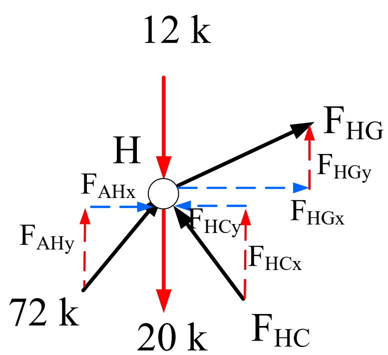
Node B:



$\sum F_x = 0 \qquad \boxed{F_{BC} := 43.5kip}$ Tension

$\sum F_y = 0 \qquad \boxed{F_{BH} := 20kip}$ Tension

Node H:



$\sum F_x = 0 \qquad F_{AHx} - F_{HCx} - F_{HGx} = 0$

$$43.5kip - \frac{24ft}{\sqrt{(24ft)^2 + (32ft)^2}} \cdot F_{HC} - \frac{24ft}{\sqrt{(24ft)^2 + (10ft)^2}} \cdot F_{HG} = 0$$

$\sum F_y = 0 \qquad F_{AHy} + F_{HCy} - 12kip - F_{HGy} - 20kip = 0:$

$$58kip + \frac{32ft}{\sqrt{(24ft)^2 + (32ft)^2}} \cdot F_{HC} - \frac{10ft}{\sqrt{(24ft)^2 + (10ft)^2}} \cdot F_{HG} - 12kip - 20kip = 0$$

This can be most conveniently written as

$$\begin{bmatrix} \frac{24ft}{\sqrt{(24ft)^2 + (32ft)^2}} & \frac{24ft}{\sqrt{(24ft)^2 + (10ft)^2}} \\ -\frac{32ft}{\sqrt{(24ft)^2 + (32ft)^2}} & \frac{10ft}{\sqrt{(24ft)^2 + (10ft)^2}} \end{bmatrix} \cdot \begin{pmatrix} F_{HC} \\ F_{HG} \end{pmatrix} = \begin{pmatrix} 43.5kip \\ 26kip \end{pmatrix}$$

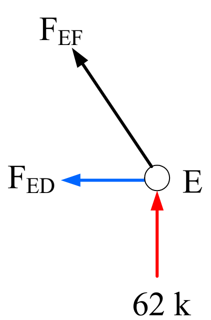
Solving we obtain

$$\begin{pmatrix} F_{HC} \\ F_{HG} \end{pmatrix} := \begin{bmatrix} \dfrac{24ft}{\sqrt{(24ft)^2 + (32ft)^2}} & \dfrac{24ft}{\sqrt{(24ft)^2 + (10ft)^2}} \\ -\dfrac{32ft}{\sqrt{(24ft)^2 + (32ft)^2}} & \dfrac{10ft}{\sqrt{(24ft)^2 + (10ft)^2}} \end{bmatrix}^{-1} \cdot \begin{pmatrix} 43.5kip \\ 26kip \end{pmatrix} = \begin{pmatrix} -7.5 \\ 52 \end{pmatrix} \cdot kip$$

$F_{HC} = -7.5 \cdot kip$ Compression

$F_{HG} = 52 \cdot kip$ Tension

Node E:



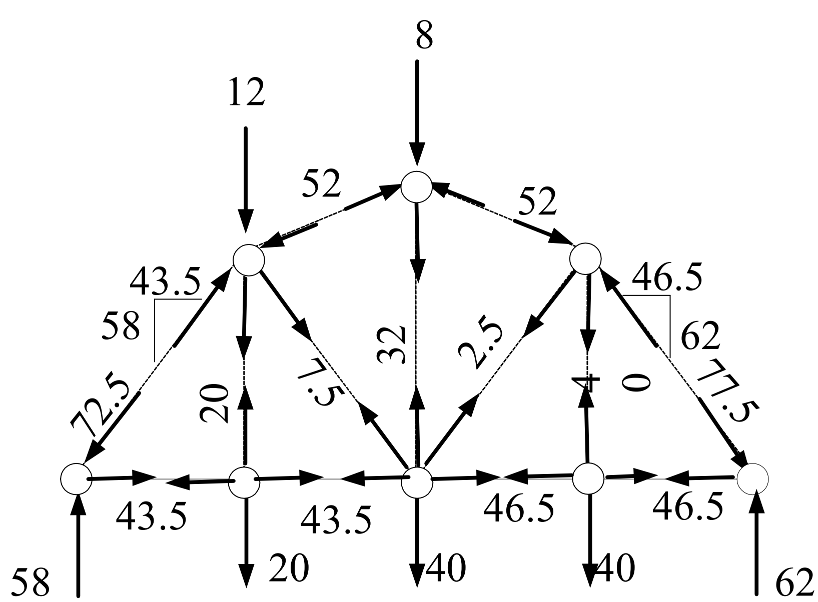
$\sum F_y = 0$ $\quad F_{EFy} := 62kip$

$$F_{EF} := \frac{\sqrt{(24ft)^2 + (32ft)^2}}{32ft} \cdot 62kip = 77.5 \cdot kip$$ Compression

$\sum F_x = 0$ $\quad F_{ED} = F_{EFx}$

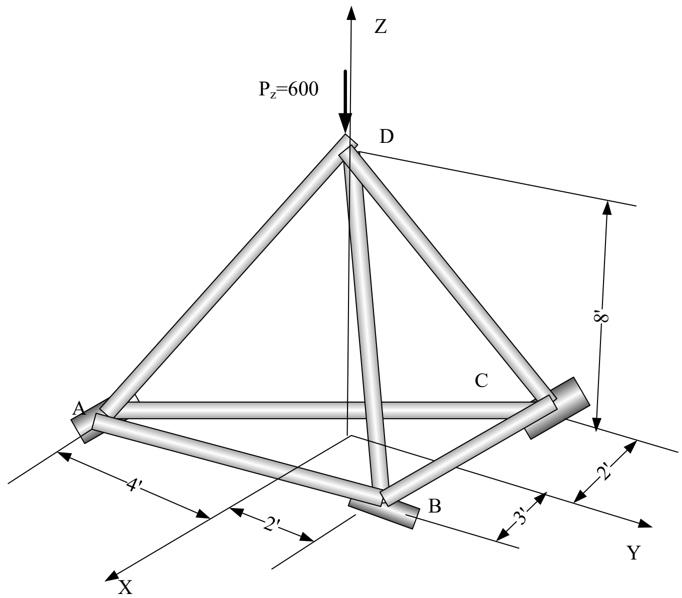
$$F_{ED} := 24\frac{ft}{32ft} \cdot 62kip = 46.5 \cdot kip$$ Tension

The results of the analysis are summarized below



4. We would check our calculations by verifying equilibrium of forces at a node not previously used, such as *D*.

7.2 3D Truss



Solution:

1. Consider the free body diagram of the entire truss

$\sum M_{AB} = 0 \qquad C_z \cdot 5ft - 600 \cdot 3ft = 0 \qquad C_z := \frac{600 \cdot 3ft}{5ft}$

$\boxed{C_z = 360}$

$\sum M_{CB} = 0 \qquad 600 \cdot 2ft - A_z \cdot 6ft = 0 \qquad A_z := \frac{600 \cdot 2ft}{6ft}$

$\boxed{A_z = 200}$

$\sum F_z = 0 \qquad B_z + 200 + 360 - 600 = 0 \qquad B_z := 600 - 200 - 360$

$\boxed{B_z = 40}$

$\sum F_x = 0 \qquad \boxed{B_x := 0}$

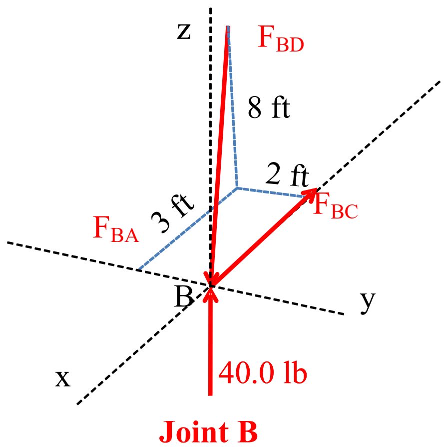
$\sum F_y = 0 \qquad A_y - C_y = 0$

$\sum M_z = 0 \qquad A_y \cdot 3ft + C_y \cdot 2ft = 0 \qquad A_y = C_y = 0$

$\boxed{A_y := 0}$

$\boxed{C_y := 0}$

2. Consider the free body diagram of joint B



$$L_{BD} := \sqrt{(2ft)^2 + (3ft)^2 + (8ft)^2} = 8.775 \cdot ft$$

$\sum F_z = 0$ $\quad \frac{-8ft}{L_{BD}} \cdot F_{BD} + 40 = 0$

$$\boxed{F_{BD} := 40 \cdot \frac{-L_{BD}}{8ft} = -43.875}$$ Compression

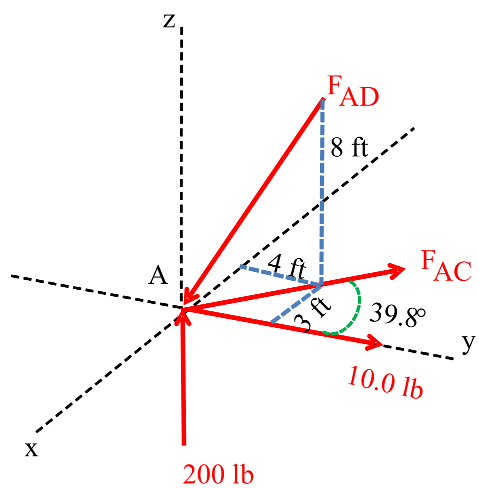
$\sum F_x = 0$ $\quad F_{BDx} - F_{BC} = 0$

$$\boxed{F_{BDx} := \frac{-3ft}{L_{BD}} \cdot F_{BD} = 15}$$ Tension

$\sum F_y = 0$ $\quad F_{BDy} - F_{BA} = 0$

$$\boxed{F_{BDy} := \frac{-2ft}{L_{BD}} \cdot F_{BD} = 10}$$ Tension

3. FBD of joint A



$L_{AD} := \sqrt{(8ft)^2 + (3ft)^2 + (4ft)^2} = 9.434 \cdot ft$

$\sum F_z = 0 \qquad \frac{-8ft}{L_{AD}} \cdot F_{AD} + 200 = 0$

$F_{AD} := 200 \cdot \frac{-L_{AD}}{8ft} = -235.85$ Compression

$\sum F_x = 0 \qquad F_{ADx} - F_{ACx} = 0$

$\tan(\alpha) = \frac{5ft}{6ft} \qquad \alpha := \operatorname{atan}\left(\frac{5}{6}\right) = 39.8 \cdot deg$

$\frac{3ft}{L_{AD}} \cdot F_{AD} - F_{AC} \cdot \sin(\alpha) = 0$

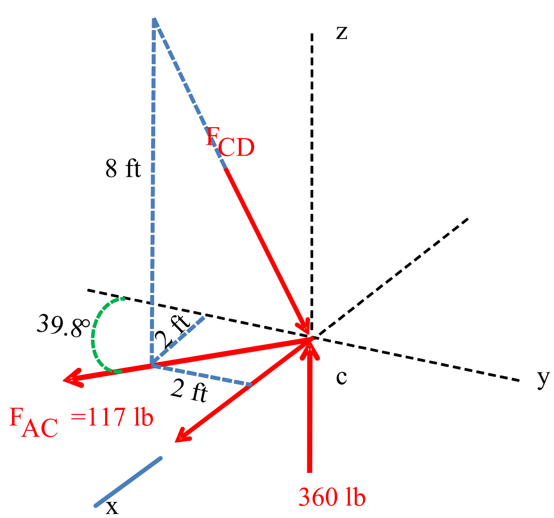
$F_{AC} := \frac{\frac{3ft}{L_{AD}} \cdot -F_{AD}}{\sin(39.8deg)} = 117.2$ Tension

$\sum F_y = 0 \qquad F_{BDy} - F_{BA} = 0$

$\tan(\alpha) = \frac{5ft}{6ft} \qquad \alpha := \operatorname{atan}\left(\frac{5}{6}\right) = 39.8 \cdot deg$

$F_{BA} := \frac{-2ft}{L_{BD}} \cdot F_{BD} = 10$ Tension

5. Joint C



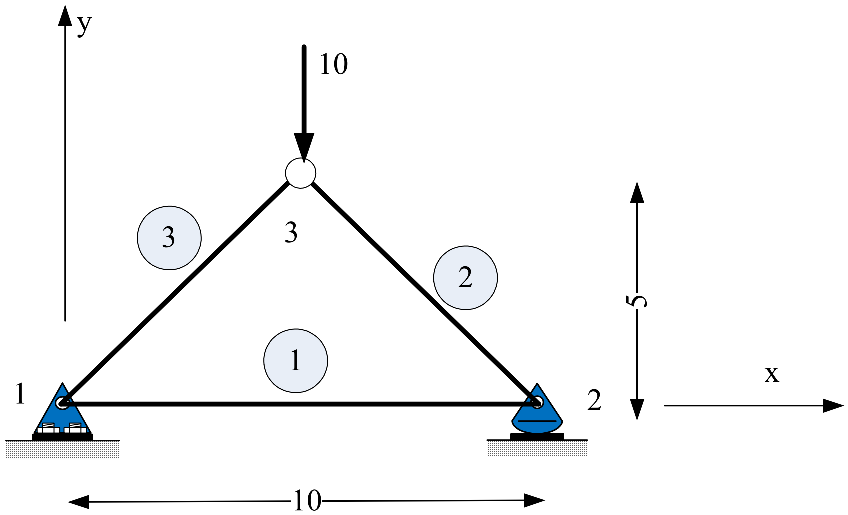
$L_{CD} := \sqrt{(8ft)^2 + (2ft)^2 + (2ft)^2} = 8.485 \cdot ft$

$\sum F_z = 0 \qquad F_{CDz} + 360 = 0 \qquad \frac{-8ft}{L_{CD}} \cdot F_{CD} + 360 = 0$

$F_{CD} := -360 \cdot \frac{-L_{CD}}{-8ft} = -381.8$ Compression

7.3 Truss I, Matrix Method

Determine all member forces for the following truss



Solution:

1. We first determine the direction cosines

Member 1 (Nodes 1-2)

Node1: $\alpha_{11} := 1$ $\beta_{11} := 0$ Node2: $\alpha_{21} := -1$ $\beta_{21} := 0$

Member 2 (Nodes 2-3)

Node2: $\alpha_{22} := \frac{-\sqrt{2}}{2} = -0.707$ $\beta_{22} := \frac{\sqrt{2}}{2} = 0.707$ Node3: $\alpha_{32} := \frac{\sqrt{2}}{2} = 0.707$ $\beta_{32} := \frac{-\sqrt{2}}{2} = -0.707$

Member 3 (Nodes 3-1)

Node2: $\alpha_{33} := \frac{-\sqrt{2}}{2} = -0.707$ $\beta_{33} := \frac{-\sqrt{2}}{2} = -0.707$ Node3: $\alpha_{31} := \frac{\sqrt{2}}{2} = 0.707$ $\beta_{31} := \frac{\sqrt{2}}{2} = 0.707$

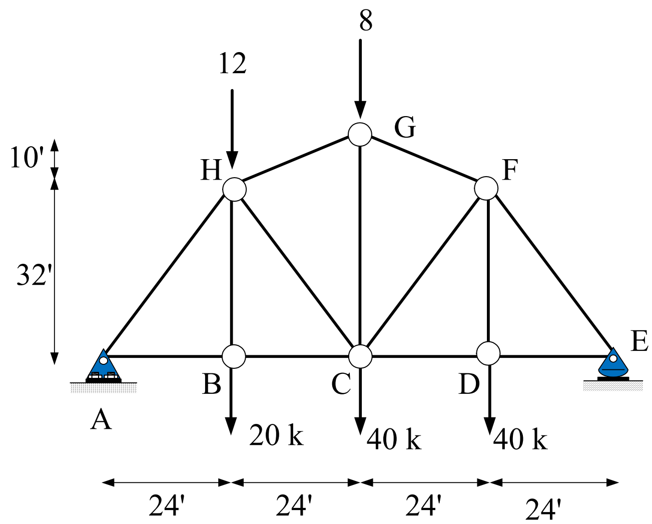
2. Next we write the equations of equilibrium

$$\begin{pmatrix} \alpha_{11} & 0 & \alpha_{31} & 1 & 0 & 0 \\ \beta_{11} & 0 & \beta_{31} & 0 & 1 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 0 & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 & 0 & 0 & 1 \\ 0 & \alpha_{32} & \alpha_{33} & 0 & 0 & 0 \\ 0 & \beta_{32} & \beta_{33} & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ R_{1x} \\ R_{1y} \\ R_{2y} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \end{pmatrix} = 0$$

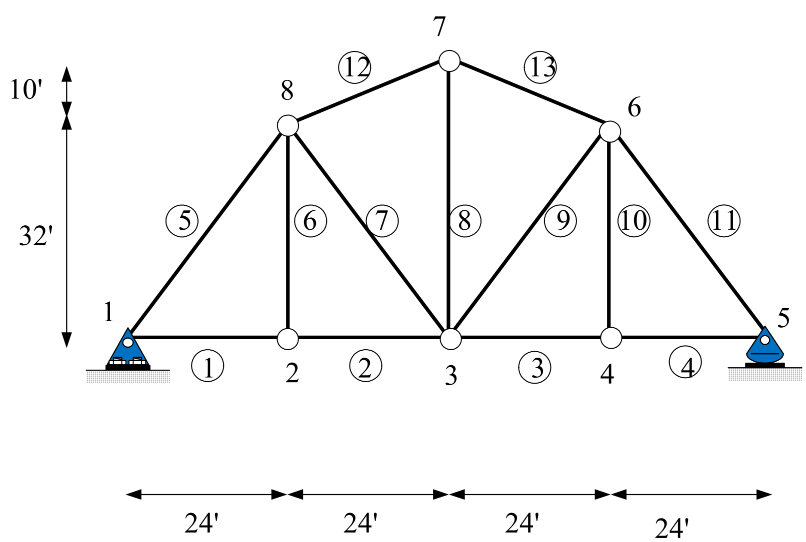
$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ R_{1x} \\ R_{1y} \\ R_{2y} \end{pmatrix} := \begin{pmatrix} \alpha_{11} & 0 & \alpha_{31} & 1 & 0 & 0 \\ \beta_{11} & 0 & \beta_{31} & 0 & 1 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 0 & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 & 0 & 0 & 1 \\ 0 & \alpha_{32} & \alpha_{33} & 0 & 0 & 0 \\ 0 & \beta_{32} & \beta_{33} & 0 & 0 & 0 \end{pmatrix}^{-1} \cdot - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \end{pmatrix} = \begin{pmatrix} 5 \\ -7.071 \\ -7.071 \\ 0 \\ 5 \\ 5 \end{pmatrix}$$

7.4 Truss II, Matrix Method

Set up the statics matrix for the truss shown below



Solution:



1. We first determine the direction cosines

$$\frac{24\text{ft}}{\sqrt{(24\text{ft})^2 + (32\text{ft})^2}} = 0.6$$

$$\frac{32\text{ft}}{\sqrt{(24\text{ft})^2 + (32\text{ft})^2}} = 0.8$$

$$\frac{10\text{ft}}{\sqrt{(24\text{ft})^2 + (10\text{ft})^2}} = 0.385$$

$$\frac{24\text{ft}}{\sqrt{(24\text{ft})^2 + (10\text{ft})^2}} = 0.923$$

2. Next we write the equations of equilibrium

2. Next we write the equations of equilibrium

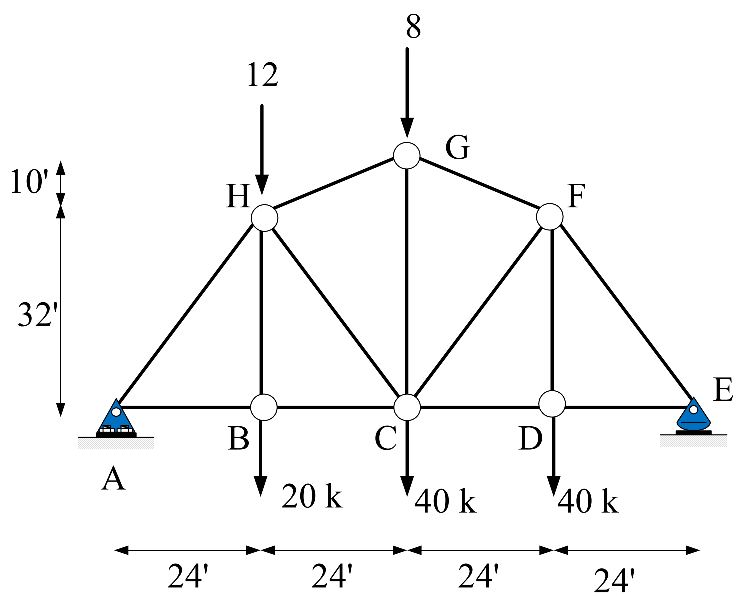
$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & -0.6 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 1 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6 & 0 & 0.6 & 0 & -0.923 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.8 & -1 & -0.8 & 0 & 0.385 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.923 & 0.923 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -0.385 & -0.385 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.6 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0.923 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.8 & -1 & -0.8 & 0 & 0 & 0 & 0 & 0.385 & 0 & 0 & 0 & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \\ F_{13} \\ R_{1x} \\ R_{1y} \\ R_{5y}
\end{pmatrix}
+
\begin{pmatrix}
0 \\ 0 \\ 0 \\ -20 \\ 0 \\ -40 \\ 0 \\ -40 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -8 \\ 0 \\ -12
\end{pmatrix}
= 0$$

$$
\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \\ F_{13} \\ R_{1x} \\ R_{1y} \\ R_{5y} \end{pmatrix} :=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & -0.6 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 1 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6 & 0 & 0.6 & 0 & -0.923 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.8 & -1 & -0.8 & 0 & 0.385 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.923 & 0.923 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -0.385 & -0.385 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.6 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0.923 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.8 & -1 & -0.8 & 0 & 0 & 0 & 0 & 0.385 & 0 & 0 & 0 & 0
\end{pmatrix}^{-1} \cdot -
\begin{pmatrix} 0 \\ 0 \\ 0 \\ -20 \\ 0 \\ -40 \\ 0 \\ -40 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -8 \\ 0 \\ -12 \end{pmatrix} =
$$

	1
1	43.5
2	43.5
3	46.5
4	46.5
5	-72.5
6	20
7	7.479
8	32.033
9	2.479
10	40
11	-77.5
12	-51.991
13	-51.991
14	0
15	58
16	62

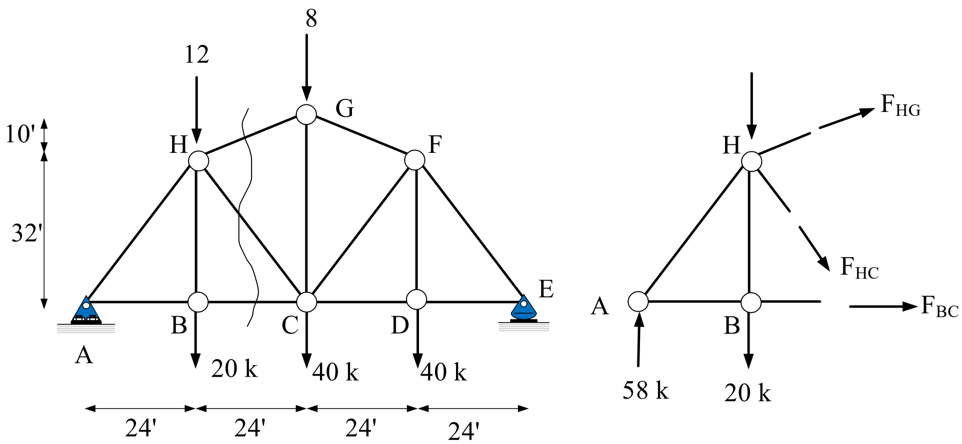
7.5 Truss, Method of Sections

Determine F_{BC} and F_{HG} in the truss below



Solution:

Cutting through members HG, HC, and BC, we first take the summation of forces with respect to H:



$\sum M_z^{H} = 0 \qquad R_{Ay} \cdot 24ft - F_{BC} \cdot 32ft = 0$

$R_{Ay} := 58kip$ (From previous example)

$F_{BC} := \frac{24ft}{32ft} \cdot R_{Ay} = 43.5 \cdot kip$ Tension

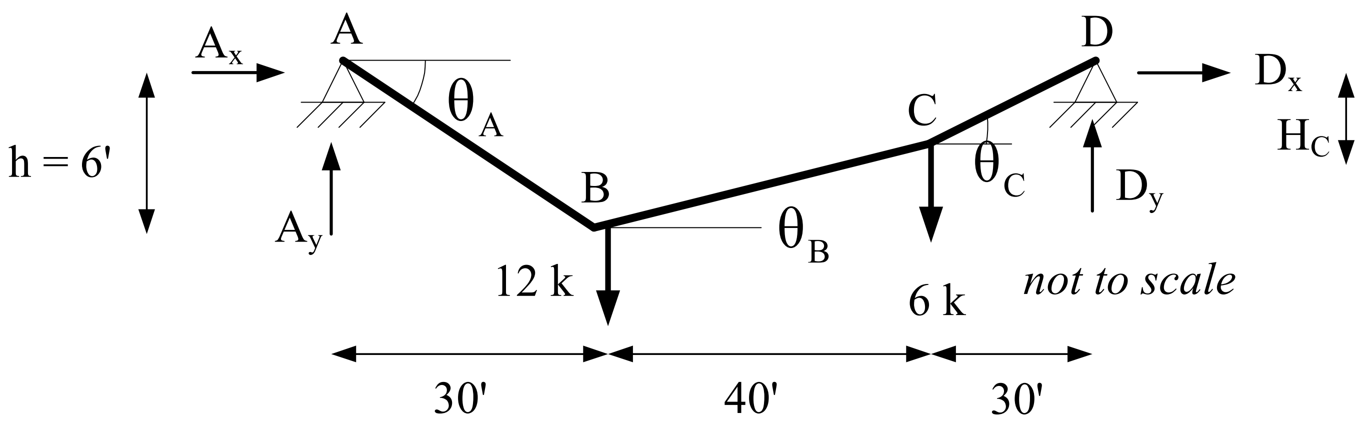
$\sum M_z^{C} = 0 \qquad 58kip \cdot 48ft - (20kip + 12kip) \cdot 24ft - F_{HGx} \cdot 32ft - F_{HGy} \cdot 24ft = 0$

$$58kip \cdot 48ft - (20kip + 12kip) \cdot 24ft - 32ft \cdot F_{HG} \cdot \frac{24ft}{\sqrt{(24ft)^2 + (10ft)^2}} - 24ft \cdot F_{HG} \cdot \frac{10ft}{\sqrt{(24ft)^2 + (10ft)^2}} = 0$$

$\boxed{F_{HG} := 52kip}$ Compression

8.1 Funicular Cable Structures

Determine the reactions and the tensions for the cable structure shown below



Solution:

We have 4 external reactions, however the horizontal ones are equal and we can use any one of a number of equations of conditions in additiona to the three equations of equilibrium. First, we solve for the vertical reactions A_y, D_y and then for the horizontal ones (which are equal and opposite ($|H| = A_x = -D_x$). For this problem we could use the following 3 equations of static equilibrium $\Sigma F_x = \Sigma F_y = \Sigma M = 0$, however since we do not have any force in the x direction, the first equation is of no avail. Instead, we will consider the following set $\Sigma F_y = \Sigma M_A = \Sigma M_B = 0$. Alternatively, we can consider the problem as one with 8 unknowns (A_x, A_y, D_x, D_y, θ_A, θ_B, θ_C, and h_C), to be solved through the 2 equations of equilibrium expressed at each of the four points of interesect equations of equilibrium expressed at each of the four points of interest.

1. Solve for D_y

$$\sum M_z^{A} = 0 \qquad 12kip \cdot 30ft + 6kip \cdot 70ft - D_y \cdot 100ft = 0$$

$$\boxed{D_y := \frac{12kip \cdot 30ft + 6kip \cdot 70ft}{100ft} = 7.8 \cdot kip}$$

2. Solve for A_y

$$\sum F_y = 0 \qquad A_y - 12kip - 6kip + D_y = 0$$

$$\boxed{A_y := 12kip + 6kip - D_y = 10.2 \cdot kip}$$

3. Solve for the horizontal force by isolating the free body diagram AB

$$\sum M_z^{B} = 0 \qquad A_y \cdot 30ft - H \cdot 6ft = 0$$

$$\boxed{H := \frac{A_y \cdot 30ft}{6ft} = 51 \cdot kip}$$

4. Solve for the sag at point C by isolating the free body diagram CD

$$\sum M_z^{C} = 0 \qquad -D_y \cdot 30ft + H \cdot h_c = 0$$

$$\boxed{h_c := \frac{30ft\, D_y}{H} = 4.6 \cdot ft}$$

5. Solve for the cable internal forces or tractions in this case

$$\tan(\theta_A) = \frac{6ft}{30ft} \qquad \theta_A := \operatorname{atan}\left(\frac{6ft}{30ft}\right) = 11.31 \cdot deg$$

$$\boxed{T_{AB} := \frac{H}{\cos(\theta_A)} = 52.01 \cdot kip}$$

$$\tan(\theta_B) = \frac{6ft - h_c}{40ft} \qquad \theta_B := \operatorname{atan}\left(\frac{6ft - h_c}{40ft}\right) = 2.02 \cdot deg$$

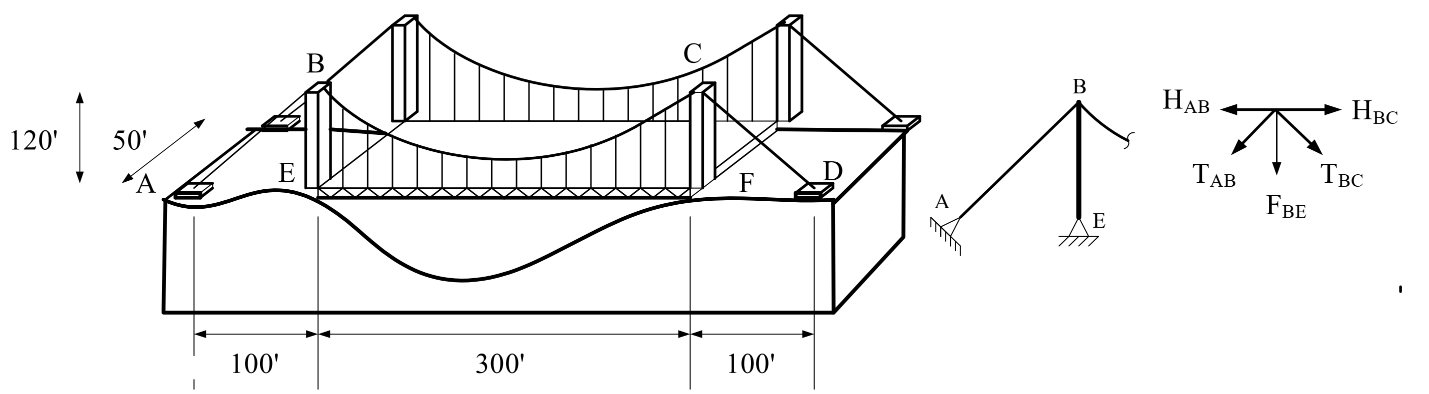
$$\boxed{T_{BC} := \frac{H}{\cos(\theta_B)} = 51.03 \cdot kip}$$

$$\tan(\theta_C) = \frac{h_c}{30ft} \qquad \theta_C := \operatorname{atan}\left(\frac{h_c}{30ft}\right) = 8.7 \cdot deg$$

$$\boxed{T_{CD} := \frac{H}{\cos(\theta_C)} = 51.59 \cdot kip}$$

8.2 Design of Suspension Bridge

Design the following 4 lane suspension bridge by selecting the cable diameters assuming an allowable cable strength of $\sigma_{all} := 190ksi$.. The bases of the tower are hinged in order to avoid large bending moments. The total dead load is estimated at 200 psf. Assume a sag to span ratio of 1/5.



Solution:

1. The dead load is carried by each cable with one half the total deal load or $p_1 := \frac{1}{2} \cdot 200psf \cdot 50ft = 5 \cdot \frac{kip}{ft}$

2. Using the HS 20 truck (or its distributed equivalent load of 0.64kip/ft per lane), the uniform additional load per cable is

$p_2 := 2 \cdot 0.64 \frac{kip}{ft} = 1.28 \cdot \frac{kip}{ft}$

Thus the total design load is

$p := p_1 + p_2 = 6.28 \cdot \frac{kip}{ft}$

3. The thrust H is determined by

$L := 300ft$

$h := 60ft$

$H := \frac{p \cdot L^2}{8 \cdot h} = 1177.5 \cdot kip$

4. The maximum tension is

$r := \frac{1}{5}$ sag to span ratio

$$T_{max} := H \cdot \sqrt{1 + 16r^2} = 1508 \cdot kip$$

5. Note that if we used the approximate formula we woul dhave obtained

$$T_{maxApp} := H \cdot \left(1 + 8 \cdot r^2\right) = 1554.3 \cdot kip$$

6. The required cross sectional area of the cable along the main span should be equal to

$$A := \frac{T_{max}}{\sigma_{all}} = 7.94 \cdot in^2$$

which corresponds to a diameter

$$d := \sqrt{\frac{4 \cdot A}{\pi}} = 3.18 \cdot in$$

7. We seek to determine the cable force in AB. Since the pylon cannot take any horizontal force, we should have the horizontal component of T_{max} equal and opposite to the horizontal component of T_{AB} or

$$T_{AB} := H \cdot \frac{\sqrt{(100ft)^2 + (120ft)^2}}{100ft} = 1839 \cdot kip$$

The cable area should be

$$A := \frac{T_{AB}}{\sigma_{all}} = 9.68 \cdot in^2$$

which corresponds to a diameter

$$d := \sqrt{\frac{4 \cdot A}{\pi}} = 3.51 \cdot in$$

8. To determine the vertical load acting on the pylon, we must add the vertical components of T_{max} and T_{AB} (V_{BC} and V_{AB} respectively). We can determine V_{BC} from H and T_{max}, thus

$$P := \frac{120ft}{100ft} \cdot H + \sqrt{T_{max}^2 - H^2} = 2355 \cdot kip$$

Using A36 steel with an allowable stress of $\sigma_{A36} := 21ksi$, the cross sectional area of the tower should be

$$A := \frac{P}{\sigma_{A36}} = 112 \cdot in^2$$

Note that buckling of such a tower might govern the final dimensions

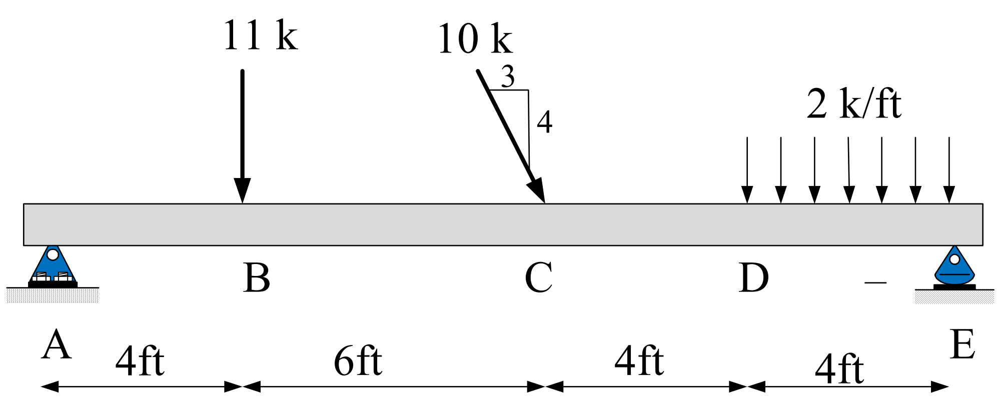
Note that buckling of such a tower might govern the final dimensions.

9. If the cables were to be anchored to a concrete block, the volume of the block shoul dbe at least equal to

$$V := \frac{\frac{120ft}{100ft} \cdot H}{150 \frac{lbf}{ft^3}} = 9420 \cdot ft^3$$

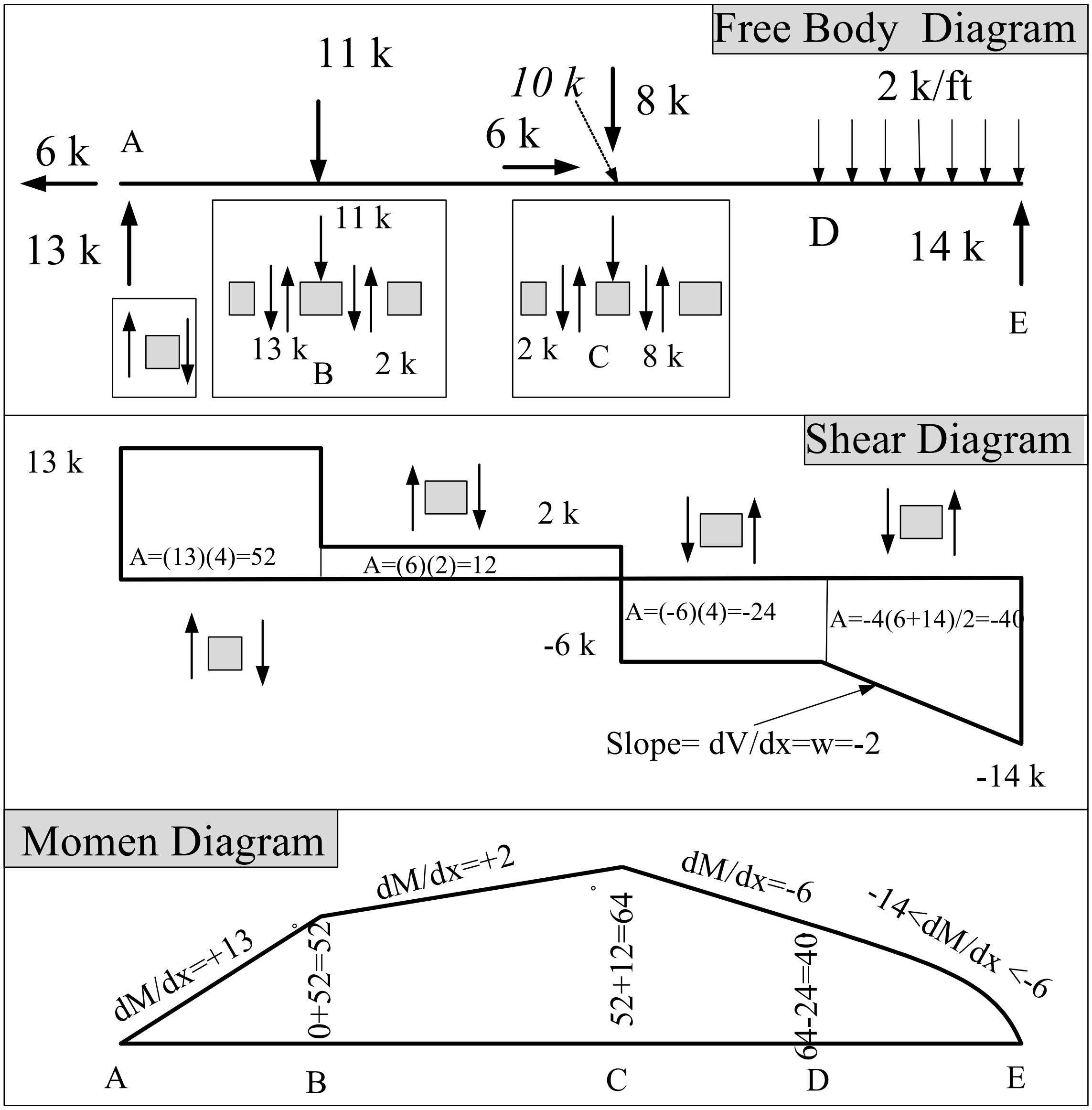
10.1 Simple Shear and Moment Diagram

Draw the shear and moment diagram for the beam shown below.



Solution:

The free body diagram is drawn below



Reactions are determined from the equilibrium equations

$\sum F_x = 0 \qquad -A_x + 6kip = 0 \qquad A_x := 6kip$

$\sum M_A = 0 \qquad 11kip\cdot 4ft + 10kip\cdot\frac{3}{5}\cdot 10ft + 2\frac{kip}{ft}\cdot 4ft\cdot 16ft - E_y\cdot 18ft = 0$

$$E_y := \frac{11kip\cdot 4ft + 10kip\cdot\frac{4}{5}\cdot 10ft + 2\frac{kip}{ft}\cdot 4ft\cdot 16ft}{18ft} = 14\cdot kip$$

$\sum F_y = 0 \qquad A_y - 11kip - 10kip\cdot\frac{4}{5} - 2\frac{kip}{ft}\cdot 4ft + E_y = 0$

$$A_y := 11kip + 8kip + 2\frac{kip}{ft}\cdot 4ft - E_y = 13\cdot kip$$

Shear are determined next

1. At A the shear is equal to the reaction and is positive

1. At A the shear is equal to the reaction and is positive.
2. At B the shear drops (negative load) by 11 k to 2 k
3. At C it drops again by 8 k to -6 k
4. It stays constant up to D and then it decreases (constant negative slope since the load is uniform and negative) by 2 k per linear foot up to -14 k
5. As a check, -14 k is also the reaction previously determined at E.

Moment is determined last

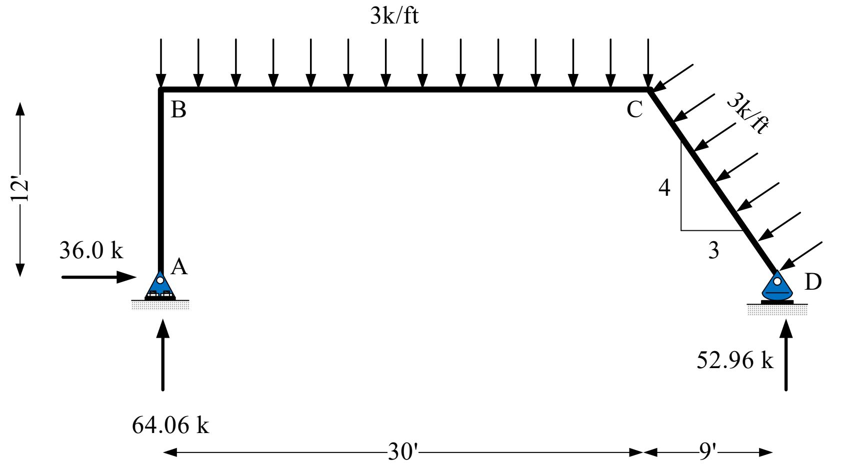
1. The moment at A is zero (hinge support)
2. The change in moment between A and B is equal to the area under the corresponding shear diagram, or

$$\Delta M_{BA} := 13\text{kip}\cdot 4\text{ft} = 52\cdot\text{kip}\cdot\text{ft}$$

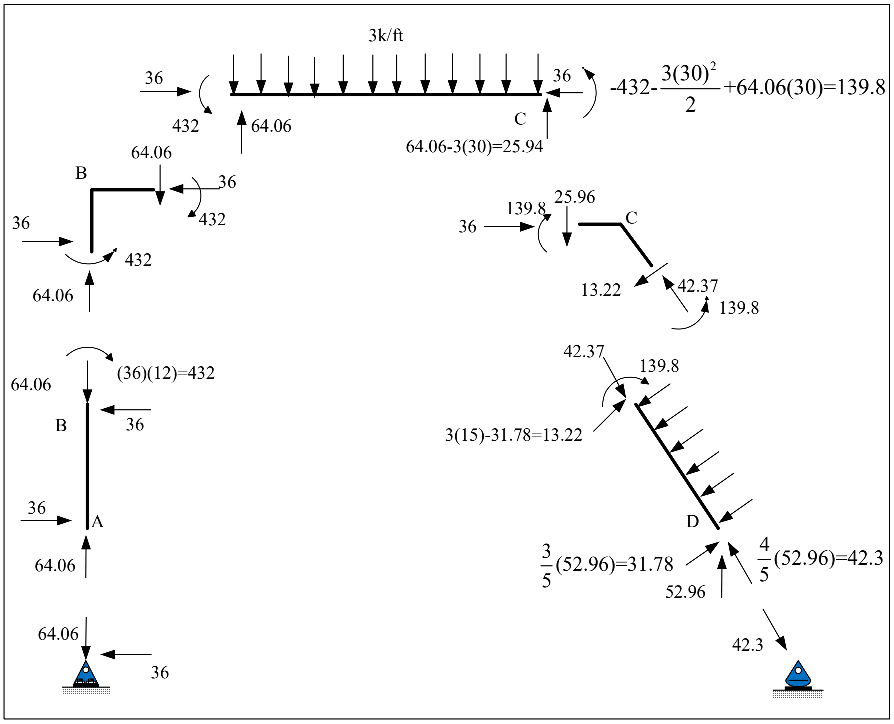
3. Changes between other points are determined the same by taking the area under the shear diagram

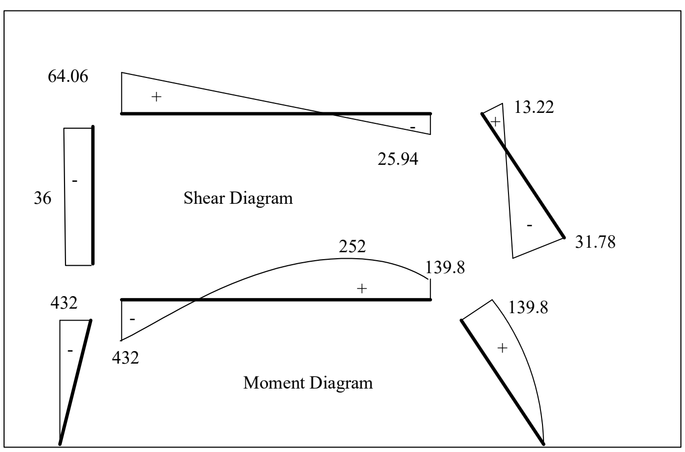
10.2 Frame Shear and Moment Diagram

Draw the shear and moment diagram of the following frame


3k/ft
B
C
3k/ft
12'
4
3
36.0 k
A
D
52.96 k
64.06 k
30'
9'

Solution:


3k/ft
36
432
64.06
36
C
-432-3(30)^2/2+64.06(30)=139.8
64.06-3(30)=25.94
64.06
B
36
432
36
432
64.06
139.8
25.96
C
36
13.22
42.37
139.8
64.06
(36)(12)=432
B
36
42.37
139.8
3(15)-31.78=13.22
36
A
64.06
D
3/5(52.96)=31.78
4/5(52.96)=42.3
52.96
64.06
36
42.3


64.06
+
-
25.94
13.22
+
36
-
Shear Diagram
-
31.78
252
139.8
+
432
-
-
432
139.8
+
Moment Diagram

Reactions are determined first

$$\sum F_x = 0 \qquad R_{Ax} - \frac{4}{5}\cdot 3\,\frac{kip}{ft}\cdot 15ft = 0 \qquad R_{Ax} := \frac{4}{5}\cdot 3\,\frac{kip}{ft}\cdot 15ft = 36\cdot kip$$

$$\sum M_A = 0 \qquad 3\,\frac{kip}{ft}\cdot 30ft\cdot 15ft + \frac{3}{5}\cdot 3\,\frac{kip}{ft}\cdot 15ft\cdot 34.5ft - \frac{4}{5}\cdot 3\,\frac{kip}{ft}\cdot 15ft\cdot 6ft - R_{Dy}\cdot 39ft = 0$$

$$R_{Dy} := \frac{3\,\frac{kip}{ft}\cdot 30ft\cdot 15ft + \frac{3}{5}\cdot 3\,\frac{kip}{ft}\cdot 15ft\cdot 34.5ft - \frac{4}{5}\cdot 3\,\frac{kip}{ft}\cdot 15ft\cdot 6ft}{39ft} = 52.96\cdot kip$$

$$\sum F_y = 0 \qquad R_{Ay} - 3\,\frac{kip}{ft}\cdot 30ft - \frac{3}{5}\cdot 3\,\frac{kip}{ft}\cdot 15ft + R_{Dy} = 0$$

$$R_{Ay} := 3\,\frac{kip}{ft}\cdot 30ft + \frac{3}{5}\cdot 3\,\frac{kip}{ft}\cdot 15ft - R_{Dy} = 64.04\cdot kip$$

We isolate each member and draw its free body diagram for each force component.

Shear

1. For A-B the shear is constant, equal to the horizontal reaction at A and negative according to our previously defined sign convention, $V_A := -36kip$
2. For member B-C at B, the shear must be equal to the vertical force which was transmitted along A-B, and which is equal to the vertical reaction at A, $V_B := R_{Ay} = 64.04\cdot kip$
3. Since B-C is subjected to a uniform load, the shear along B-C will have a slope equal to -3 and in terms of x (measured from B to C) is equal to

   $$V_{BC}(x) := R_{Ay} - 3x$$
4. The shear along C-D is obtained by decomposing the vertical reaction at D into axial and shear components. Thus, at D the shear is equal to $\frac{3}{5}\cdot R_{Dy} = 31.78\cdot kip$ and is negative. Based on our sign convention for the load, the slope of the shear must be equal to -3 along C-D. Thus the shear at point C is such that $V_c - \frac{5}{3}\cdot 9ft\cdot 3\,\frac{kip}{ft} = -\frac{3}{5}\cdot R_{Dy}$ or

   $V_c := \frac{5}{3}\cdot 9ft\cdot 3\,\frac{kip}{ft} - \frac{3}{5}\cdot R_{Dy} = 13.22\cdot kip$. The equation for shear is given by (for x going from C to D)

   $$V_{CD}(x) := V_c - 3x$$

Moment

1. Along A-B, the moment is zero at A (since we have a hinge) and its slope is equal to the shear, thus at B the moment is equal to $-36kip\cdot 12ft = -432\cdot kip\cdot ft$
2. Along B-C, the moment is equal todx

   $$M_{BC} = M_B + \int_0^x V_{BC}(x)\,dx = -432kip\cdot ft + \int_0^x \left(R_{Ay} - 3x\right)dx = -432kip\cdot ft + R_{Ay}\cdot x - \frac{3x^2}{2}$$

   which is a parabola. Substituting for x=30, we obtain at node C:

   $$M_C := -432kip\cdot ft + R_{Ay}\cdot 30ft - \frac{3\,\frac{kip}{ft}\cdot(30ft)^2}{2} = 139.2\cdot kip\cdot ft$$
3. If we need to determine the maximum moment along B-C, we know that $\frac{dM_{BC}}{dx} = 0$ at the point where where V_{BC}=0,

   that is $V_{BC}(x) = R_{Ay} - 3x = 0$, $x_{max} := \frac{R_{Ay}}{3\,\frac{kip}{ft}} = 21.3\cdot ft$. In other words, the maximum moment occurs where the shear is zero. Thus

$$M_{BCmax} := -432\text{kip}\cdot\text{ft} + R_{Ay}\cdot x_{max} - \frac{3\frac{\text{kip}}{\text{ft}}\cdot\left(x_{max}\right)^2}{2} = 251.5\cdot\text{kip}\cdot\text{ft}$$

4. Finally, along C-D the moment varies quadratically (since we had a linear shear), the moment first increases (positive shear), and then decreases (negative shear). The moment along C-D is given by

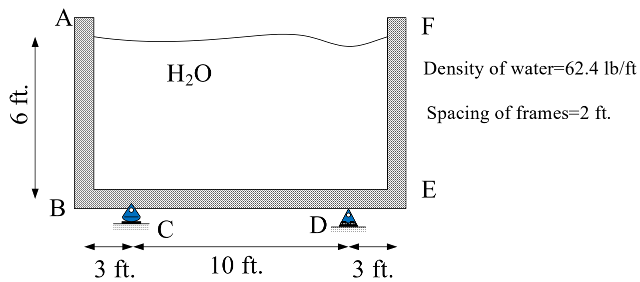
$M_{CD} = M_C + \int_0^x V_{CD}(x)\,dx = M_C + \int_0^x \left(V_c - 3x\right)dx = M_C + V_c\cdot x - \frac{3x^2}{2}$ which is a parabola. Substituting for x =15ft, we obtain at node D

$$M_D := M_C + V_c\cdot 15\text{ft} - 3\,\frac{\frac{\text{kip}}{\text{ft}}\cdot(15\text{ft})^2}{2} = -0\cdot\text{kip}\cdot\text{ft} \quad \text{OK!}$$

10.3 Frame Shear and Moment Diagram: Hydrostatic Load

The frame shown below is the structural support of a flume. Assuming that the frames are spaced 2 ft apart along the length of the flume,

1. Determine all internal member end actions
2. Draw the shear and moment diagrams
3. Locate and compute maximum internal bending moments
4. If this is a reinforced concrete frame, show the location of reinforcement

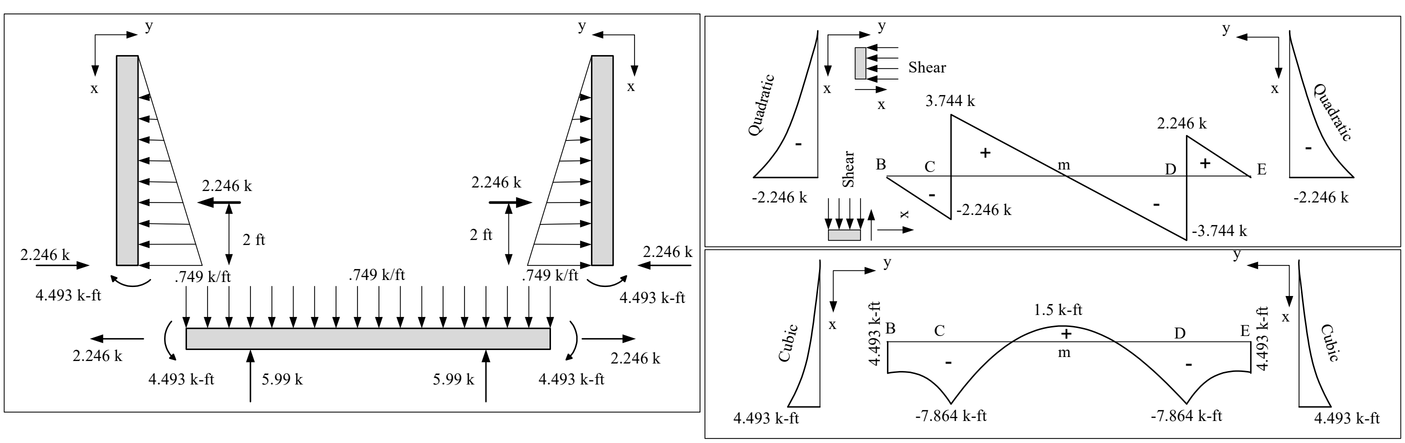


Solution:

The hydrostatic pressure causes lateral forces on the vertical members which can be treated as cantilevers fixed at the lower end. The pressure is linear and is given by $p(\gamma,h) := \gamma \cdot h$. Since each frame supports a 2 ft wide section of the flume, the equation for *w* (pounds/ft) is

$$w(h) := 2ft \cdot 62.4\frac{lbf}{ft^3} \cdot h \rightarrow \frac{124.8 \cdot h \cdot lbf}{ft^2}$$

At the base $w_{base} := w(6ft) = 0.749 \cdot \frac{kip}{ft}$. Note that this is both the lateral pressure on the end walls as well as the uniform load on the horizontal members.



End Actions

1. Base force at B is $F_{Bx} := \frac{1}{2} w_{base} \cdot 6ft = 2.246 \cdot kip$
2. Base moment at B is $M_B := \frac{1}{3} F_{Bx} \cdot 6ft = 4.493 \cdot kip \cdot ft$
3. End forces at B for member B-E are equal and opposite
4. Reaction at C is $R_{Cy} := \frac{1}{2} w_{base} \cdot 16ft = 5.99 \cdot kip$

Shear forces

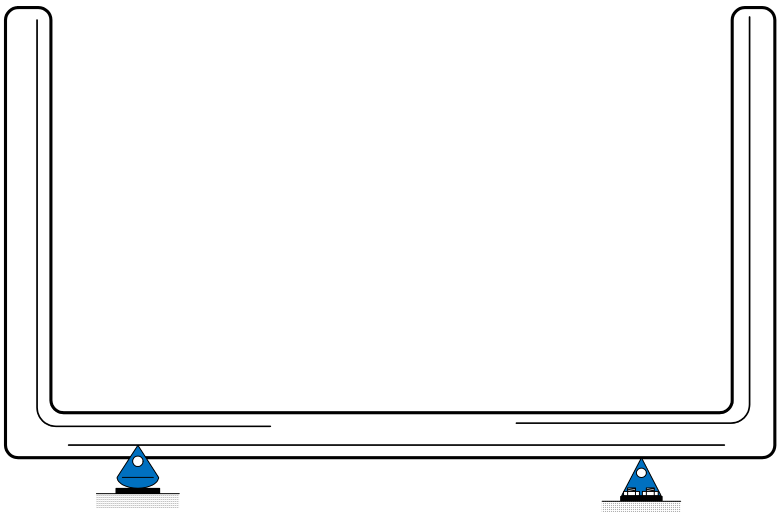
1. Base at B the shear force was determined eariler and was equal to $F_{Bx} = 2.246 \cdot kip$. Based on the orientation of the x-y axis, this is a negative shear.

2. The vertical shear at B is zero (neglecting the weight of A-B)
3. The shear to the left of C is $V_{Cleft} := 0 - w_{base} \cdot 3ft = -2.246 \cdot kip$
4. The shear to the right of C is $V_{Cright} := V_{Cleft} + R_{Cy} = 3.744 \cdot kip$

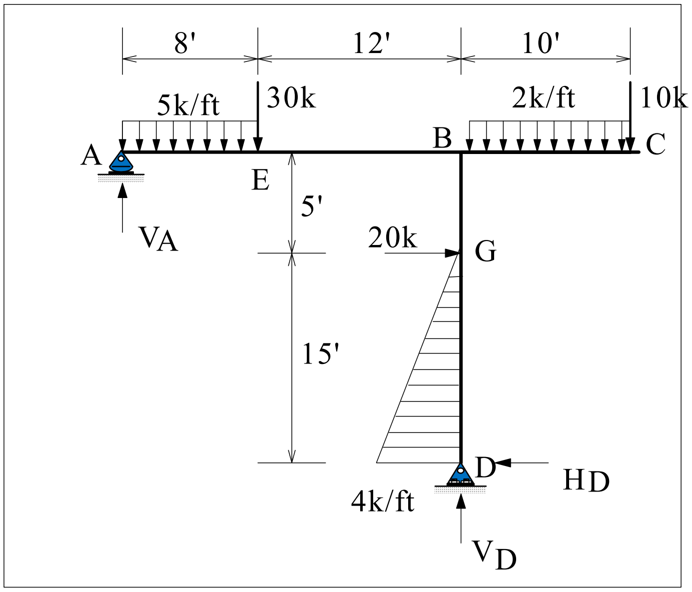
Moment diagrams

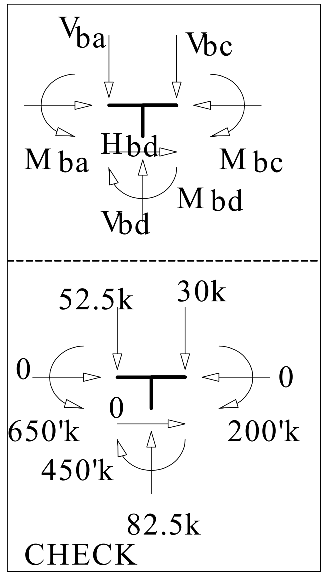
1. At the base: B $M_B = 4.493 \cdot kip \cdot ft$
2. At the support C, $M_C := -M_B - w_{base} \cdot 3ft \cdot \frac{3ft}{2} = -7.862 \cdot kip \cdot ft$
3. The maximum moment is equal to $M_{max} := M_C + w_{base} \cdot 5ft \cdot \frac{5ft}{2} = 1.498 \cdot kip \cdot ft$

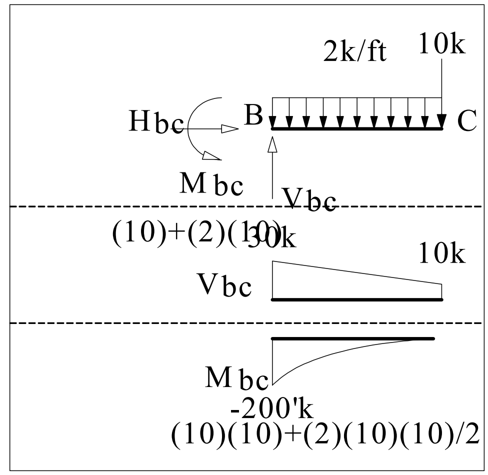
Design: Reinforcement should be placed along the fibers which are under tension, that is on the side of the negative moment. The figure below schematically illustrates the location of the flexural reinforcement.

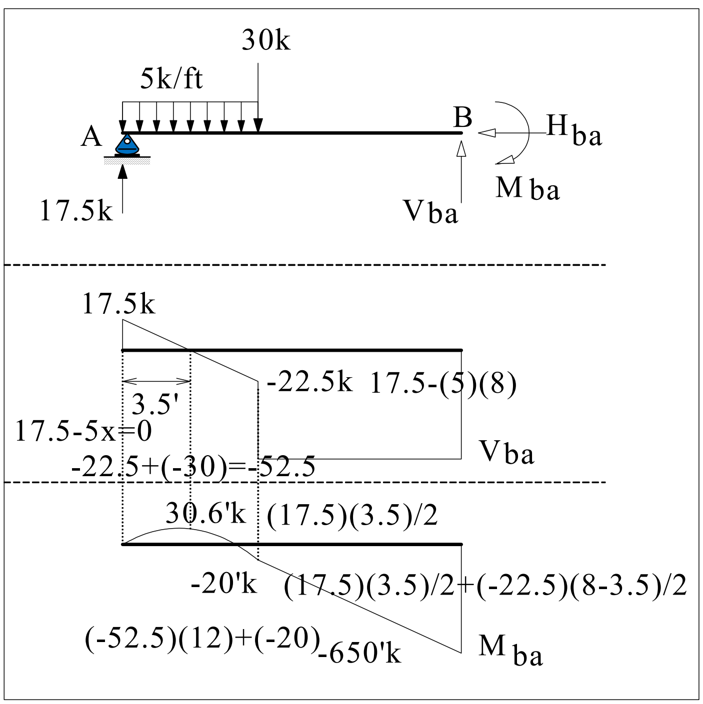


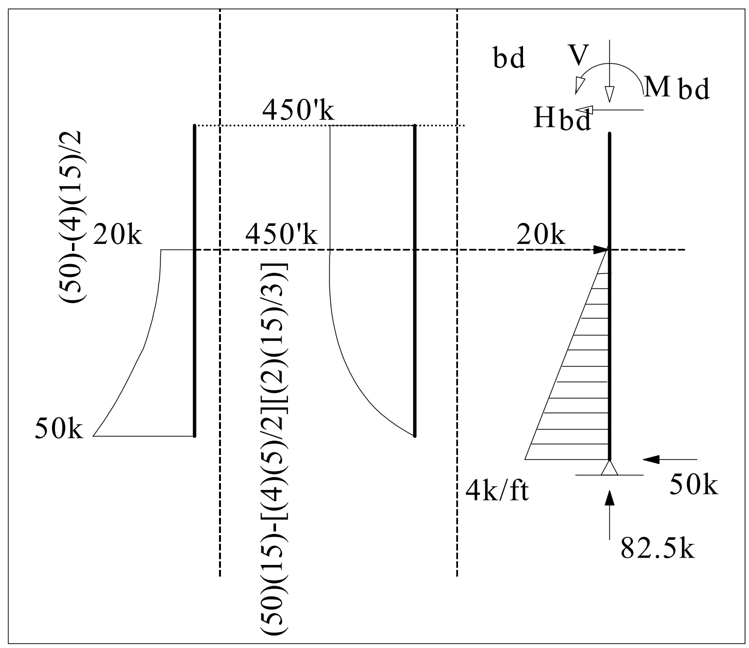
10.4 Shear Moment Diagrams for Frame



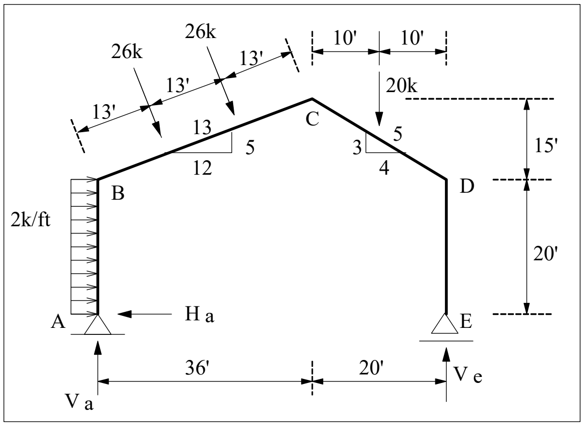


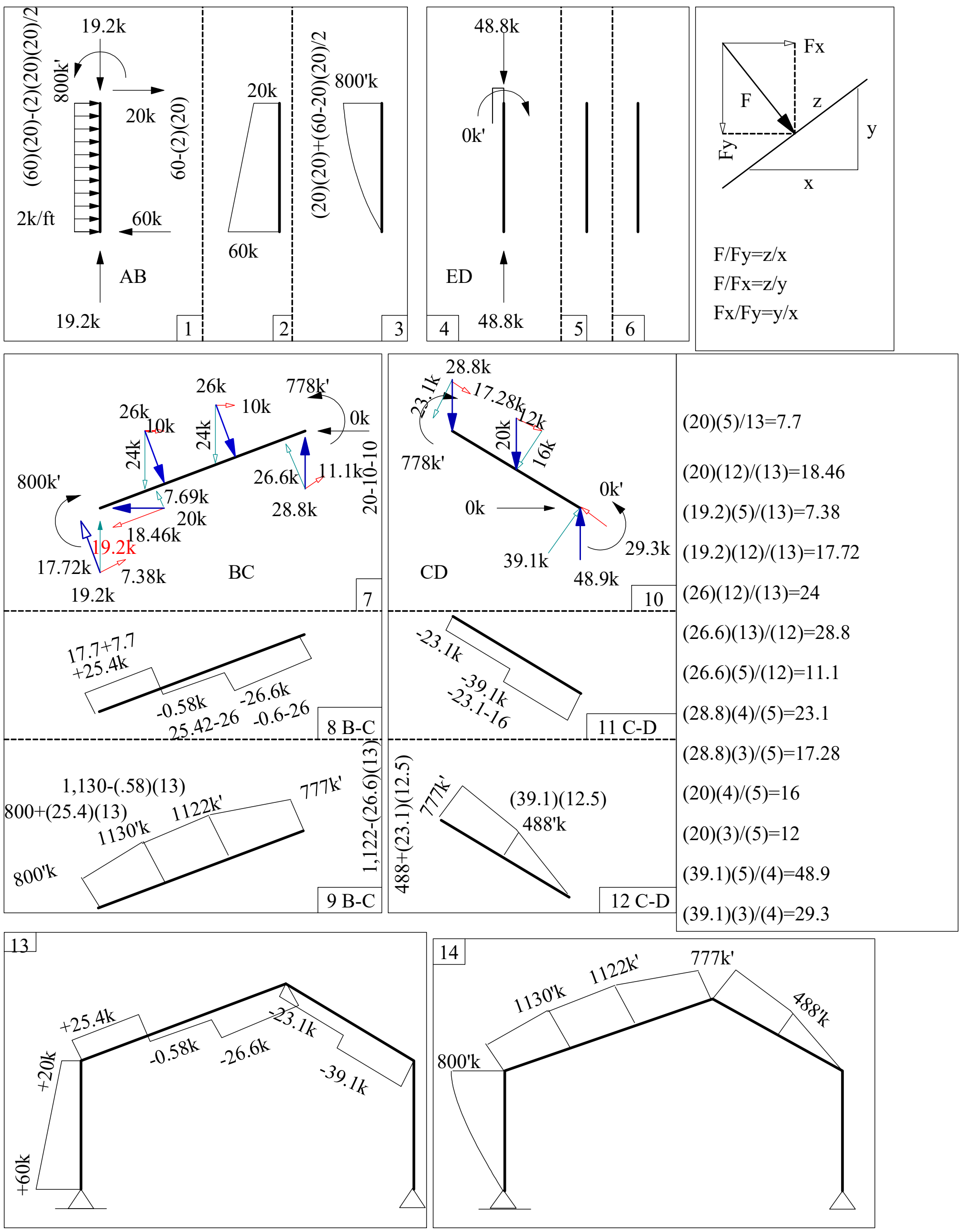




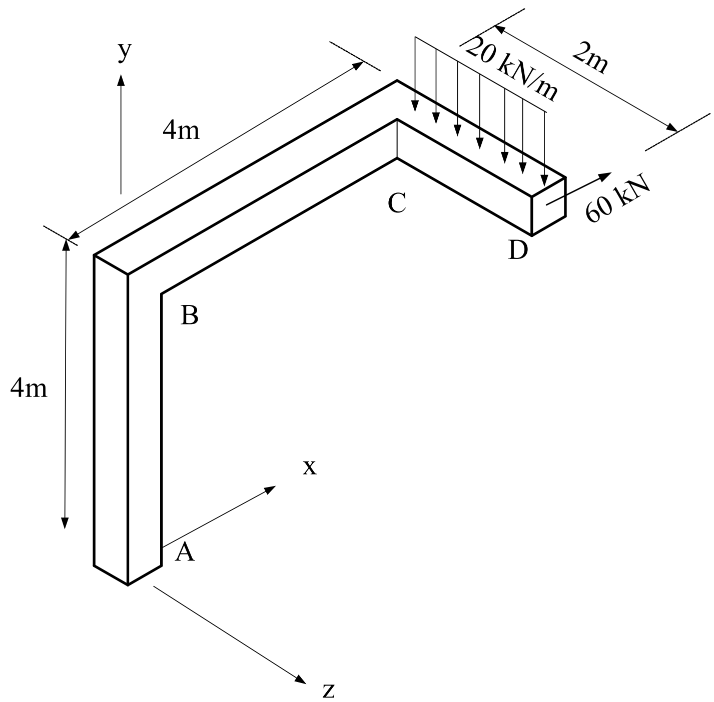


10.5 Shear Moment Diagrams for Inclined Frame





10.6 3D Frame



1. The frame has a total of 6 reactions (3 forces and 3 moments) at the support, and we have a total of 6 equations of equilibrium, thus it is staticall determinate.
2. Each member has the following internal forces (defined in terms of the local coordinate system of each member x' - y'- z' such that x is along the member)

Member	Internal Forces					
Member	Axial	Shear		Moment		Torsion
	$N_{x'}$	$V_{y'}$	$V_{z'}$	$M_{y'}$	$M_{z'}$	$T_{x'}$
C - D		✓	✓	✓	✓	
B - C	✓		✓	✓	✓	✓
A - B	✓	✓		✓	✓	✓

3. The numerical calculations for the analysis of the three dimensional frame are quite simple, however the main complexity stems from the difficulty in visualizing the inter-relations between internal forces of adjacent members.
4. In this particular problem, rather than starting by determining the reactions, it is easier to determined the internal forces at the end of each member starting with member C-D. Note that temporarily we adopt a sign convention which is compatible with the local coordinate systems.

C-D

$$\sum F_{y'} = 0 \qquad V_{y'C} := 20\frac{kN}{m}\cdot 2m = 40\cdot kN$$

$$\sum F_{z'} = 0 \qquad V_{z'C} := 60kN$$

$$\sum M_{y'} = 0 \qquad M_{y'C} := -60kN\cdot 2m = -120\cdot kN\cdot m$$

$$\sum M_{z'} = 0 \qquad M_{z'C} := 20\frac{kN}{m}\cdot 2m\cdot\frac{2m}{2} = 40\cdot kN\cdot m$$

B-C

$\sum F_{x'} = 0 \qquad N_{x'B} := V_{z'C} = 60 \cdot kN$

$\sum F_{y'} = 0 \qquad V_{y'B} := V_{y'C} = 40 \cdot kN$

$\sum M_{y'} = 0 \qquad M_{y'B} := M_{y'C} = -120 \cdot kN \cdot m$

$\sum M_{z'} = 0 \qquad M_{z'B} := V_{y'C} \cdot 4m = 160 \cdot kN \cdot m$

$\sum T_{x'} = 0 \qquad T_{x'B} := -M_{z'C} = -40 \cdot kN \cdot m$

A-B

$\sum F_{x'} = 0 \qquad N_{x'A} := V_{y'B} = 40 \cdot kN$

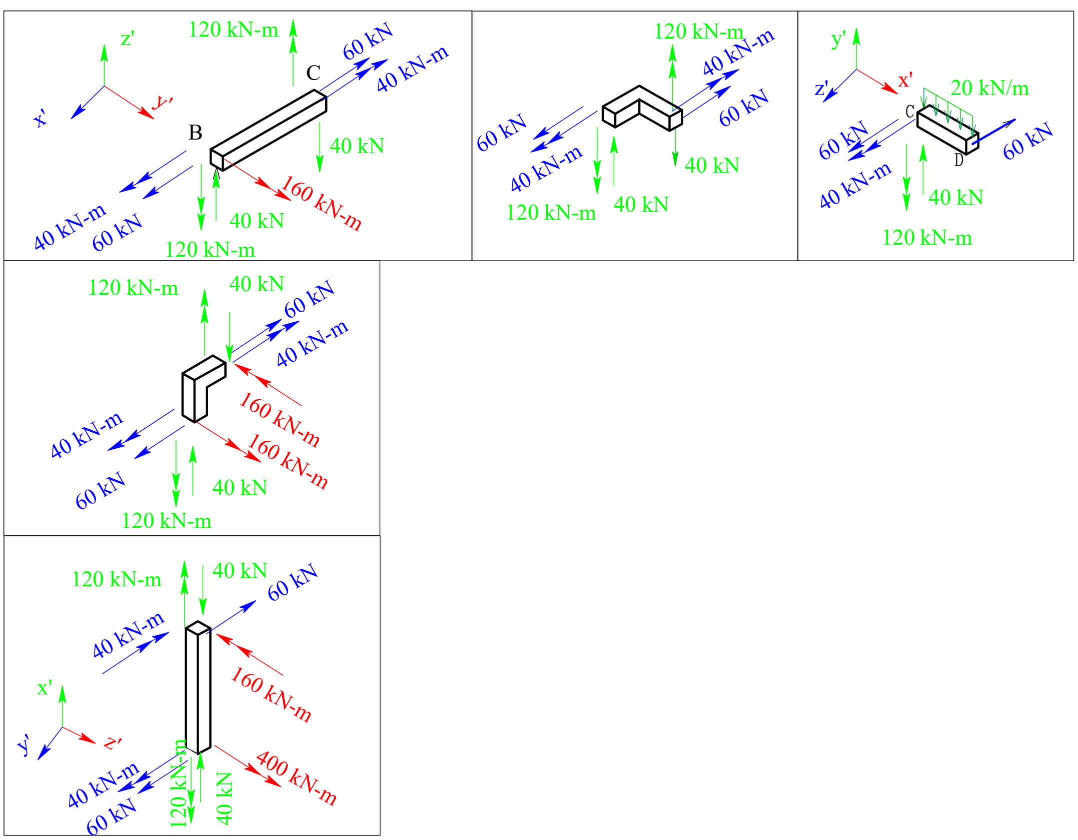
$\sum F_{y'} = 0 \qquad V_{y'A} := N_{x'B} = 60 \cdot kN$

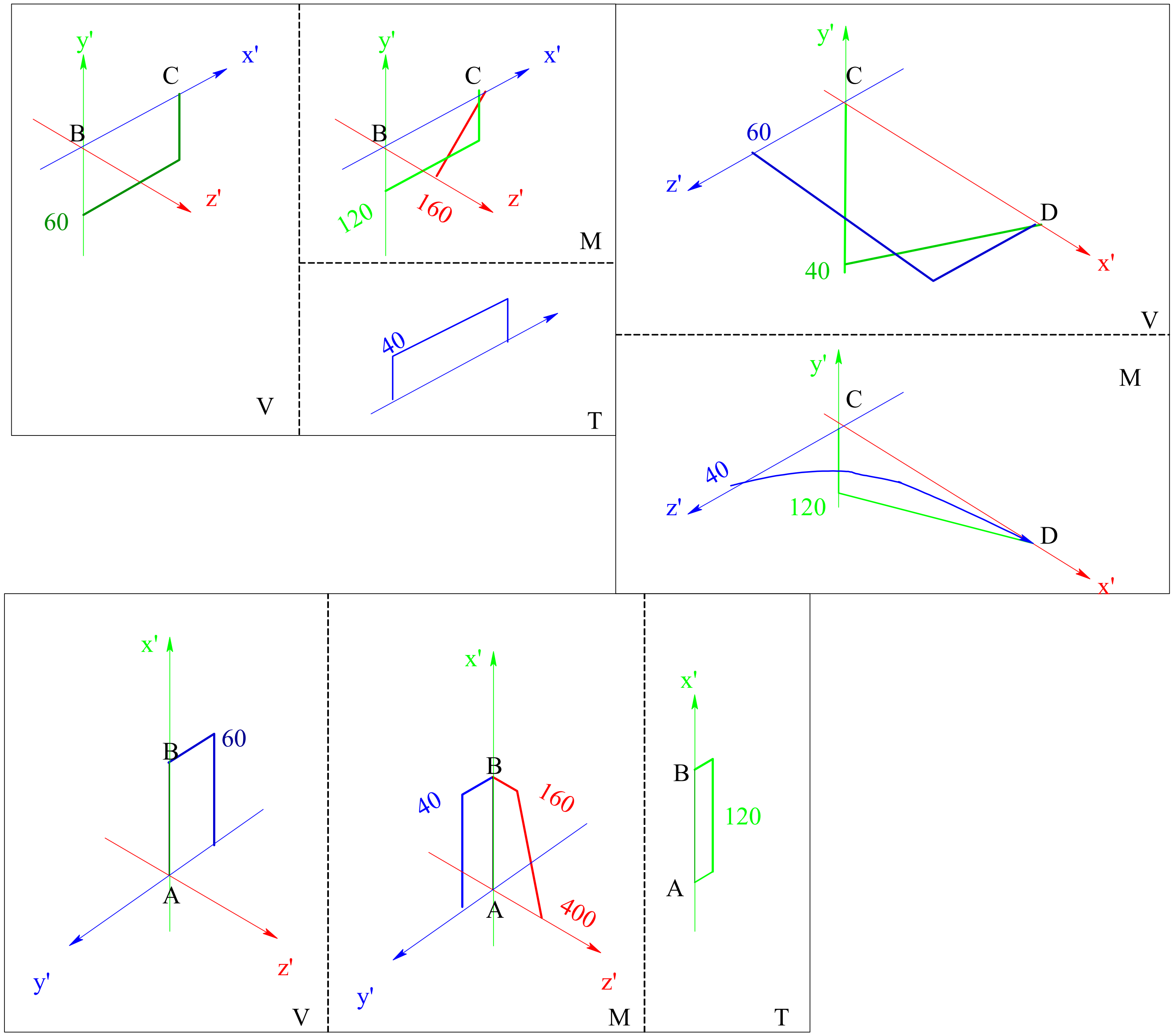
$\sum M_{y'} = 0 \qquad M_{y'A} := -T_{x'B} = 40 \cdot kN \cdot m$

$\sum M_{z'} = 0 \qquad M_{z'A} := M_{z'B} + N_{x'B} \cdot 4m = 400 \cdot kN \cdot m$

$\sum T_{x'} = 0 \qquad T_{x'A} := M_{y'B} = -120 \cdot kN \cdot m$

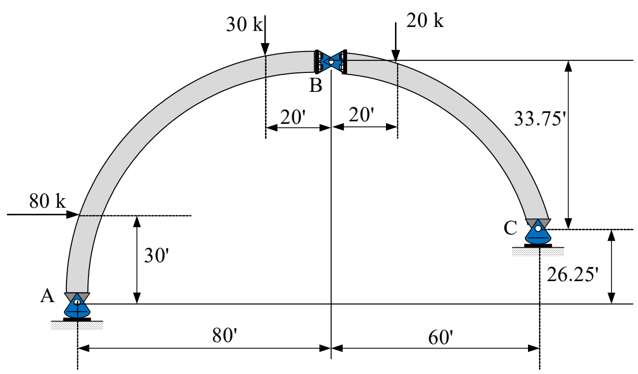
The interaction between axial forces N and shear V as well as between moments M and torsion T is clearly highlighted by this example.

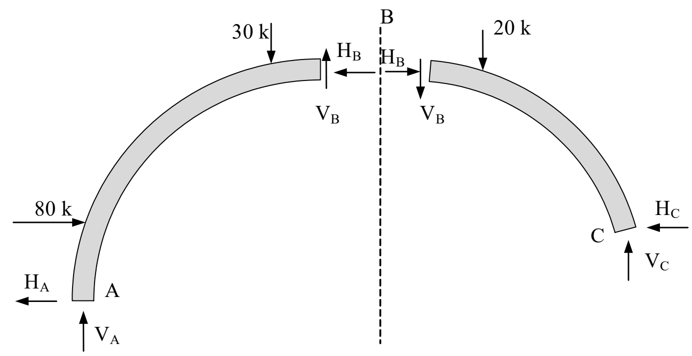



y'
x'
C
B
z'
60
V
y'
x'
C
B
z'
120
160
M
40
T
y'
C
60
z'
D
40
x'
V
y'
M
C
40
z'
120
D
x'
x'
B
60
A
y'
z'
V
x'
B
40
160
A
400
y'
z'
M
x'
B
120
A
T

11.1 Three Hinged Arch, Point Loads

Determine the reactions of the three-hinged arch shown below





Solution:

Four unknowns, three equations of equilibrium, one equation of condition - staticall determinate

$$\sum M_z{}^C = 0$$

$$R_{Ay} \cdot 140ft + 80kip \cdot (30ft - 26.25ft) - 30kip \cdot (60ft + 20ft) - 20kip \cdot (60ft - 20ft) + R_{Ax} \cdot 26.25ft = 0$$

$$R_{Ay} \cdot 140ft + R_{Ax} \cdot 26.25ft = 30kip \cdot 80ft + 20kip \cdot 40ft - 80kip \cdot 3.75ft$$

$$\boxed{R_{Ay} \cdot 140ft + R_{Ax} \cdot 26.25ft = 2900kip \cdot ft}$$

$$\sum F_x = 0 \qquad 80kip - R_{Ax} - R_{Cx} = 0$$

$$\boxed{R_{Ax} + R_{Cx} = 80kip}$$

$$\sum F_x = 0 \qquad R_{Ay} + R_{Cy} - 30kip - 20kip = 0$$

$$\boxed{R_{Ay} + R_{Cy} = 50kip}$$

$$\sum M_z{}^B = 0 \qquad R_{Ax} \cdot 60ft - 80kip \cdot 30ft - 30kip \cdot 20ft + R_{Ay} \cdot 80ft = 0$$

$$R_{Ay} \cdot 80ft + R_{Ax} \cdot 60ft = 80kip \cdot 30ft + 30kip \cdot 20ft$$

$$\boxed{R_{Ay} \cdot 80ft + R_{Ax} \cdot 60ft = 3000kip \cdot ft}$$

Solving those four equations simultaneously we have:

$$\begin{pmatrix} 140 & 26.25 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 80 & 60 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} R_{Ay} \\ R_{Ax} \\ R_{Cy} \\ R_{Cx} \end{pmatrix} = \begin{pmatrix} 2900 \\ 80 \\ 50 \\ 3000 \end{pmatrix}$$

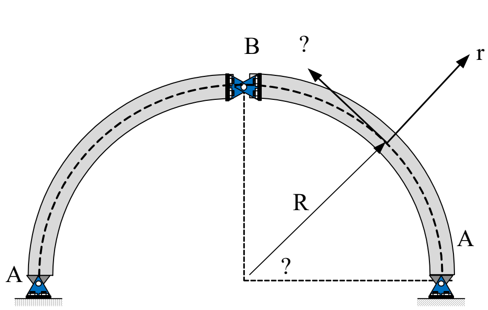
$$\begin{pmatrix} Ay \\ Ax \\ Cy \\ Cx \end{pmatrix} := \begin{pmatrix} 140 & 26.25 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 80 & 60 & 0 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2900 \\ 80 \\ 50 \\ 3000 \end{pmatrix} = \begin{pmatrix} 15.1 \\ 29.8 \\ 34.9 \\ 50.2 \end{pmatrix} \qquad \begin{pmatrix} R_{Ay} \\ R_{Ax} \\ R_{Cy} \\ R_{Cx} \end{pmatrix} := \begin{pmatrix} Ay \\ Ax \\ Cy \\ Cx \end{pmatrix} \cdot kip = \begin{pmatrix} 15.1 \\ 29.8 \\ 34.9 \\ 50.2 \end{pmatrix} \cdot kip$$

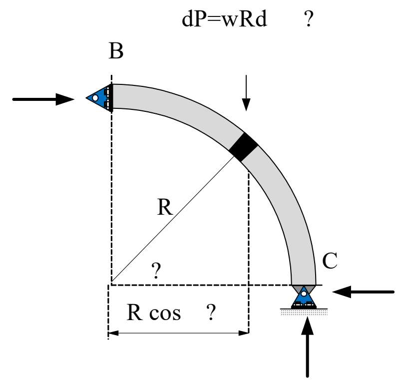
We can check our results by considering the summation with respect to B from the right:

$$\sum M_z{}^B = 0 \qquad -20kip \cdot 20ft - R_{Cx} \cdot 33.75ft + R_{Cy} \cdot 60ft = -0 \cdot kip \cdot ft$$

11.2 Semi-Circular Arch

Determine the reactions of the three-hinged staticall determinate semi-circular arch under its own dead weight *w* (per unit arc length s, where ds=rdθ





Solution:

Reactions The reactions can be determined by **integrating** the load over the entire structure

1. Vertical Reaction is determined first

$$\sum M_A = 0 \qquad -C_y \cdot 2R + \int_{\theta=0}^{\theta=\pi} wR\, d\theta \cdot R(1 + \cos(\theta)) = 0$$

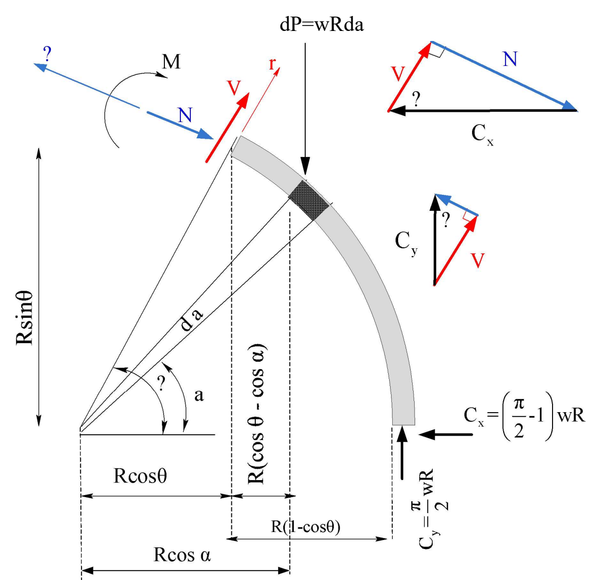
$$C_y = \frac{wR}{2} \cdot \int_{\theta=0}^{\theta=\pi} (1 + \cos(\theta))\, d\theta = \frac{\pi wR}{2}$$

2. Horizontal Reactions are determined next

$$\sum M_B = 0 \qquad -C_x \cdot R + C_y \cdot R - \int_{\theta=0}^{\theta=\frac{\pi}{2}} wR\, d\theta \cdot R\cos(\theta) = 0$$

$$C_z = \frac{\pi wR}{2} - \frac{wR}{2} \cdot \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos(\theta)\, d\theta = \left(\frac{\pi}{2} - 1\right) \cdot wR$$

Internal Forces can now be determined



1. Shear Forces: Considering the free body diagram of the arch, and summing the forces in the radial direction ($\Sigma F_R=0$)

$$-\left(\frac{\pi}{2}-1\right)\cdot wR\cos(\theta)+\frac{\pi}{2}\cdot wR\sin(\theta)-\int_{\alpha=0}^{\theta} wR\,d\alpha\cdot\sin(\theta)+V=0$$

$$\boxed{V=wR\cdot\left[\left(\frac{\pi}{2}-1\right)\cdot\cos(\theta)+\left(\theta-\frac{\pi}{2}\right)\cdot\sin(\theta)\right]}$$

2. Axial Forces: Similarly, if we consider the summation of forces in the axial direction ($\Sigma F_\theta=0$)

$$\left(\frac{\pi}{2}-1\right)\cdot wR\sin(\theta)+\frac{\pi}{2}\cdot wR\cos(\theta)-\int_{\alpha=0}^{\theta} wR\,d\alpha\cdot\cos(\theta)+N=0$$

$$\boxed{N=wR\cdot\left[\left(\theta-\frac{\pi}{2}\right)\cdot\cos(\theta)-\left(\frac{\pi}{2}-1\right)\cdot\sin(\theta)\right]}$$

2. Moment: Now we can consider the third equation of equilibrium ($\Sigma M_z=0$)

$$\left(\frac{\pi}{2}-1\right)\cdot wR\cdot R\sin(\theta)+\frac{\pi}{2}\cdot wR\cdot R\cdot(1-\cos(\theta))+\int_{\alpha=0}^{\theta} wR\,d\alpha\cdot R(\cos(\alpha)-\cos(\theta))+M=0$$

$$\boxed{M=wR^2\cdot\left[\frac{\pi}{2}\cdot(1-\sin(\theta))+\left(\theta-\frac{\pi}{2}\right)\cdot\cos(\theta)\right]}$$

Deflection are determined last

1. The real curvature ϕ is obtained by dividing the moment by EI

$$\phi=\frac{M}{EI}=\frac{wR^2}{EI}\cdot\left[\frac{\pi}{2}\cdot(1-\sin(\theta))+\left(\theta-\frac{\pi}{2}\right)\cdot\cos(\theta)\right]$$

2. The virtual force $\delta\cdot\overline{P}$ will be a unit vertical point load in the direction of the desired deflection, causing a virtual internal moment

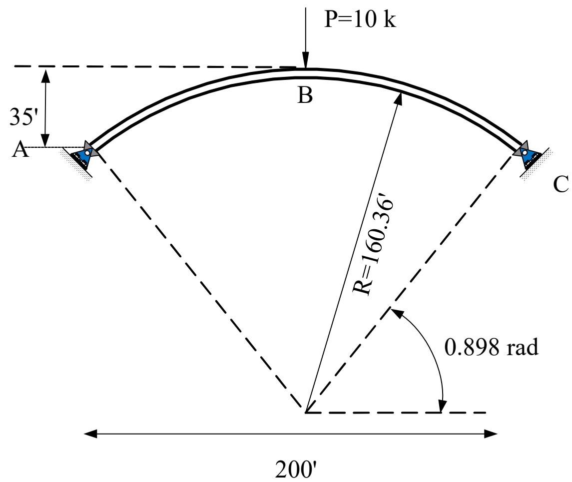
$$\delta\cdot\overline{M}=\frac{R}{2}\cdot(1-\cos(\theta)-\sin(\theta))\qquad 0\le\theta\le\frac{\pi}{2}$$

3. Hence, application of the virtual work equation yields:

$$1\cdot\Delta=2\cdot\int_{\theta=0}^{\frac{\pi}{2}}\frac{wR^2}{EI}\cdot\left[\frac{\pi}{2}\cdot(1-\sin(\theta))+\left(\theta-\frac{\pi}{2}\right)\cdot\cos(\theta)\right]\cdot\left[\frac{R}{2}\cdot(1-\cos(\theta)-\sin(\theta))\right]\cdot R\,d\theta=\frac{wR^4}{16EI}\cdot\left(7\cdot\pi^2-18\cdot\pi-12\right)$$

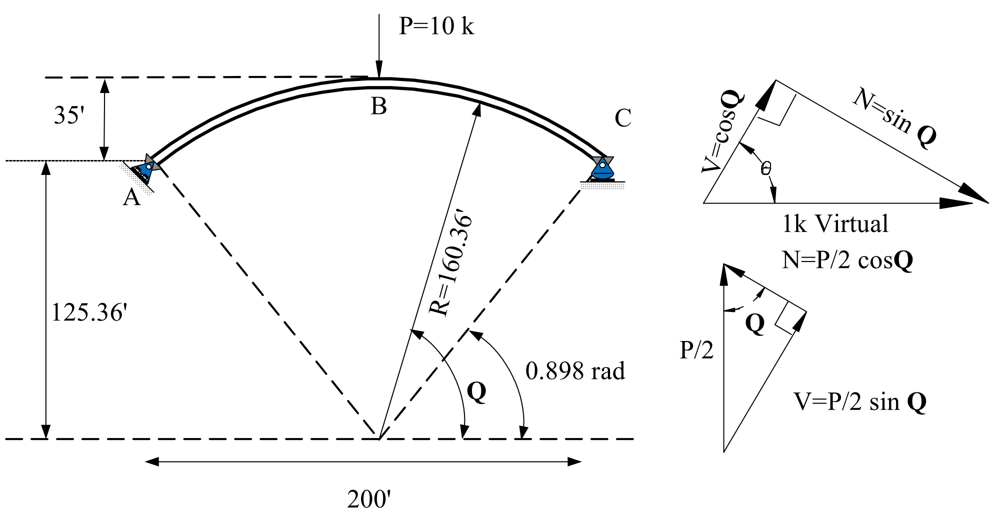
11.3 Statically Indeterminate Arch

Determine the value of the horizontal reaction component of the indicated two-hinged solid rib arch, as caused by a concentrated vertical load of 10 k at the center line of the span. Consider shearing, axial, and flexural strains. Assume the rib is a W24x130 with a total area of 38.21 in^2, that it has a web area of 13.70 in^2, a moment of inertia equal to 4,000 in^4, E of 30,000 k/in^2, and a shearinf modulus of 13,000 k/in^2.



Solution:

1. Consider that end C is placed on rollers, as shown below.



A unit ficticious horizontal force is applied at C. The axial and shearinf components of this ficticious force and of the vertical reaction at C, acting on any section θ in the right half of the rib, are shown at the right end of the rib in the figure above.

2. The expression for the horizontal displacement of C is

$$1\cdot\Delta_{Ch} = 2\cdot\int_C^B \delta\cdot\overline{M}\cdot\frac{M}{EI}\,ds + 2\cdot\int_C^B \delta\cdot\overline{V}\cdot\frac{V}{A_w\cdot G}\,ds + 2\cdot\int_C^B \delta\cdot\overline{N}\cdot\frac{N}{AE}\,ds$$

3. From the figure above, for the rib from C to B

$$M = \frac{P}{2}\cdot(100 - R\cdot\cos\theta)$$

$$\delta\cdot\overline{M} = 1\cdot(R\cdot\sin\theta - 125.36ft)$$

$$V = \frac{P}{2}\cdot\sin\theta$$

$$\delta\cdot\overline{V} = \cos\theta$$

$$N = \frac{P}{2}\cdot\cos\theta$$

$$\delta\cdot\overline{N} = -\sin\theta$$

$$ds = Rd\theta$$

4. If the above values are substituted in the equation for the horizontal displacement of C and integrated between the limits of 0.898 and π/2, the result will be

$$\Delta_{Ch} = 22.55in + 0.023in - 0.003in$$

$$\Delta_{Ch} := 22.57in$$

5. The load P is now assumed to be removed fro mthe rib, and a real horizontal force of 1 k is assumed to act toward the right at C in conjunction with the ficticious force of 1 k acting to the right at the same point. The horizontal displacement of C will be given by

$$\delta_{ChCh} = 2\cdot\int_C^B \delta\cdot\overline{M}\cdot\frac{\overline{M}}{EI}\,ds + 2\cdot\int_C^B \delta\cdot\overline{V}\cdot\frac{\overline{V}}{A_w\cdot G}\,ds + 2\cdot\int_C^B \delta\cdot\overline{N}\cdot\frac{\overline{N}}{AE}\,ds = 2.309in + 0.002in + 0.002in$$

$$\delta_{ChCh} := 2.313in$$

6. The value of the horizontal reaction component will be

$$\boxed{H_C := \frac{\Delta_{Ch}}{\delta_{ChCh}}\cdot kip = 9.76\cdot kip}$$

7. If only flexural strains are considered, the result would be

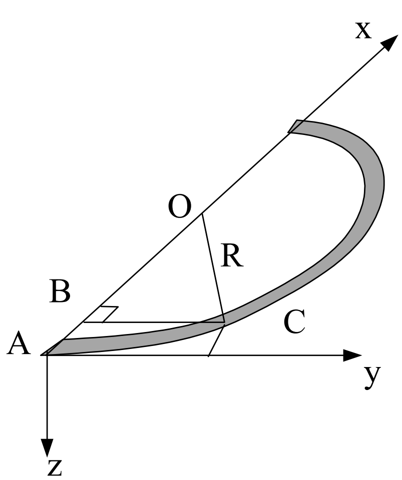
$$\boxed{H_C := \frac{22.55in}{2.309in}\cdot kip = 9.77\cdot kip}$$

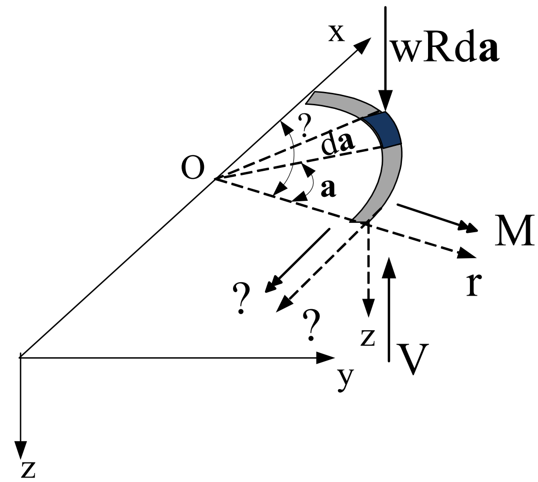
Comments

1. For the given rib and the single concentrated load at the center of the span it is obvious that the effects of shearing and axial strains are insignificant and can be disregarded.
2. Eroneous conclusions as to the relative importance of shearing and axial strains in the usual solid rib may be drawn, however, from the values in the equation for Δ_{Ch}. These indicated that the effects of the shearing strains are much more significant thatn those of the axial strains. This is actually the case for the single concentrated load chosen for the demonstration, but only because the rib does not approximate the funicular polygon for the single load. As a result, the shearing components on most sections of the rib are more important than would otherwise be the case.
3. The usual arch encountered in practice, however, is subjected to a series of loads, and the axis of the rib will approximate the funicular polygon for these loads. In other words, the line of pressure is nearly perpindicular to the right section at all points along the rib. Consequently, the shearing components are so small that the shearing strains are insignificant and are neglected
4. Axial strains, resulting in rib shortening, become increasingly important as the rise-to-span ratio of the arch decreases. It is advisable to determine the effects of the rib by considering flexural strains only, and then to check for effects or rib shortening.

11.4 Semi-Circular Box Girder

Determine the reactions of the semi-circular cantilevered box girder subjected to its own weight w.





Solution:

Reactions are again determined first

From geometry we have OA=R, OB=Rcosθ, CD = BA = OA - OB = R - Rcosθ, EB = R(1+cosθ) and BC = Rsinθ. The moment arms for the moments with respect to the x and y axis are BC and EB respectively. Applying three equations of equilibrium we obtain

$$F_z^A - \int_{\theta=0}^{\theta=\pi} wR\, d\theta = 0 \qquad \boxed{F_z^A = wR\pi}$$

$$M_z^A - \int_{\theta=0}^{\theta=\pi} wR\, d\theta \cdot R \cdot \sin\theta = 0 \qquad \boxed{M_z^A = 2 \cdot w \cdot R^2}$$

$$M_y^A - \int_{\theta=0}^{\theta=\pi} wR\, d\theta \cdot R \cdot (1 + \cos\theta) = 0 \qquad \boxed{M_y^A = -w \cdot R^2 \cdot \pi}$$

Internal Forces are determined next

1. Shear Force:

$$\sum F_z = 0 \qquad V - \int_0^{\theta} wR\, d\alpha = 0$$

$$\boxed{V = wR\theta}$$

2. Bending Moment:

$$\sum M_R = 0 \qquad M - \int_0^{\theta} wR\, d\alpha \cdot R \cdot \sin\alpha = 0$$

$$\boxed{M = wR^2 \cdot (1 - \cos\theta)}$$

3. Torsion:

$$\sum M_\theta = 0 \qquad T + \int_0^{\theta} wR\, d\alpha \cdot R \cdot (1 - \cos\alpha) = 0$$

$$\boxed{T = -wR^2 \cdot (\theta - \sin\theta)}$$

Deflection are determined last

We assume a rectangular cross-section of width b and heigh d=2b and a Poisson's ratio of $\nu := 0.3$

1. Noting that the member will be subjected to both flexural and torsional deformations, we seek to determine the two stiffnesses.
2. The flexural stiffness EI is given by $EI = E\cdot\frac{b\cdot d^3}{12} = \frac{2Eb^4}{3}$
3. The torsional stiffness of solid rectangular sections $J = kb^3\cdot d$ where b is the shorter side of the section, d is the longer, and k is a factor equal to 0.229 for d/b=2. Hence $G = \frac{E}{2\cdot(1+\nu)} = \frac{E}{2.6}$, and $GJ = \frac{E}{2.6}\cdot 0.229\cdot b^4 = 0.176E\cdot b^4$.
4. Considering both flexural and torsional deformations, and replacing dx by rdθ:

$$\delta\cdot\overline{M}\cdot\Delta = \int_0^{\pi} \delta\cdot\overline{M}\cdot\frac{M}{EI_z}\cdot R\,d\theta + \int_0^{\pi} \delta\cdot\overline{T}\cdot\frac{T}{GJ}\cdot R\,d\theta$$

   where the real moments were given above.
5. Assuming a virtual downward force $\delta\cdot\overline{P} = 1$, we have

$$\delta\cdot\overline{M} = R\cdot\sin\theta$$

$$\delta\cdot\overline{T} = -R\cdot(1-\cos\theta)$$

6. Substituting these expressions into the equation for displacement (in 4)

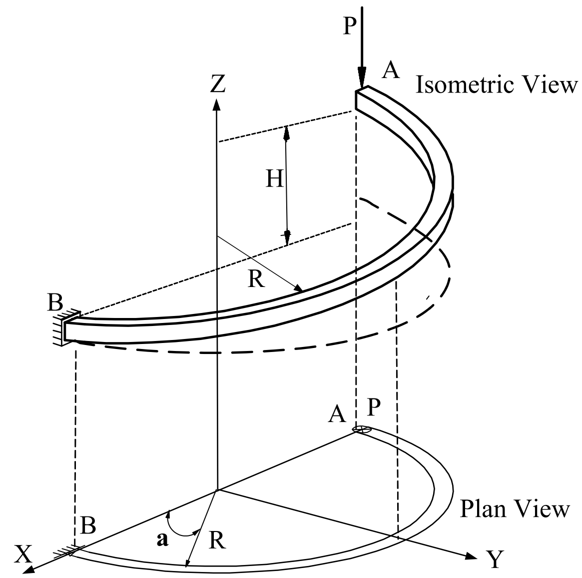
$$1\cdot\Delta = \frac{w\cdot R^2}{EI}\cdot\int_0^{\pi}(R\cdot\sin\theta)\cdot(1-\cos\theta)\cdot R\,d\theta + \frac{w\cdot R^2}{GJ}\cdot\int_0^{\pi}(\theta-\sin\theta)\cdot R\cdot(1-\cos\theta)\cdot R\,d\theta$$

$$1\cdot\Delta = \frac{w\cdot R^4}{EI}\cdot\int_0^{\pi}\left[(\sin\theta - \sin\theta\cdot\cos\theta) + \frac{1}{0.265}\cdot(\theta - \theta\cdot\cos\theta - \sin\theta + \sin\theta\cdot\cos\theta)\right]d\theta$$

$$\boxed{\Delta = 20.56\cdot\frac{w\cdot R^4}{EI}}$$

11.5 Internal Forces in an Helicoidal Cantilevered Girder, Point Load

Determine the internal forces N, V_s, and V_w and the internal moments T, M, and M_w along the helicoidal cantilevered girder.



Solution:

1. We first determine the geometry in terms of the angle θ

$$x(R,\theta) := R\cdot\cos(\theta) \qquad y(R,\theta) := R\cdot\sin(\theta) \qquad z(H,\theta) := \frac{H}{\pi}\cdot\theta$$

2. To determine the unit vector **n** at any point we need the derivatives

$$dx(R,\theta) := \frac{\partial}{\partial\theta}x(R,\theta) \rightarrow -R\cdot\sin(\theta) \qquad dy(R,\theta) := \frac{\partial}{\partial\theta}y(R,\theta) \rightarrow R\cdot\cos(\theta) \qquad dz(H,\theta) := \frac{\partial}{\partial\theta}z(H,\theta) \rightarrow \frac{H}{\pi}$$

and then determine the unit vector

$$\mathbf{n} = \frac{-R\cdot\sin(\theta)\mathbf{i} + R\cdot\cos(\theta)\cdot\mathbf{j} + \frac{H}{\pi}\cdot\mathbf{k}}{\left[R^2\cdot(\sin\theta)^2 + R^2\cdot(\cos\theta)^2 + \left(\frac{H}{\pi}\right)^2\right]^{\frac{1}{2}}} = \frac{1}{\left[1 + \left(\frac{H}{\pi\cdot R}\right)^2\right]^{\frac{1}{2}}}\cdot\left[-\sin\theta\cdot\mathbf{i} + \cos\theta\cdot\mathbf{j} + \left(\frac{H}{\pi\cdot R}\right)\cdot\mathbf{k}\right]$$

Since the denominator depends only on the geometry, it will be designated K.

3. The strong bending axis lies in a horizontal plane, and its unit vector can thus be determined

$$\mathbf{n}\times\mathbf{k} = \frac{1}{K}\cdot\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & \frac{H}{\pi\cdot R} \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{K}\cdot(\cos\theta\cdot\mathbf{i} + \sin\theta\cdot\mathbf{j})$$

and the absolute magnitude of this vector $|\mathbf{n}\times\mathbf{k}| = \frac{1}{K}$, and thus

$$\mathbf{s} = \cos\theta\cdot\mathbf{i} + \sin\theta\cdot\mathbf{j}$$

4. The unit vector along the weak axis is determined

$$\mathbf{w} = \mathbf{s}\times\mathbf{n} = \frac{1}{K}\cdot\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & \frac{H}{\pi\cdot R} \end{pmatrix} = \frac{1}{K}\cdot\left(\frac{H}{\pi\cdot R}\cdot\sin\theta\cdot\mathbf{i} - \frac{H}{\pi\cdot R}\cdot\cos\theta\cdot\mathbf{j} + \mathbf{k}\right)$$

5. With the geometry definition completed, we now examine the equilibrium equations.

$$\sum F = 0 \qquad \mathbf{F} = -\mathbf{P}$$

$$\sum M_b = 0 \qquad \mathbf{M} = -\mathbf{L} \times \mathbf{P}$$

where

$$\mathbf{L} = (R - R\cdot\cos\theta)\cdot\mathbf{i} + (0 - R\cdot\sin\theta)\cdot\mathbf{j} + \left(0 - \frac{\theta}{\pi}\cdot H\right)\cdot\mathbf{k}$$

and

$$\mathbf{M} = \mathbf{L} \times \mathbf{P} = R\cdot\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 - \cos\theta & -\sin\theta & \frac{-\theta}{\pi}\cdot\frac{H}{R} \\ 0 & 0 & P \end{pmatrix} = PR\cdot\left[-\sin\theta\cdot\mathbf{i} - (1 - \cos\theta)\cdot\mathbf{j}\right]$$

and

$$\mathbf{M} = PR\cdot\left[\sin\theta\cdot\mathbf{i} + (1 - \cos\theta)\cdot\mathbf{j}\right]$$

6. Finally, the components of the force $\mathbf{F} = -Pk$ and the moment $\mathbf{M}$ are obtained by appropriate dot products with the unit vectors

$$N = \mathbf{F}\cdot\mathbf{n} = -\frac{1}{K}\cdot P\cdot\frac{H}{\pi\cdot R}$$

$$V_s = \mathbf{F}\cdot\mathbf{s} = 0$$

$$V_w = \mathbf{F}\cdot\mathbf{w} = -\frac{1}{K}\cdot P$$

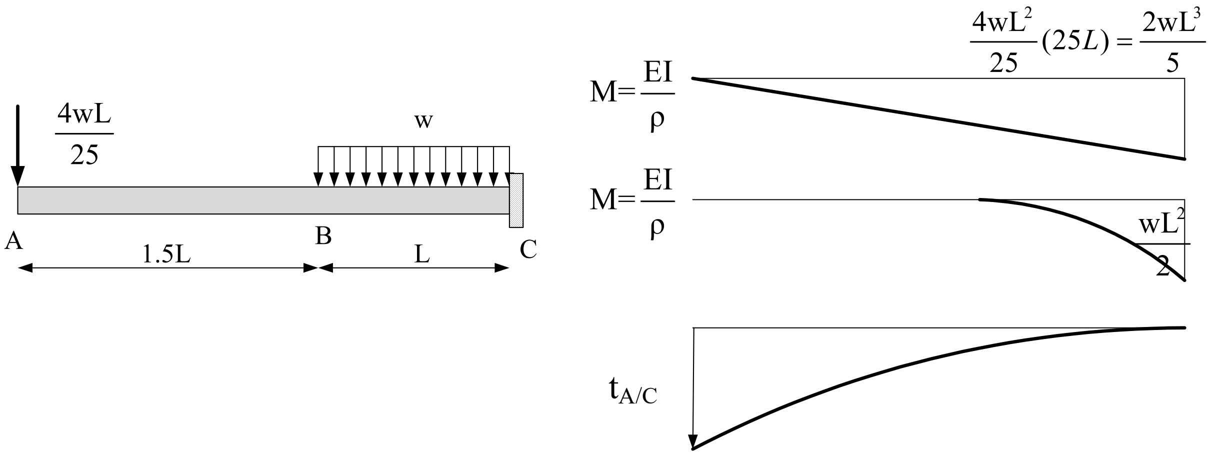
$$T = \mathbf{M}\cdot\mathbf{n} = -\frac{PR}{K}\cdot(1 - \cos\theta)$$

$$M_s = \mathbf{M}\cdot\mathbf{s} = PR\cdot\sin\theta$$

$$M_w = \mathbf{M}\cdot\mathbf{w} = -\frac{PH}{\pi\cdot K}\cdot(1 - \cos\theta)$$

12.1 Moment Area, Cantilevered Beam

Determine the deflection at Point A



Solution:

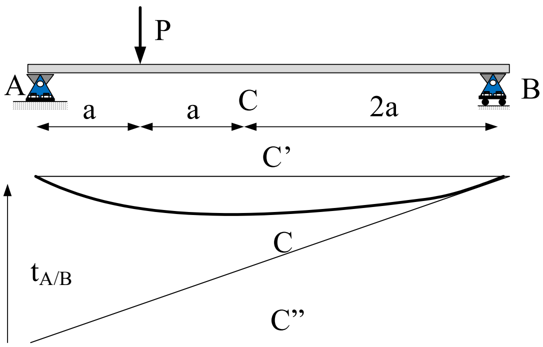
$$EIt_{AC} = \frac{1}{2}\cdot\left(\frac{-2w\cdot L^2}{5}\right)\cdot\left(\frac{5L}{2}\right)\cdot\left(\frac{2}{3}\cdot\frac{5L}{2}\right) + \frac{1}{3}\cdot\left(\frac{-w\cdot L^2}{2}\right)\cdot L\cdot\left(\frac{9L}{4}\right) = \frac{-29w\cdot L^4}{24}$$

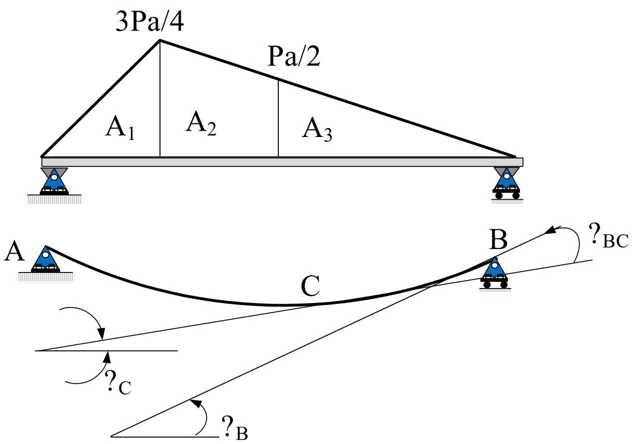
Thus,

$$\boxed{\Delta_A = \frac{-29w\cdot L^4}{24\cdot E\cdot I}}$$

12.2 Moment Area, Simply Supported Beam

Determine Δ_C and θ_C for the following example





Solution:

Deflection Δ_C is determined from $\Delta_C = c'\cdot c = c'\cdot c'' - c''\cdot c,\ c''\cdot c = t_{CB}$, and $c'\cdot c'' = \dfrac{t_{AB}}{2}$

$$t_{AB} = \frac{1}{EI}\cdot\left[\left(\frac{3P\cdot a}{4}\right)\cdot\left(\frac{a}{2}\right)\cdot\left(\frac{2a}{3}\right) + \left(\frac{3P\cdot a}{4}\right)\cdot\left(\frac{3a}{2}\right)\cdot\left(a + \frac{3a}{3}\right)\right] = \frac{5P\cdot a^3}{2\cdot E\cdot I}$$

This is positive, thus above tangent from B

$$t_{CB} = \frac{1}{E\cdot I}\cdot\left(\frac{P\cdot a}{2}\right)\cdot\left(\frac{2a}{2}\right)\cdot\left(\frac{2a}{3}\right) = \frac{P\cdot a^3}{3\cdot E\cdot I}$$

Positive, thus above tangent from B. Finally,

$$\boxed{\Delta_C = \frac{1}{2}\cdot\left(\frac{5P\cdot a^3}{2\cdot E\cdot I}\right) - \frac{P\cdot a^3}{3\cdot E\cdot I} = \frac{11}{12}\cdot\frac{P\cdot a^3}{E\cdot I}}$$

Rotationa θ_C is

$$\theta_{BC} = \theta_B - \theta_C \Rightarrow \theta_C = \theta_B - \theta_{BC}$$

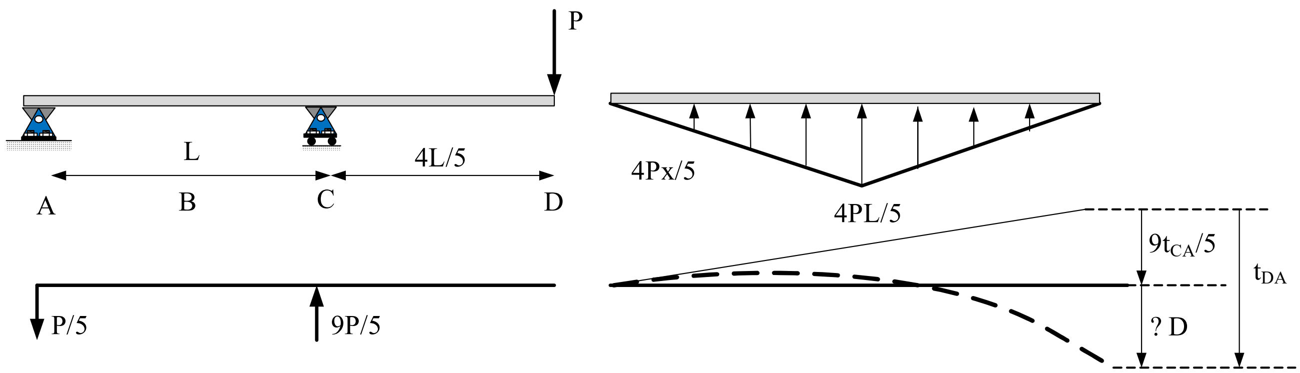
$$\theta_{BC} = A_3$$

$$\theta_B = \frac{t_{AB}}{L}$$

$$\boxed{\theta_C = \frac{5P\cdot a^3}{2\cdot E\cdot I}\cdot\frac{1}{4a} - \left(\frac{P\cdot a}{2}\right)\cdot\left(\frac{2a}{2}\right) = \frac{P\cdot a^2}{8E\cdot I}}$$

12.3 Maximum Deflection

Determine the deflection at D and the maximum deflection at B



Solution:

Deflection at D:

$$\Delta_D = t_{DA} - \frac{9}{5}\cdot t_{CA}$$

$$EIt_{CA} = \frac{1}{2}\cdot\left(\frac{-4P\cdot L}{5}\right)\cdot L\cdot\left(\frac{L}{3}\right) = \frac{-2P\cdot L^3}{15}$$

$$t_{CA} = \frac{-2P\cdot L^3}{15E\cdot I}$$

$$EIt_{DA} = \frac{1}{2}\cdot\left(\frac{-4P\cdot L}{5}\right)\cdot L\cdot\left(\frac{17L}{15}\right) + \frac{1}{2}\cdot\left(\frac{-4P\cdot L}{5}\right)\cdot\left(\frac{4L}{5}\right)\cdot\left(\frac{8L}{15}\right) = \frac{-234P\cdot L^3}{375}$$

$$t_{DA} = \frac{-234P\cdot L^3}{375E\cdot I}$$

Substituting we obtain

$$\boxed{\Delta_D = \frac{-48}{125}\cdot\frac{P\cdot L^3}{E\cdot I}}$$

Maximum Deflection at B:

$$t_{CA} = \frac{-2P\cdot L^3}{15E\cdot I}$$

$$\theta_A = \frac{t_{CA}}{L} = \frac{-2P\cdot L^2}{15E\cdot I}$$

$$\theta_{AB} = \frac{1}{E\cdot I}\cdot\left[\frac{1}{2}\cdot\left(\frac{4P\cdot x}{5}\right)\cdot x\right] = \frac{-2}{5}\cdot\frac{P\cdot x^2}{E\cdot I}$$

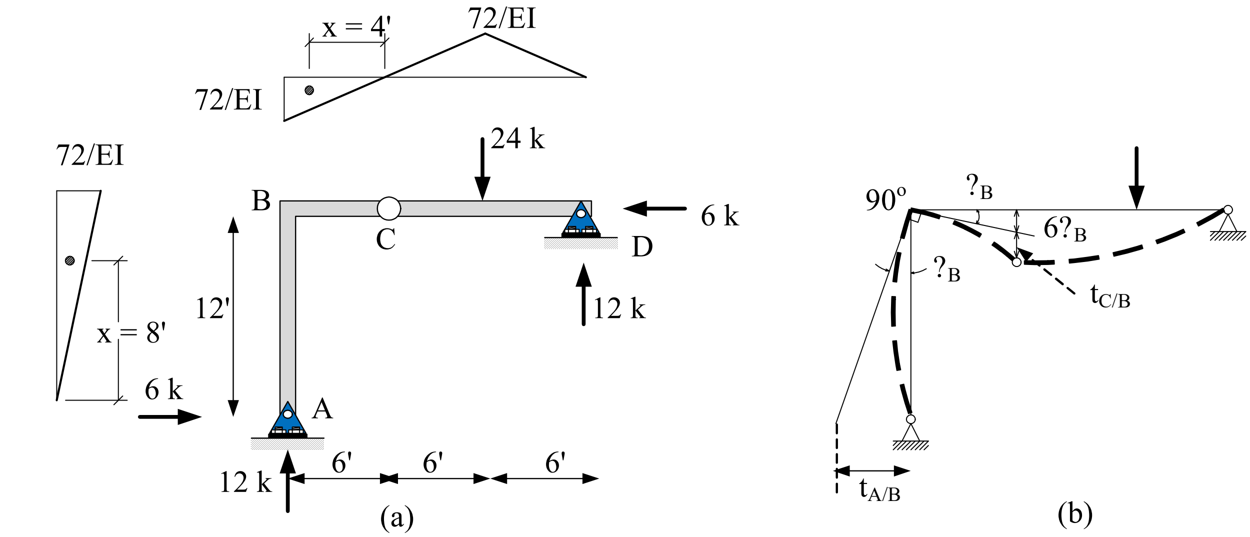
$$\theta_A = \theta_{AB} \Rightarrow \frac{-2P\cdot L^2}{15E\cdot I} = \frac{-2}{5}\cdot\frac{P\cdot x^2}{E\cdot I} \quad x = \frac{L}{\sqrt{3}}$$

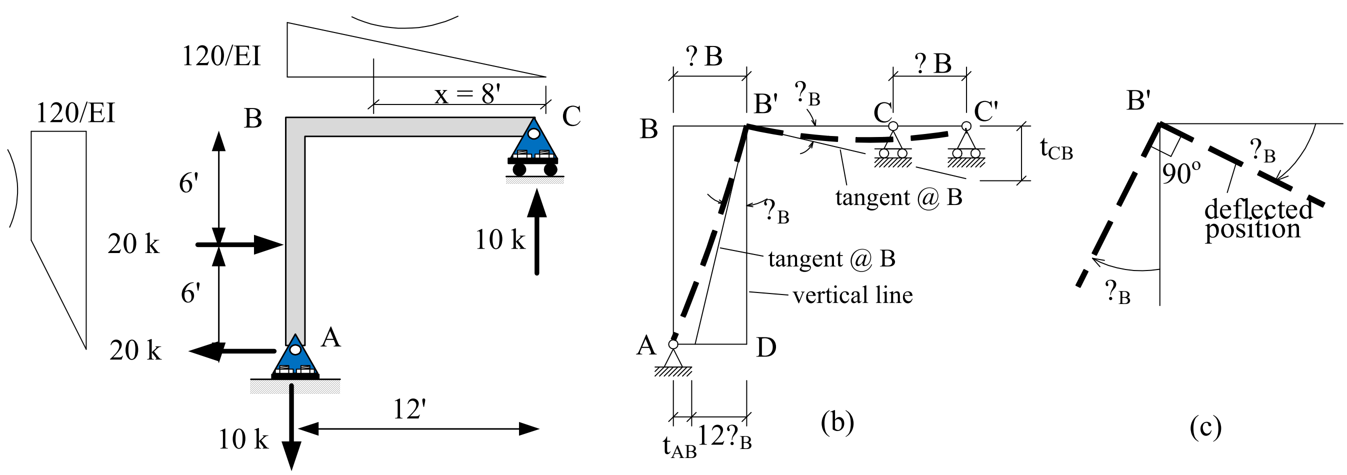
$$\Delta_{max} = \left(\frac{4P\cdot x}{5\cdot E\cdot I}\right)\cdot\left(\frac{x}{2}\right)\cdot\left(\frac{2x}{3}\right) \text{ at } x = \frac{L}{\sqrt{3}}$$

$$\boxed{\Delta_{max} = \frac{4\cdot P\cdot L^3}{45\cdot\sqrt{3}\cdot E\cdot I}}$$

12.4 Frame Deflection

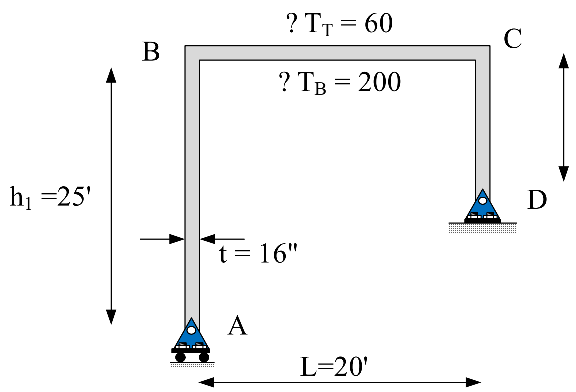
Complete the following example problem

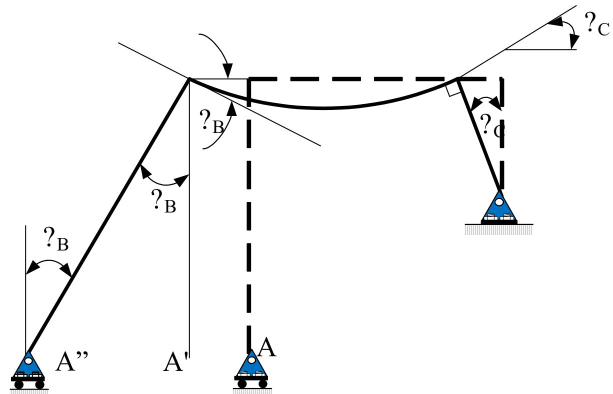




12.5 Frame Subjected to Temperature Loading

Neglecting axial deformation, compute displacement at A for the following frame





Solution:

1. First let us sketch the deformed shape
2. BC flexes => $\theta_B = \theta_C \neq 0$
3. Rigid hinges at B and C with no load on AB and CD
4. Deflection at A

$$\Delta_A = A{\cdot}A'' = A{\cdot}A' + A'{\cdot}A''$$

$$A{\cdot}A' = \Delta_B = \Delta_C = \left|\theta_C\right|\cdot h_2$$

$$A'{\cdot}A'' = \left|\theta_B\right|\cdot h_1$$

5. We need to compute θ_B and θ_C

$$\theta_B = \frac{t_{CB}}{L}$$

$$\theta_C = \theta_{CB} + \theta_B \quad \text{OR} \quad \theta_C = \frac{t_{BC}}{L}$$

6. In order to apply the curvature area theorem, we need a curvature (or moment diagram)

$$\frac{1}{\rho} = \alpha\cdot\left(\frac{T_B - T_T}{h}\right) = \frac{M}{E\cdot I}$$

7.

$$t_{CB} = A\cdot\frac{L}{2} \;\Rightarrow\; \left|\theta_B\right| = A\cdot\left(\frac{L}{2}\right)\cdot\left(\frac{1}{L}\right) = \frac{A}{2} \quad \text{OR} \quad \theta_B = \frac{-A}{2}$$

8.

$$\theta_{CB} = A$$

$$\theta_C = \theta_{CB} + \theta_B$$

$$\theta_C = A - \frac{A}{2} = \frac{A}{2}$$

9. From above

$$\Delta_A = \left|\theta_C\right|\cdot h_2 + \left|\theta_B\right|\cdot h_1 = \frac{A}{2}\cdot h_2 + \frac{A}{2}\cdot h_1 = \frac{A}{2}\left(h_1 + h_2\right)$$

$$A = \alpha\cdot\left(\frac{T_B - T_T}{h}\right)\cdot L$$

$$\Delta_A = \alpha\cdot\left(\frac{T_B - T_T}{h}\right)\cdot L\cdot\left(\frac{1}{2}\right)\cdot\left(h_1 + h_2\right)$$

10. Substitute

$$\Delta_A := \left(6.5\cdot10^{-6}\right)\cdot(200-60)\cdot\frac{20ft}{16in}\cdot\frac{1}{2}\cdot35ft$$

$$\boxed{\Delta_A = 2.867\cdot in}$$

11. Other numerical values

$$\theta_B = \theta_C = \frac{A}{2}$$

$$\theta_B := \frac{1}{2}\left(6.5\cdot10^{-6}\right)\cdot\left(\frac{200-60}{16in}\right)\cdot20ft = 0.006825\cdot rad$$

$$\theta_C := \theta_B = 0.006825$$

$$\frac{M}{EI} = \frac{1}{\rho} = \alpha\cdot\left(\frac{T_B - T_T}{h}\right)$$

$$\rho := \frac{1}{\left[\left(6.5\cdot10^{-6}\right)\cdot\left(\frac{200-60}{16in}\right)\right]} = 1465.2\cdot ft$$

12. In order to get M, we need E and I. Note the difference with other statically determinate structures; the stiffer the beam, the higher the moment; the higher the moment, the higher the stress? NO!!

13. $$\sigma = \frac{M\cdot y}{I} = \frac{E\cdot I}{\rho}\cdot\frac{y}{I} = \frac{E\cdot y}{\rho}$$

14. ρ is constant => BC is on arc of circle M is constant and $\frac{M}{EI} = \frac{d^2\cdot y}{dx^2}$ => $\frac{d^2\cdot y}{dx^2} = \frac{M}{EI} = c$ => $y = \frac{cx^2}{2} + dx + e$

15. The slope is a parabola

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{\frac{d^2\cdot y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

Let us get curvature from the parabola slope and compare it with ρ

$$y = \frac{cx^2}{2} + dx + e$$

$$\frac{dy}{dx} = cx + d$$

$$\frac{d^2\cdot y}{dx^2} = c$$

at x=0, y=e, thus e=0

at x=0, y=0, thus e=0

at x=0, $\frac{dy}{dx} = \theta_B = -0.006825\text{rad}$, thus $d = -0.006825\text{rad}$

at x=20 ft, $\frac{dy}{dx} = \theta_C = 0.006825\text{rad}$, thus $c(20\text{ft}) - 0.006825 = 0.006825$, thus $c = 6.825\cdot 10^{-4}$

$$y = 6.825\cdot 10^{-4}\cdot\left(\frac{x^2}{2} - 10x\right)$$

$$\frac{dy}{dx} = 6.825\cdot 10^{-4}\cdot(x - 10)$$

$$\frac{d^2\cdot y}{dx^2} = 6.825\cdot 10^{-4}$$

$$\phi := 6.825\cdot 10^{-4}$$

$\rho := \frac{1}{\phi}\cdot\text{ft} = 1465.2\cdot\text{ft}$ as expected!

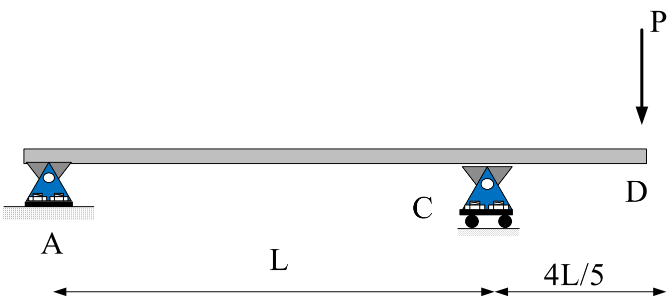
If we were to use the exact curvature formula $\frac{\frac{d^2\cdot y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$

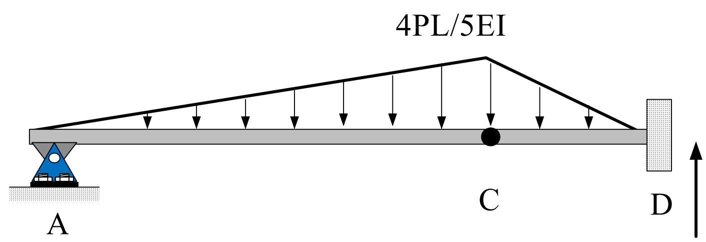
$$\phi := \frac{6.825\cdot 10^{-4}}{\left[1 + (0.006825)^2\right]^{\frac{3}{2}}} = 0.000682$$

$$\rho := \frac{1}{\phi}\cdot\text{ft} = 1465.3\cdot\text{ft}$$

12.6 Conjugate Beam

Analyze the following beam





Solution:

3 equations of equilibrium and 1 equation of condition = 4 = number of reactions. Deflection at D = Shear at D of the corresponding conjugate beam (Reaction at D) Take AC and ΣM with respect to C

$$R_A \cdot (L) - \left(\frac{4P \cdot L}{5E \cdot I}\right) \cdot \left(\frac{L}{2}\right) \cdot \left(\frac{L}{3}\right) = 0$$

$$R_A = \frac{2 \cdot P \cdot L^3}{15E \cdot I}$$

(Slope in real beam at A) As computed before. Let us dram the moment diagram for the conjugate beam

$$M = \frac{P}{EI} \cdot \left(\frac{2}{15} \cdot L^2 \cdot x - \frac{2}{15} \cdot x^3\right) = \frac{2 \cdot P}{15 \cdot E \cdot I} \cdot \left(L^2 \cdot x - x^3\right)$$

Point of maximum moment (Δ_{max}) occurs when $\frac{dM}{dx} = 0$

$$\frac{dM}{dx} = \frac{2 \cdot P}{15 \cdot E \cdot I} \cdot \left(L^2 - 3x^2\right) = 0 \; => \; x = \frac{L}{\sqrt{3}}$$

as previously determined

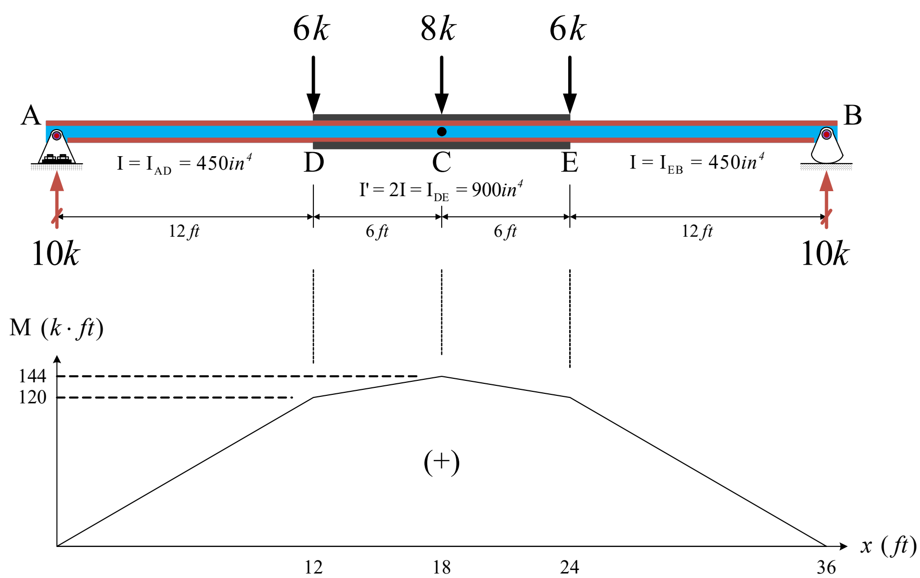
$$x = \frac{L}{\sqrt{3}}$$

$$M = \frac{2 \cdot P}{15 \cdot E \cdot I} \cdot \left[L^2 \cdot \left(\frac{L}{\sqrt{3}}\right) - \left(\frac{L}{\sqrt{3}}\right)^3\right]$$

$$M = \frac{4P \cdot L^3}{45 \cdot \sqrt{3} \cdot E \cdot I}$$ as before

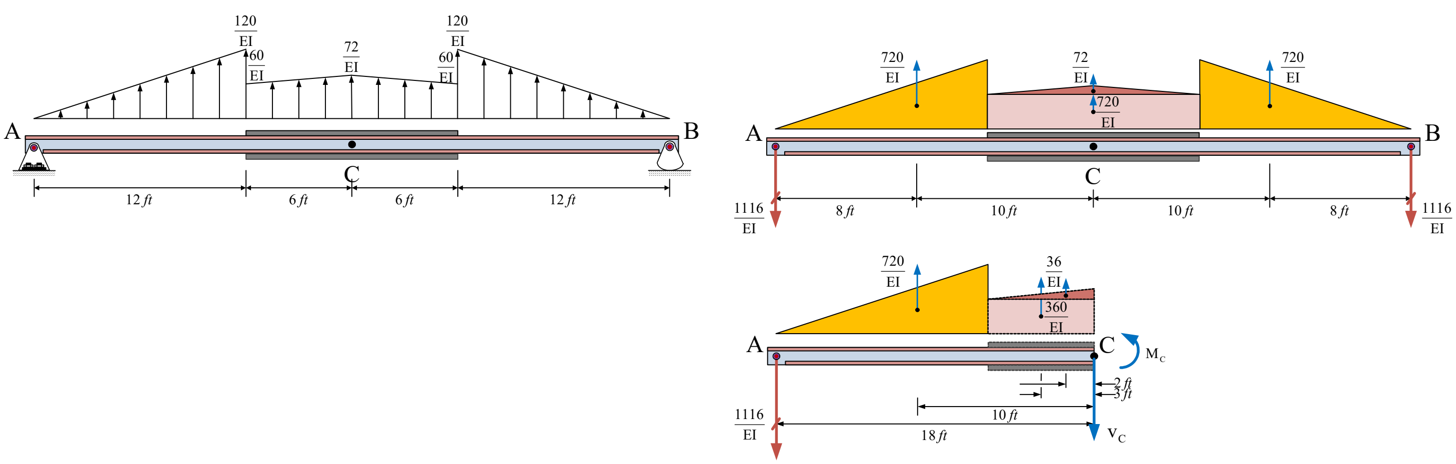
12.7 Conjugate Beam

Analyze the following beam



Solution:

From simple observation, the reactions at A and B are equal to 10 k. The elastic load on the conjugate beam is then shown below.



We next seek to determine the internal moment at C' in the conjugate beam. It is obtained from equilibrium:

$$\sum M_z^{B} = 0 \qquad \frac{1116\cdot kip\cdot ft^2}{E\cdot I}\cdot 18ft - \frac{720\cdot kip\cdot ft^2}{E\cdot I}\cdot 10ft - \frac{360\cdot kip\cdot ft^2}{E\cdot I}\cdot 3ft - \frac{36\cdot kip\cdot ft^2}{E\cdot I}\cdot 2ft + M_{C'} = 0$$

$$E := 29\cdot 10^3 ksi$$

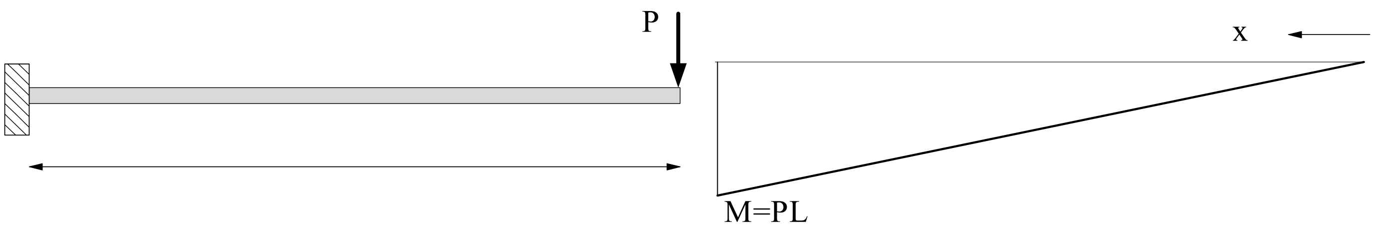
$$I := 450in^4$$

$$\Delta_C = M_{C'}$$

$$\boxed{\Delta_{C'} := \frac{720\cdot kip\cdot ft^2}{E\cdot I}\cdot 10ft + \frac{360\cdot kip\cdot ft^2}{E\cdot I}\cdot 3ft + \frac{36\cdot kip\cdot ft^2}{E\cdot I}\cdot 2ft - \frac{1116\cdot kip\cdot ft^2}{E\cdot I}\cdot 18ft = -1.554\cdot in}$$

13.1 Deflection of a Cantilever Beam

Determine the deflection of the cantilever beam with span L under a point load P at its free end. Assume constant EI.



Solution:

$$W_e = \frac{1}{2}\cdot P\cdot \Delta_f$$

$$U = \int_0^L \frac{M^2(x)}{2\cdot E\cdot I}\,dx$$

$$M = -P\cdot x$$

$$U(P,L,EI) := \frac{P^2}{2\cdot EI}\cdot\int_0^L x^2\,dx \rightarrow \frac{L^3\cdot P^2}{6\cdot EI}$$

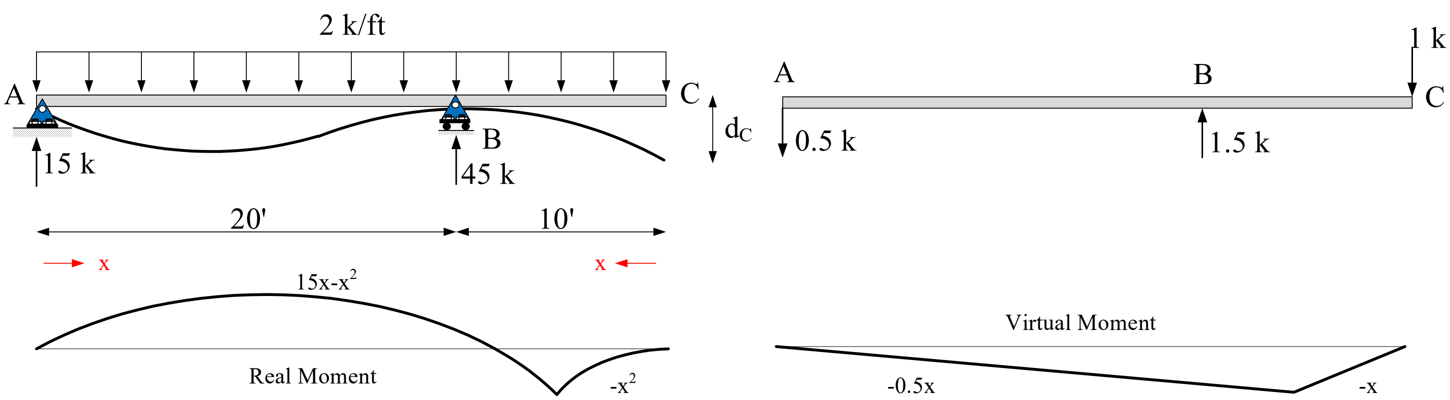
$$\frac{1}{2}\cdot P\cdot \Delta_f = \frac{L^3\cdot P^2}{6\cdot EI}$$

$$\boxed{\Delta_f = \frac{P\cdot L^3}{3\cdot EI}}$$

Note that the solution of this problem was facilitated by the fact that Δ_f is co-alligned with P.

13.2 Beam Deflection

Determine the deflection at point C. $E := 29000ksi$, $I := 100in^4$



Solution:

For the virtual force method, we need to have two expressions for the moment, one due to the real load, and the other to the (unit) virtual one.

Element	x = 0	M	δM
AB	A	$15x - x^2$	$- 0.5x$
BC	C	$- x^2$	$- x$

For the virtual force method, we need to have two expressions for the moment, one due to the real load, and the other to the (unit) virtual one.

$$\Delta_C \cdot \delta \cdot \overline{P} = \int_0^L \delta \cdot \overline{M} \cdot \frac{M(x)}{EI_z}\, dx$$

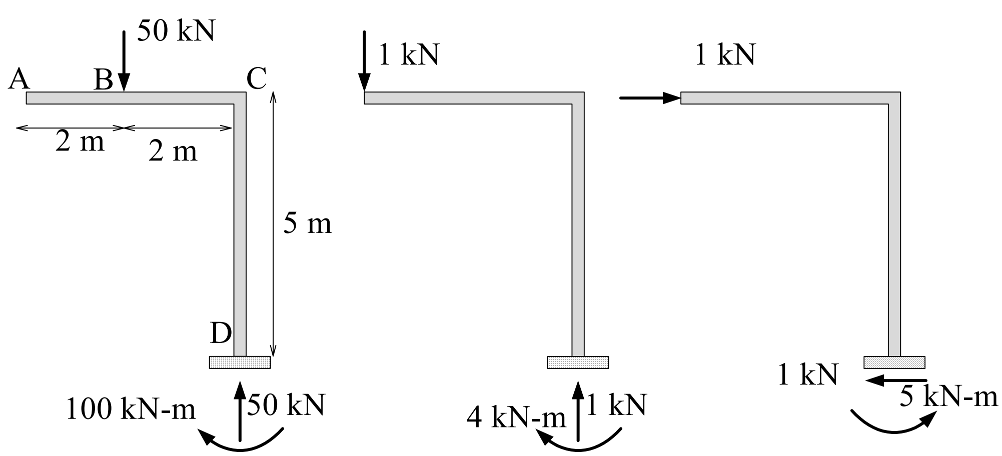
$$1 \cdot \Delta_C = \int_0^{20} (-0.5x) \cdot \left(\frac{15x - x^2}{EI}\right) dx + \int_0^{10} (-x) \cdot \left(\frac{-x^2}{EI}\right) dx$$

$$\Delta_C(EI) := \int_0^{20} (-0.5x) \cdot \left(\frac{15x - x^2}{EI}\right) dx + \int_0^{10} (-x) \cdot \left(\frac{-x^2}{EI}\right) dx \rightarrow \frac{2500}{EI}$$

$$\boxed{\Delta_C := \frac{2500 kip \cdot ft^3}{E \cdot I} = 1.49 \cdot in}$$

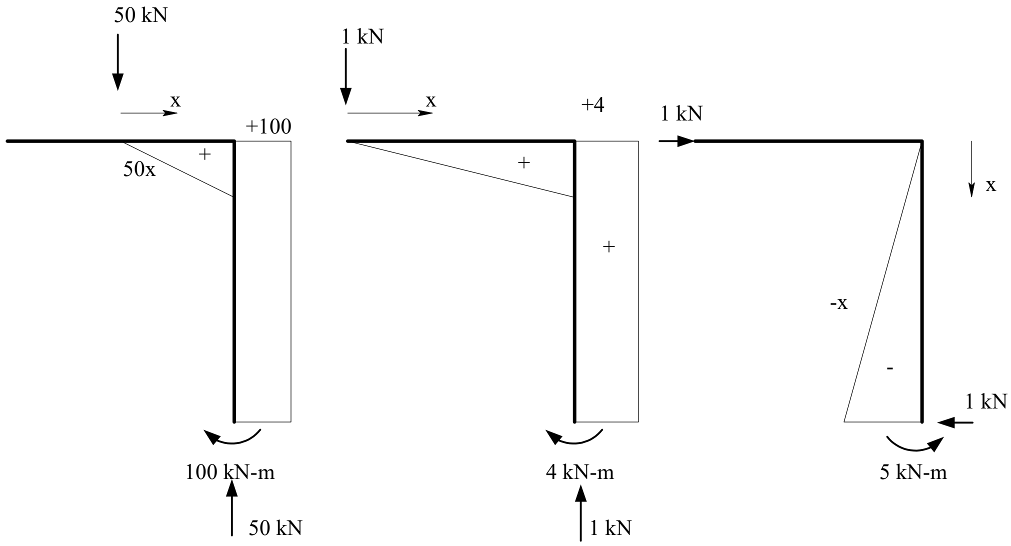
13.3 Deflection of a Frame

Determine both the vertical and horizontal deflection at A for the frame shown. $E := 200 \cdot 10^6 \frac{kN}{m^2}$, $I := 200 \cdot 10^6 mm^4$



Solution:

To analyze the frame we must determine analytical expressions for the moments along each member for the real load and the two virtual ones. One virtual load is a unit horizontal load at A, and the other a unit vertical one at A



Element	x = 0	M	δM_v	δM_h
AB	A	0	- x	0
BC	B	- 50x	- 2 - x	0
CD	C	100	4	- x

Note that moments are considered positive when they produce compression on the inside of the frame. Substitution yields:

$$\Delta_\nu \cdot \delta \cdot \bar{P} = \int_0^L \delta \cdot \overline{M}(x) \cdot \frac{M(x)}{EI_z}\, dx$$

$$1\Delta_\nu = \int_0^2 -x \cdot \frac{(0)}{EI}\, dx + \int_0^2 (-2 - x) \cdot \frac{(-50x)}{EI}\, dx + \int_0^5 4 \cdot \frac{100}{EI}\, dx$$

$$\Delta_\nu(EI) := \int_0^2 -x \cdot \frac{(0)}{EI}\, dx + \int_0^2 (-2 - x) \cdot \frac{(-50x)}{EI}\, dx + \int_0^5 4 \cdot \frac{100}{EI}\, dx \rightarrow \frac{7000}{3 \cdot EI}$$

$$\Delta_\nu := \frac{7000 kN \cdot m^3}{3 \cdot E \cdot I} = 5.833 \cdot cm$$

Similarly for the horizontal displacement

$$\Delta_h \cdot \delta \cdot \bar{P} = \int_0^L \delta \cdot \overline{M}(x) \cdot \frac{M(x)}{EI_z}\, dx$$

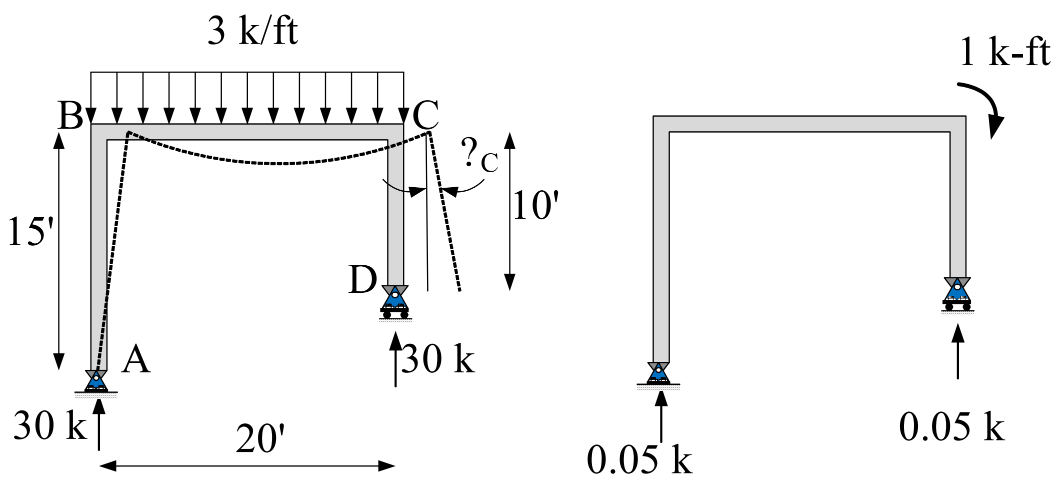
$$1\Delta_h = \int_0^2 0\cdot\frac{(0)}{EI}\,dx + \int_0^2 (0)\cdot\frac{(-50x)}{EI}\,dx + \int_0^5 -x\cdot\frac{100}{EI}\,dx$$

$$\Delta_h(EI) := \int_0^2 0\cdot\frac{(0)}{EI}\,dx + \int_0^2 (0)\cdot\frac{(-50x)}{EI}\,dx + \int_0^5 -x\cdot\frac{100}{EI}\,dx \rightarrow -\frac{1250}{EI}$$

$$\Delta_h := \frac{-1250kN\cdot m^3}{E\cdot I} = -3.125\cdot cm$$

13.4 Rotation of a Frame

Determine the rotation of joint C for the frame shown. $E := 29000ksi$, $I := 240in^4$



Solution:

In this problem the virtual force is a unit moment applied at joint C, $\delta\cdot\overline{M_e}$. It will cause an internal moment $\delta\cdot\overline{M_i}$.

Element	x = 0	M	δM
AB	A	0	0
BC	B	$30x - 1.5x^2$	- 0.05x
CD	D	0	0

Note that moments are considered positive when they produce compression on the outside of the frame. Substitution yields:

$$\theta_C\cdot\delta\cdot\overline{M_e} = \int_0^L \delta\cdot\overline{M}\cdot\frac{M(x)}{EI_z}\,dx$$

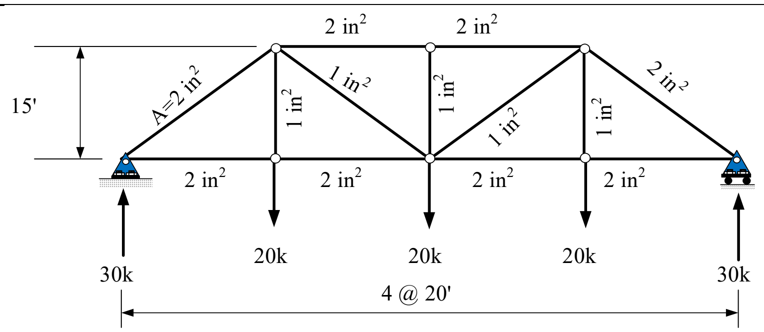
$$1\cdot\theta_C = \int_0^{20} (-0.05x)\cdot\left(\frac{30x - 1.5x^2}{EI}\right)dx$$

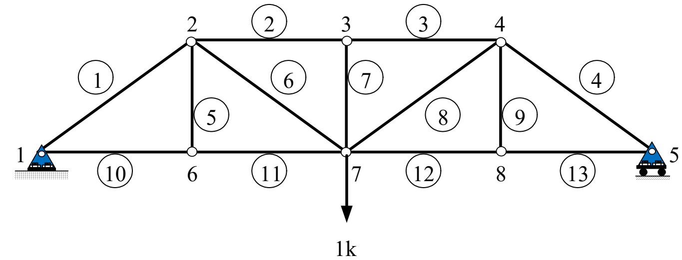
$$\theta_C(EI) := \int_0^{20} (-0.05x)\cdot\left(\frac{30x - 1.5x^2}{EI}\right)dx \rightarrow -\frac{1000}{EI}$$

$$\boxed{\theta_C := \frac{-1000\text{kip}\cdot\text{ft}^2}{E\cdot I} = -0.021\cdot\text{rad}}$$

13.5 Truss Deflection

Determine the vertical deflection of joint 7 in the truss shown $E := 30000\text{ksi}$





Solution:

Two analyses are required. One with the real load, and the other using a unit vertical load at joint 7. Results for those analyses are summarized below. Note that advantage was taken of the symmetric load and structure.

Member	A in^2	L ft	P^e kip	δP kip	δPPL/A k-ft/in^2	n	nδPPL/A k-ft/in^2
1 & 4	2	25	-50	-0.083	518.75	2	1037.5
10 & 13	2	20	40	0.67	268	2	536
11 & 12	2	20	40	0.67	268	2	536
5 & 9	1	15	20	0	0	2	0
6 & 8	1	25	16.7	0.83	346.5	2	693
2 & 3	2	20	-53.3	-1.33	708.9	2	1417.8
7	1	15	0	0	0	1	0
Total							4220.3

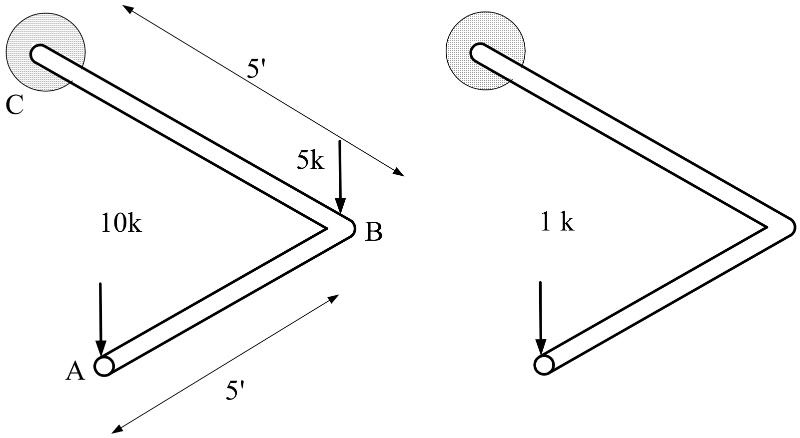
$$\Delta\cdot\delta\cdot\overline{P} = \int_0^L \delta\cdot\overline{P}\cdot\frac{P}{A\cdot E}\,dx$$

$$1\cdot\Delta = \Sigma\delta\cdot\overline{P_e}\cdot\frac{P_e\cdot L}{A\cdot E}$$

$$\Delta := \frac{4220.3\,\frac{kip\cdot ft}{in^2}}{E} = 1.688\cdot in$$

13.6 Torsional and Flexural Deformation

Determine the vertical deflection at A in the structure shown. $E := 30000ksi,\ I := 144in^4,\ G := 12000ksi,\ J := 288in^4$



Solution:

1. In this problem we have both flexural and torsional deformation. Hence we should determine the internal moment and torsion distribution for both the real and the unit virtual load.
2. Then we will use the following relation

$$\delta \cdot \bar{P} \cdot \Delta_A = \int \delta \cdot \bar{M} \cdot \frac{M}{EI} dx + \int \delta \cdot \bar{T} \cdot \frac{T}{GJ} dx$$

3. The moment and torsion expressions are given by

Element	x = 0	M	δM	T	δT
AB	A	10x	x	0	0
BC	B	15x	x	50	5

4. Substituting

$$\delta \cdot \bar{P} \cdot \Delta_A = \int \delta \cdot \bar{M} \cdot \frac{M}{EI} dx + \int \delta \cdot \bar{T} \cdot \frac{T}{GJ} dx$$

$$1 \cdot \Delta_A = \int_0^5 x \cdot \frac{10x}{EI} dx + \int_0^5 x \cdot \frac{15x}{EI} dx + \int_0^5 5 \cdot \frac{50}{GJ} dx$$

$$\Delta_A(E, I, G, J) := \int_0^5 x \cdot \frac{10x}{E \cdot I} dx + \int_0^5 x \cdot \frac{15x}{E \cdot I} dx + \int_0^5 5 \cdot \frac{50}{G \cdot J} dx \rightarrow \frac{3125}{3 \cdot E \cdot I} + \frac{1250}{G \cdot J}$$

$$\boxed{\Delta_A := \frac{3125 kip \cdot ft^3}{3 \cdot E \cdot I} + \frac{1250 kip \cdot ft^3}{G \cdot J} = 1.042 \cdot in}$$

13.7 Flexural and Shear Deformations in a Beam

Determine the deflection of a cantilevered beam, of length L, subjected to an end force P due to both flexural and shear deformations. Assumed G=0.4E and a square solid beam cross section.

Solution:

1. The virtual work equation is

$$\delta\cdot\overline{P}\cdot\Delta = \int_0^L \delta\cdot\overline{M}\cdot(x)\cdot d\phi(x)\,dx + \int_0^L \delta\cdot\overline{V}\cdot(x)\cdot\gamma_{xy}(x)\,dx = \int_0^L \delta\cdot\overline{M}(x)\cdot\frac{M(x)}{EI}\,dx + \int_0^L \delta\cdot\overline{V}(x)\cdot\lambda\cdot\frac{V(x)}{GA}\,dx$$

2. The first integral yields for M=Px, and $\delta\cdot M = 1\cdot x$

$$\int_0^L \delta\cdot\overline{M}(x)\cdot\frac{M(x)}{EI}\,dx = \frac{P}{EI}\cdot\int_0^L x^2\,dx$$

$$\frac{P}{EI}\cdot\int_0^L x^2\,dx \rightarrow \frac{94280057135446.5\cdot N\cdot ft^3}{EI}$$

3. The second integral represents the contribution of the shearing action to the total internal virtual work and hence to the total displacement.
4. Both the real shear V and the virtual shear $\delta\cdot\overline{V}$ are constant along the length of the member, hence

$$\int_0^L \delta\cdot\overline{V}(x)\cdot\lambda\cdot\frac{V(x)}{GA}\,dx = \frac{\lambda}{GA}\cdot\int_0^L 1\cdot(P)\,dx$$

$$\frac{\lambda}{GA}\cdot\int_0^L 1\cdot(P)\,dx \rightarrow \frac{374606093684.841\cdot GPa\cdot N\cdot ft}{GA}$$

5. Since $\lambda := 1.2$ for a square beam; hence

$$I = \frac{h^4}{12} \quad \text{and} \quad A = h^2$$

then

$$\Delta = \frac{P\cdot L}{3\cdot E}\cdot\left(\frac{12L^2}{h^4} + \frac{9}{h^2}\right) = \frac{3P\cdot L}{E\cdot h^2}\cdot\left(\frac{1.33L^2}{h^2} + 1\right)$$

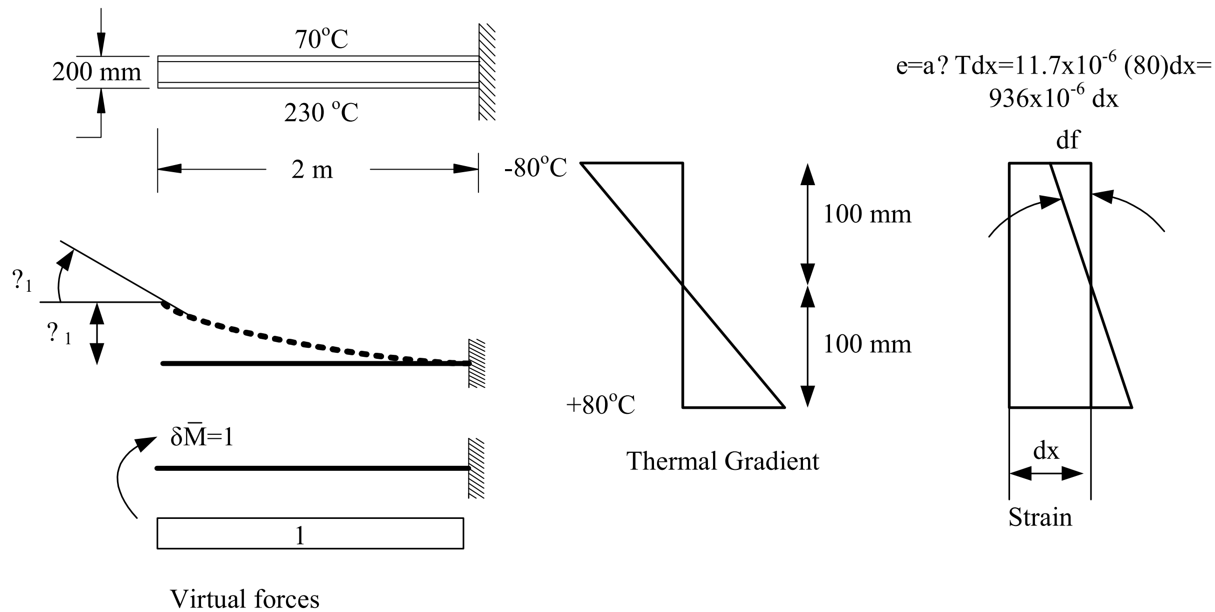
6. Choosing $L := 20ft$ and $h := 1.5ft$ (L/h=13.3)

$$\Delta = \frac{3P\cdot L}{E\cdot h^2}\cdot\left[1.33\cdot\left(\frac{20ft}{1.5ft}\right)^2 + 1\right] = \frac{3P\cdot L}{E\cdot h^2}\cdot(237 + 1)$$

Thus the flexural deformation is 237 times the shear displacement. This comparison reveals why we normally neglect shearing deformation in beams. As the beam get shorter or deeper, or as L/h decreases, the flexural deformation decreases relative to the shear displacement. At L/h=5, the flexural deformation has reduced to $1.33\cdot 5^2 = 33.25$ times the shear displacement.

13.8 Thermal Effects in a Beam

Consider the cantilever beam shown. If the beam is a steel, wide flange section, 2 m long and 200 mm deep, what is the angle of rotation, θ_1, at the end of the beam caused by the temperature effect? The original uniform temperature of the beam was 40C.



Solution:

1. The external vurtual force conforming to the desired real displacement θ_1 is a moment $\delta \cdot \overline{M} = 1$ at the tip of the cantilever, producing an external work term of moment times rotation. The internal virtual force system for this cantilever beam is a uniform moment $\delta \cdot \overline{M_{int}} = 1$.

2. The real internal deformation results from (a) the average beam temperature of 150C, which is 110C above that of the original temperature, and (b) the temperature gradient of 160C across the depth of the beam.

3. The first part of the thermal effect produces only a lengthening of the beam and does not enter into the work equation since the virtual loading produces no axial force corresponding to an axial chnge in length of the beam.

4. The second effect (thermal gradient) produces rotation dϕ, and an internal virtual work term of $\int_0^L \delta \cdot \overline{M} \cdot \phi \, dx$

5. We determine the value of dϕ by considering an extreme fiber thermal strain as shown above. The angular rotation in the length dx is the extreme fiber thermal strain divided by half the beam depth.

$$\frac{M}{EI} = \frac{d^2 \cdot y}{dx^2} = \frac{d\theta}{dx} = \frac{\varepsilon}{y} = \phi$$

$$1 \cdot \theta_1 = \int_0^L \delta \cdot \overline{M} \cdot \phi \, dx$$

$$\theta_1 = \int_0^L 1 \cdot \frac{\alpha \cdot (T_B - T_T)}{h} \, dx$$

$$\theta_1 := \frac{11.7 \cdot 10^{-6} \frac{m}{m} \cdot (230 - 70) \cdot 2m}{0.2m} = 0.01872 \cdot rad$$

6. This example raises the following points
   a) The value of θ_1 would be the same for any shape of 200 mm deep steel beam that has its neutal axis of bending at middepth

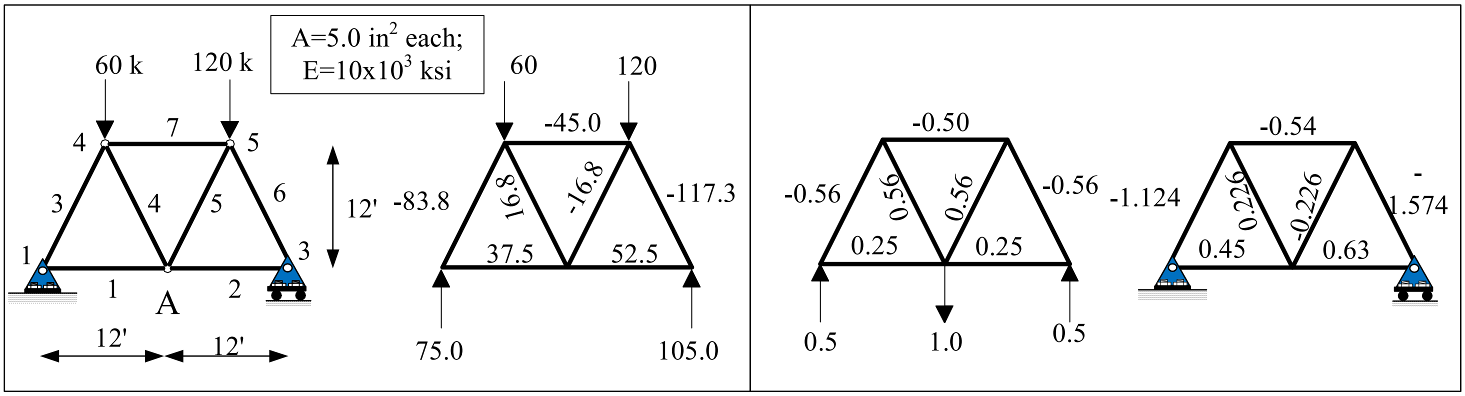
b) Curvature is produced only by thermal gradient and is independent of absolute temperature values.

c) The calculation of rotations by the method of virtual forces is simple and straightforward; the applied virtual force is a moment acting at a point where rotation is to be calculated.

d) Internal angular deformation $d\phi$ has been calculated for an effect other than load-induced stresses. This extension of the method of virtual forces to treat inelastic displacements is obvious - all we need to know is a method for determining the inelastic internal deformations.

13.9 Deflection of a Truss

Determine the deflection at node 2 for the truss shown



Solution:

Member	δP kip	P^e kip	L ft	A in^2	E ksi	δPPL/AE
1	0.25	37.5	12	5	10 x 10^3	22.5 x 10^{-4}
2	0.25	52.5	12	5	10 x 10^3	31.5 x 10^{-4}
3	-0.56	-83.8	13.42	5	10 x 10^3	125.9 x 10^{-4}
4	0.56	16.8	13.42	5	10 x 10^3	25.3 x 10^{-4}
5	0.56	-16.8	13.42	5	10 x 10^3	-25.3 x 10^{-4}
6	-0.56	-117.3	13.42	5	10 x 10^3	176.6 x 10^{-4}
7	-0.5	-45	12	5	10 x 10^3	54 x 10^{-4}
Total						410.5 x 10^{-4}

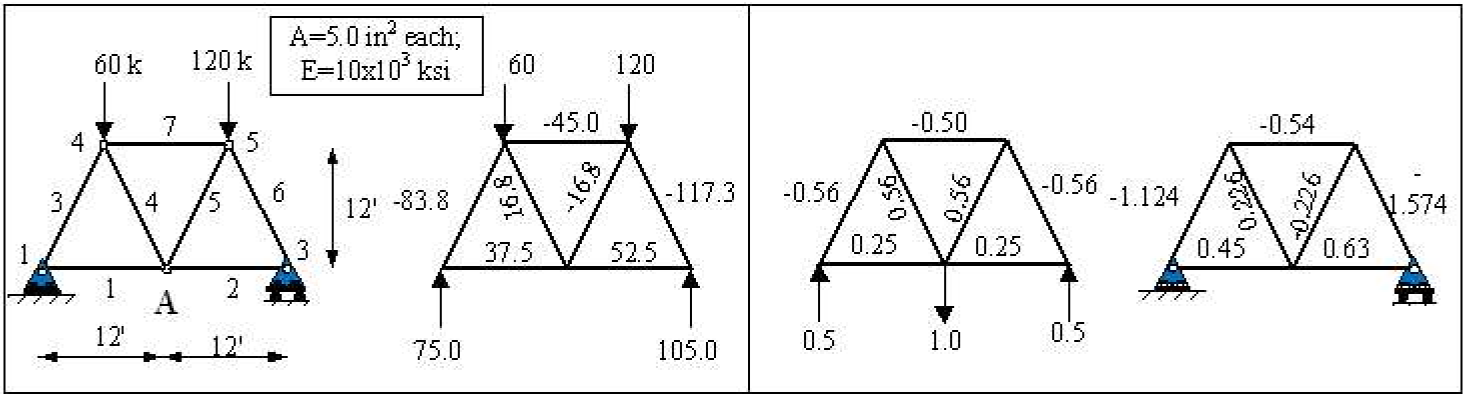
The deflection is thus given by

$$\delta \cdot \overline{P} \cdot \Delta = \sum_{n=1}^{7} \left(\delta \cdot \overline{P_e} \cdot \frac{PL}{AE} \right)$$

$$\Delta := 410.5 \cdot 10^{-4} \cdot ft = 0.493 \cdot in$$

13.10 Thermal Deflection of a Truss; I

The truss shown is built such that the lower chords are shielded from the rays of the sun. On a summer day the lower chords are 30F cooler than the rest of the truss members. What is the magnitude of the vertical displacement at joint 2 as a result of this temperature difference?



Solution:

1. The virtual force system remains identical to that of the previous example because the desired displacement component is the same.
2. The real internal displacements are made up of the shortening of those members of the truss that are shielded from the sun.
3. Both the bottom chord members 1 and 2 thus shorten by

$$\Delta L = \alpha \cdot \Delta T \cdot L$$

$$\Delta L := \left(0.0000128 \frac{\text{in}}{\text{in}\cdot\Delta°\text{F}}\right)(30\cdot\Delta°\text{F})\cdot 12\text{ft} = 0.0553\cdot\text{in}$$

4. Then,

$$1\cdot\Delta = \sum \delta\cdot\bar{P}\cdot\Delta L$$

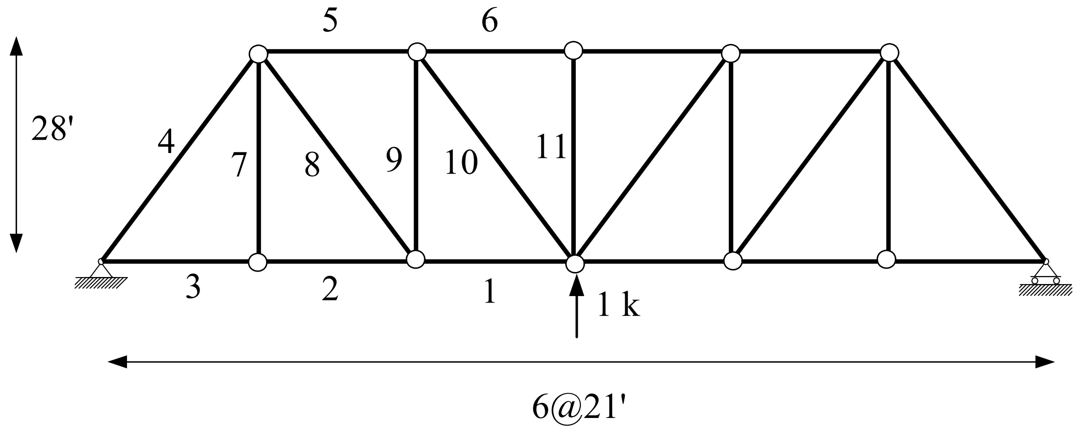
$$\Delta := 0.25(-\Delta L) + 0.25(-\Delta L)$$

$$\boxed{\Delta = -0.0276\cdot\text{in}}$$

5. The negative sign n the displacement indicates that it is in opposite sense of the displacement; the assumed direction is always identical to the direction of the applied virtual force.
6. Note that the same result would be obtained if we had considered the internal displacements to be made up of lengthening of all truss members above the bottom chord.

13.11 Thermal Deflection of a Truss; II

A six panel highway bridge is constructed with sidewalks outside the trusses so that the bottom chords are shaded. What will be the vertical deflection component of the bottom chord at the center of the bridge when the temperature of the bottom chord is 40F (ΔT) below that of the top chord, endposts, and webs? (coefficient of steel thermal expansion is $\alpha := 0.0000065 \cdot \frac{1}{\Delta°F}$



Solution:

1. The deflection is given by

$$\Delta \cdot \delta \cdot \overline{P} = \int_0^L \delta \cdot \overline{P} \cdot \frac{P}{AE} dx = \sum \delta \cdot \overline{P_i} \cdot \frac{P_i \cdot L_i}{AE} = \sum \delta \cdot \overline{P_i} \cdot \Delta L_i = \sum \delta \cdot \overline{P_i} \cdot \alpha \cdot \Delta T \cdot L$$

where ΔL is the temperature change in length of each member, and $\delta \cdot \overline{P}$ are the member virtual internal forces

2. Taking advantage of symmetry (Note that we ignor members 1-3 because we assumed that they had the reference temperature, and all other members are subjected to a relative temperature increase of 40F)

Member	L ft	αΔTL	δP kip	δPΔL
4	35	0.0091	0.625	0.00568
5	21	0.00546	0.75	0.00409
6	21	0.00546	1.13	0.00616
7	0	0	0	0
8	35	0.0091	-0.625	-0.00568
9	28	0.00728	0.5	0.00364
10	35	0.0091	-0.625	-0.00568
11	0	0	0	0
Total				0.00821

3. Hence the total deflection is

$$\Delta := 2 \cdot 0.00821 \text{ft} = 0.197 \cdot \text{in}$$

4. A more efficient solution would have consisted in considering members 1, 2, and 3 only and applying ΔT=-40F, we would obtain the same displacement.

5. Note that the forces in members 1, 2, and 3 (-0.75, -0.375, and -0.375 respectively) were not included in the table because the corresponding ΔT=0.

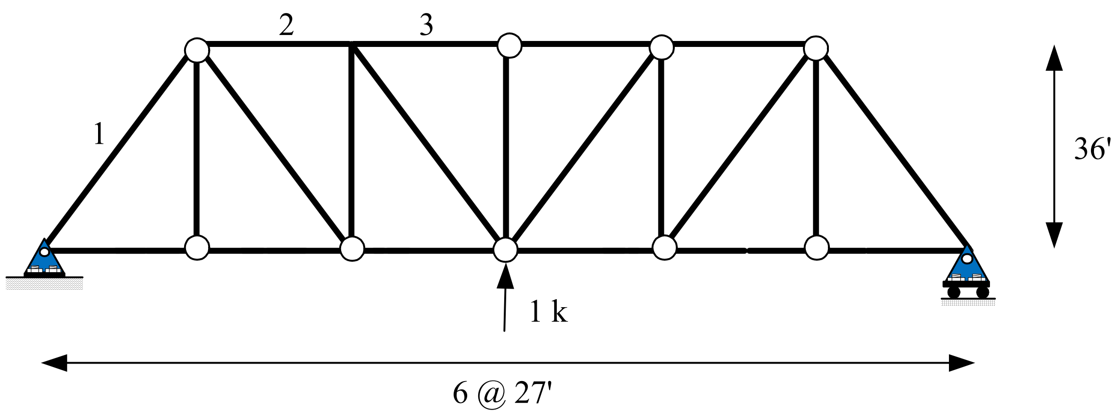
6. A simple solution would have $\Delta T := -40 \cdot \Delta°F$ in members 1, 2,and 3 thus

Member	L ft	αΔTL	δP kip	δPΔL
1	21	-0.00546	-0.75	0.004095
2	21	-0.00546	-0.375	0.0020475
3	21	-0.00546	-0.375	0.0020475
Total				0.00819

$\Delta := 2 \cdot 0.00819\text{ft} = 0.197 \cdot \text{in}$

13.12 Truss with Initial Camber

It is desired to provide 3 in. of camber at the center of the truss shown below by fabricating the endposts and top chord members additionally long. How much should the length of each endpost and each panel of the top chord be increased?



Solution:

1. Assume that eachtop chord is increased 0.1 in.

Member	δP_{int} kip	ΔL in	$\delta P \Delta L$
1	0.625	0.1	0.0625
2	0.75	0.1	0.075
3	1.125	0.1	0.1125
Total			0.25

Thus,

$$2 \cdot 0.25\text{in} = 0.5 \cdot \text{in}$$

2. Since the structure is linear and elastic, the required increase of length for each section will be

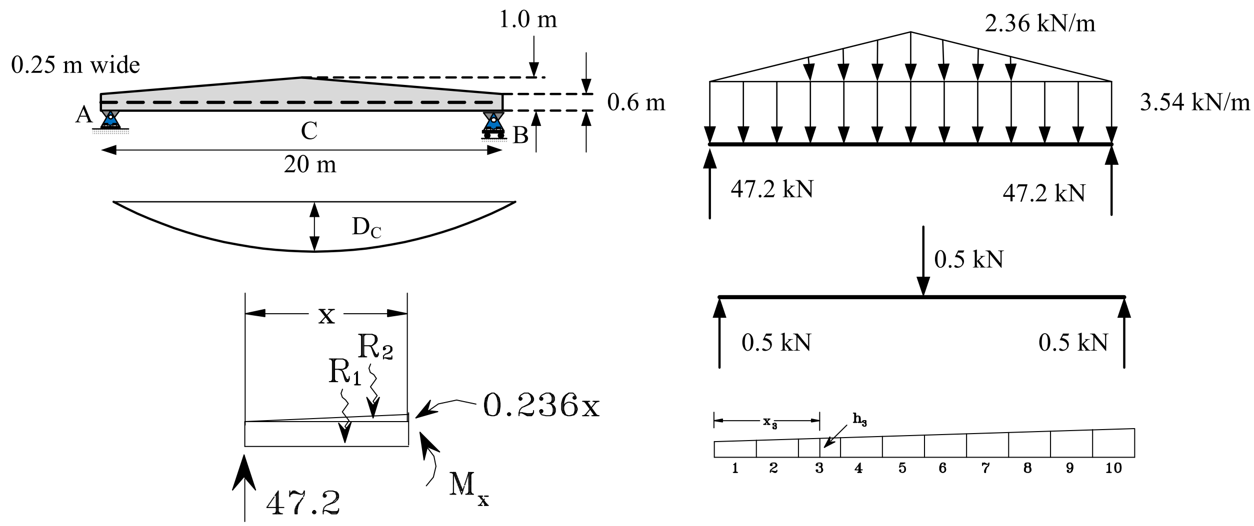
$$\frac{3\text{in}}{0.5\text{in}} \cdot 0.1\text{in} = 0.6 \cdot \text{in}$$

3. If we use the practical value of 0.625in

$$\boxed{\frac{0.625\text{in} \cdot 0.5\text{in}}{0.1\text{in}} = 3.125 \cdot \text{in}}$$

13.13 Prestressed Conrete Beam with Continuously Variable I

A prestressed concrete beam is made of variable depth for proper location of the straight pretensioning tendon. Determine the midspan displacement (point c) produced by the dead weight of the girder. The concrete weighs $w := 23.6\frac{kN}{m^3}$ and has $E := 25000MPa$. The beam is 0.25m wide.



Solution:

1. We seek an expression for the real moment M. This is accomplished by first determining the reactions and then considering the free body diagram.
2. We have the intermediary resultant forces

$$R_1(x) := 0.25m\cdot x\cdot 0.6m\cdot w \rightarrow \frac{3.54\cdot kN\cdot x}{m}$$

$$R_2(x) := \frac{1}{2}\cdot 0.25m\cdot x\cdot\frac{0.4m}{10m}\cdot x\cdot w \rightarrow \frac{0.118\cdot kN\cdot x^2}{m^2}$$

Hence,

$$M(x) := 47.2x - 3.54x\cdot\left(\frac{x}{2}\right) - 0.118x^2\cdot\left(\frac{x}{3}\right) \rightarrow 47.2\cdot x + -0.039\cdot x^3 + -1.77\cdot x^2$$

3. The moment of inertia of the rectangular beam varies continuously and is given, for the left half of the beam, by

$$I(x) = \frac{1}{12}\cdot b\cdot h^3$$

$$I(x) := \frac{1}{12}\cdot(0.25)\cdot(0.6 + 0.04x)^3 \rightarrow 0.021\cdot(0.04\cdot x + 0.6)^3$$

4. Thus, the real angle changes produced by the dead load bending are

$$d\phi = \frac{M}{EI}\cdot dx = \frac{47.2\cdot x + -0.039\cdot x^3 + -1.77\cdot x^2}{E\cdot\left(\frac{1}{48}\right)\cdot(0.6 + 0.04x)^3}$$

5. The virtual force system corresponding to the desired displacement is shown above with $\delta\cdot\overline{M} = \frac{1}{2}\cdot x$ for the left half of the span. Since the beam is symmetrical, the virtual work equations can be evaluated for only one half of the beam and the final answer is then obtained by multiplying the half-beam result by two.

6. The direct evaluation of the integral $\int \delta\cdot\overline{M}\,d\phi$ is difficult beause of the expression for dϕ. Hence we shall use a numerical procedure,

replacing the $\int \delta\cdot\overline{M}\cdot\left(\frac{M}{EI}\right)dx$ with the $\sum \delta\cdot\overline{M}\left(\frac{M}{EI}\right)\Delta x$, where each quantity in the summation is evaluated at the center of the interval Δx

and held constant over the interval length. As Δx becomes very short, the solution approaches the exact answer.
7. An interval length of 1 meter, giving 10 elements in the half length of the beam, is chosen to establish an accurate result.
8. The internal virtual work quantity is then

$$\int_0^{\frac{L}{2}} \delta\cdot\overline{M}(x)\cdot\frac{M(x)}{EI}\,dx \sim \sum\left[\delta\cdot\overline{M}(x)\cdot\frac{M(x)}{EI}\cdot\Delta x\right] = \sum \delta\cdot\overline{M}(x)\cdot\frac{M(x)}{E\left(\frac{0.25}{12}\right)\cdot h^3}\cdot\Delta x = \frac{48}{E}\cdot\sum\left[\delta\cdot\overline{M}(x)\cdot\frac{M(x)}{h^3}\cdot\Delta x\right]$$

9. The summation for the 10 elements in the left half of the beam gives

Segment	x	h	h^3	M	δM	$M\delta M/h^3$
1	0.5	0.62	0.238	23.2	0.25	24
2	1.5	0.66	0.288	66.7	0.75	174
3	2.5	0.7	0.343	106.4	1.25	388
4	3.5	0.74	0.405	150	1.75	648
5	4.5	0.78	0.475	173	2.25	820
6	5.5	0.82	0.551	200	2.75	998
7	6.5	0.86	0.636	222	3.25	1134
8	7.5	0.9	0.729	238	3.75	1224
9	8.5	0.94	0.831	250	4.25	1279
10	9.5	0.98	0.941	256	4.75	1292
Total						7981

10. The SI units should be checked for consistency. Letting the virtual force carry the units of kN, the virtual moment has the units of m-kN, and the units of the equation

$$\frac{1}{kN}\cdot\frac{(m\cdot kN)\cdot(m\cdot kN)}{\frac{MN}{m^2}\cdot m^4}\cdot m = \frac{m}{1000} = mm$$

11. Then

$$\int_0^L \frac{\delta\cdot\overline{M}\cdot M}{EI}\,dx$$

$$2\cdot\left(\frac{48}{25000}\right)\cdot 7981\cdot 1\cdot mm = 30.65\cdot mm$$

and the deflection at midspan is

$$\Delta_c := 30.65\ mm$$

12. Acceptably accurate results may be obtained with considerably fewer elements (longer intervals Δx). Using four elements with centers at 2, 5, 8, and 10, the $\sum \delta\cdot\overline{M}\left(\frac{M}{EI}\right)\Delta x$ is

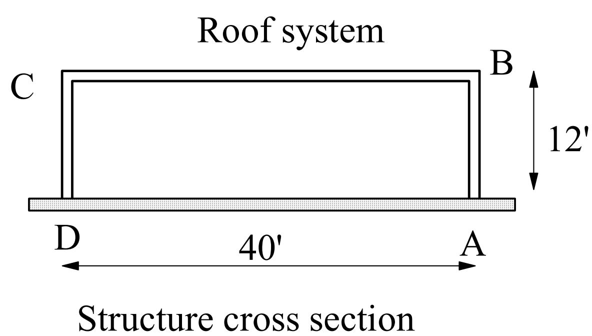
$$3(174) + 3\cdot(820) + 3\cdot(1224) + 1\cdot(1292) = 7946$$

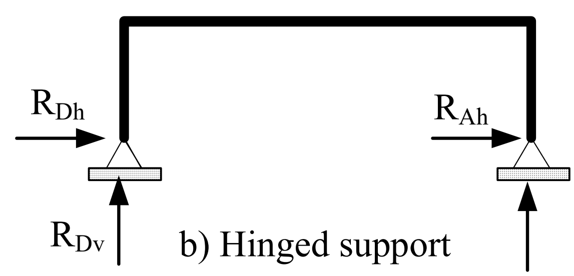
which is only 0.4% lower than the 10 element solution. If we go to two elements, 3 and 8, we obtain a summation of $5 \cdot 388 + 5 \cdot 1224 = 8060$ which is 1% high. A one element solution, with x=5m and h=0.8m, gives a summation of 9136 which is 14.4% high and much less accurate than the 2 element solution.

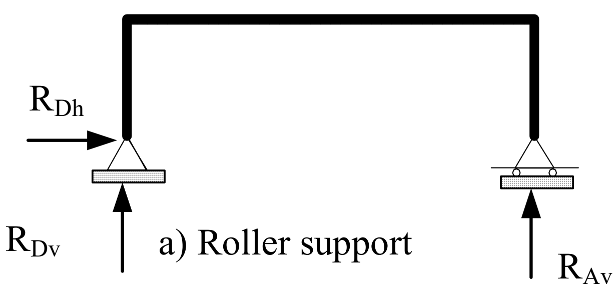
13. Finally, it should be noted that the calculations involved in this example are essentially identical to those necessary in the moment area method.

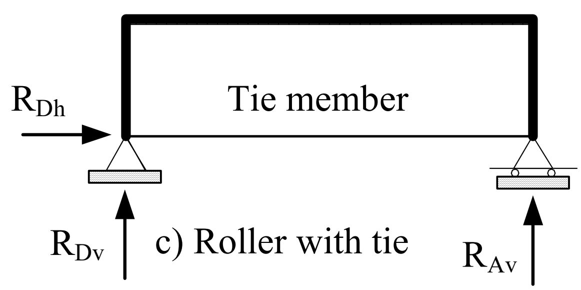
14.1 Steel Building Frame Analysis

A small, mass-produced industrial building is to be framed in structural steel with a typical cross as shown below. The engineer is considering three different designs for the frame: (a) for poor or unknown soil conditions, the foundation for the frame may not be able to develop any dependable horizontal forces at its bases. In this case the idealized base conditions are a hinge at one of the bases and a roller at the other; (b) for excellent soil conditions with properly designed foundations, the base of the frame legs will have no tendency to move horizontally, and the idealized base condition is that of hinges at both points A and D; and (c) a design intermediate to the above cases, with a steel tie member capable of carrying only tension running between points A and D in the floor of the building. The foundations would not be expected to provide any horizontal restraint for this latter case, and the hinge-roller details at points A and D would apply. Critical design loads for a frame of this type are usually the gravity loads (deal load + snow load) and the combination of dead load and wind load. We will restrict our attention to the first condition, and will use a snow load of $S := 30\text{psf}$ and an estimated total dead load of $D := 20\text{psf}$. With frames spaced at $\text{spacing} := 15\text{ft}$ on centers along the length of the building, the design load is $w_{total} := \text{spacing}\cdot(S + D) = 750\cdot\frac{\text{lbf}}{\text{ft}}$. If the frame is made of steel beam sections 21 in. deep and weight 62 lb/ft of length (W 21x62), and the member for design (c) is tentatively chosen as a 2 in^2 bar, determine the bending moment digrams for the three designs and discuss the alternate solutions



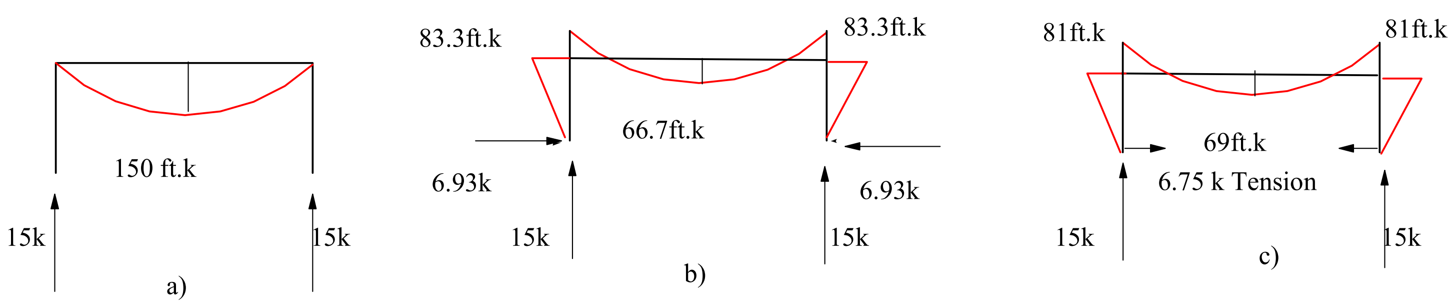






Solution:

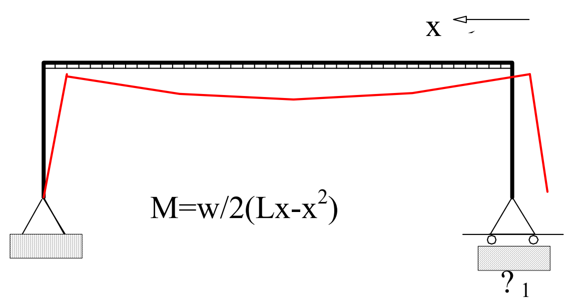
Structure a This frame is statically determinate since it has three possible unknown external forces acting on it. Final bending moments are shown below.



Structure b Hinging both legs of the frame results in another unknown force, making the structure statically indeterminate to the first degree (one redundant).

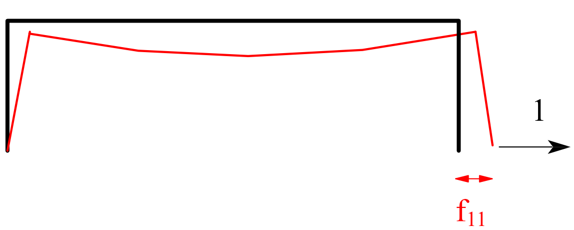
1. A lateral release at point A is chosen, with the redundant shearing force R1. The displacement Δ_1 in the primary structure, as a result of the real loading, shown in the figure below, is computed by virtual work.

2. The virtual force system produces virtual bending moment $\delta \cdot \overline{M}$, which is uniform across the top member of the frame.

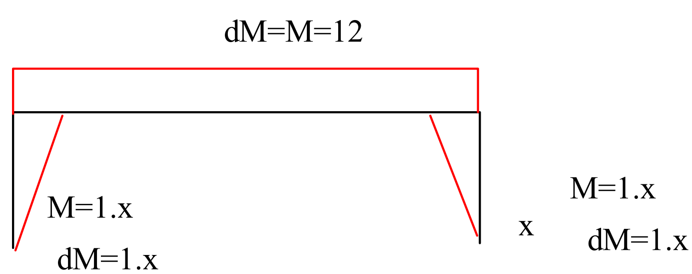


a) Moment caused by actual load on primary structure

b) Virtual forces



c) Real load for determining f_{11}



d) Moments produced by virtual forces and unit redundant

The virtual moment acting through the real angle changes given the internal work term

$$1 \cdot \Delta_1 = \int_0^{40} \delta \cdot \overline{M} \cdot \frac{M}{EI}\, dx$$

3. Equating this to the external virtual work of $1 \cdot \Delta_1$, we have

$$1 \cdot \Delta_1 = \int_0^{40} 12 \cdot \frac{\frac{1}{2} \cdot 0.75\left(40x - x^2\right)}{EI}\, dx \cdot kip \cdot ft^3$$

$$\Delta_1(EI) := \int_0^{40} 12 \cdot \frac{\frac{1}{2} \cdot 0.75\left(40x - x^2\right)}{EI}\, dx \cdot kip \cdot ft^3 \rightarrow \frac{48000 \cdot ft^3 \cdot kip}{EI}$$

4. The equation of consistent displacement is $\Delta_1 + f_{11} \cdot R_1 = 0$. The flexibility coefficient f_{11} is computed by applying a unit horizontal force at the release and determining the displacement at the same point.

5. It is seen that the real loading and the virtual loading are identical for the calculations, and

$$1\cdot f_{11} = 2\left(\int_0^{12} x\cdot\frac{x}{EI}\,dx + \int_0^{20} 12\cdot\frac{12}{EI}\,dx\right)$$

or

$$f_{11}(EI) := 2\left(\int_0^{12} x\cdot\frac{x}{EI}\,dx + \int_0^{20} 12\cdot\frac{12}{EI}\,dx\right)\cdot kip\cdot ft^3 \rightarrow \frac{6912\cdot ft^3\cdot kip}{EI}$$

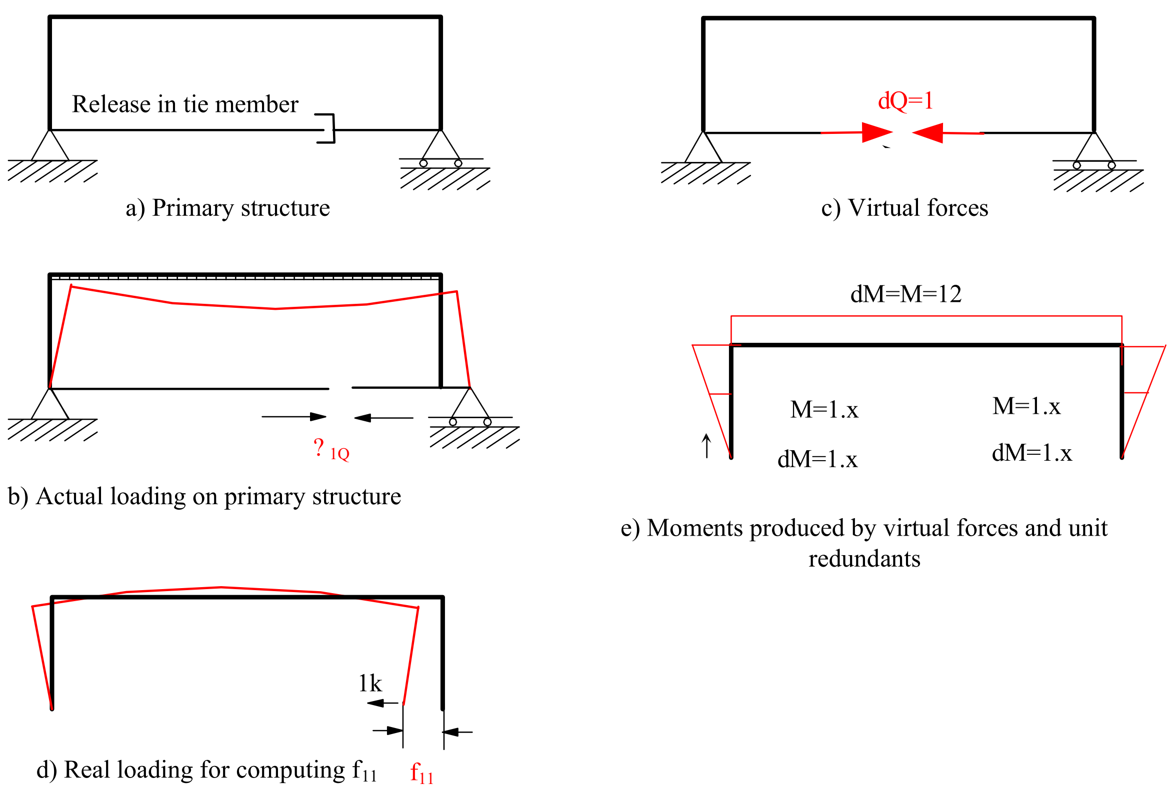
6. solving for R1

$$\frac{1}{EI}\cdot\left(48000 + 6912R_1\right) = 0$$

or

$$R_1 := \frac{48000}{6912}\cdot kip = 6.944\cdot kip$$

Structure c The frame with the horizontal tie between the points A and D has three unknown external forces. However, the structure is statically indeterminate to the first degree since the tie member provides one degree of internal redundancy.
1. The logical release to choose is a longintudinal release in the tie member, with the associated longitudinal displacement and axial force.
2. The primary structure is the frame with the tie member released.



a) Primary structure

c) Virtual forces

b) Actual loading on primary structure

e) Moments produced by virtual forces and unit redundants

d) Real loading for computing f_{11}

The compatability equation is based on the fact that the displacement at the release must be zero; that is, relative displacement of the two sections of the tie at the point of release must be zero, or

$$\Delta_1 + f_{11}\cdot R_1 = 0$$

where

Δ_1 = displacement at release 1 in the primary structure, produced by the loading

f_{11} = relative displacement at release 1 for a unit axial force in the member

R_1 = force in the tie member in the original structure

3. Virtual work is used to determine both displacement terms.
4. The value of Δ_1 is identical to the displacement for structure (b) because the tie member has no forces (and consequently no deformation) in the primary structure. Thus,

$$\Delta_1 := \frac{48000\text{kip}\cdot\text{ft}^3}{30000\text{ksi}\cdot 1327\text{in}^4} = 2.083\cdot\text{in}$$

5. The flexibility coefficient f_{11} is composed of two separate effects: a flexural displacement due to the flexibility of the frame, and the axial displacement of the stressed tie member. The virtual and real loadins for this calculation are shown in the previous figure.

$$1\cdot f_{11} = 2\left(\int_0^{12} x\cdot\frac{x}{EI}\,dx + \int_0^{20} 12\cdot\frac{12}{EI}\,dx\right) + \delta\cdot\bar{P}\cdot\frac{PL}{EA}$$

$$1f_{11} = \frac{6912}{EI} + \frac{1\cdot 1\cdot 40}{EA}$$

$$f_{11} := \left(\frac{6912\text{kip}\cdot\text{ft}^3}{30000\text{ksi}\cdot 1327\text{in}^4} + \frac{40\cdot\text{kip}\cdot\text{ft}}{30000\text{ksi}\cdot 2\text{in}^2}\right)\cdot\frac{1}{\text{kip}} = 0.308\cdot\frac{\text{in}}{\text{kip}}$$

6. The equation of consistent deformation is

$$\Delta_1 + f_{11}\cdot R_1 = 0$$

or

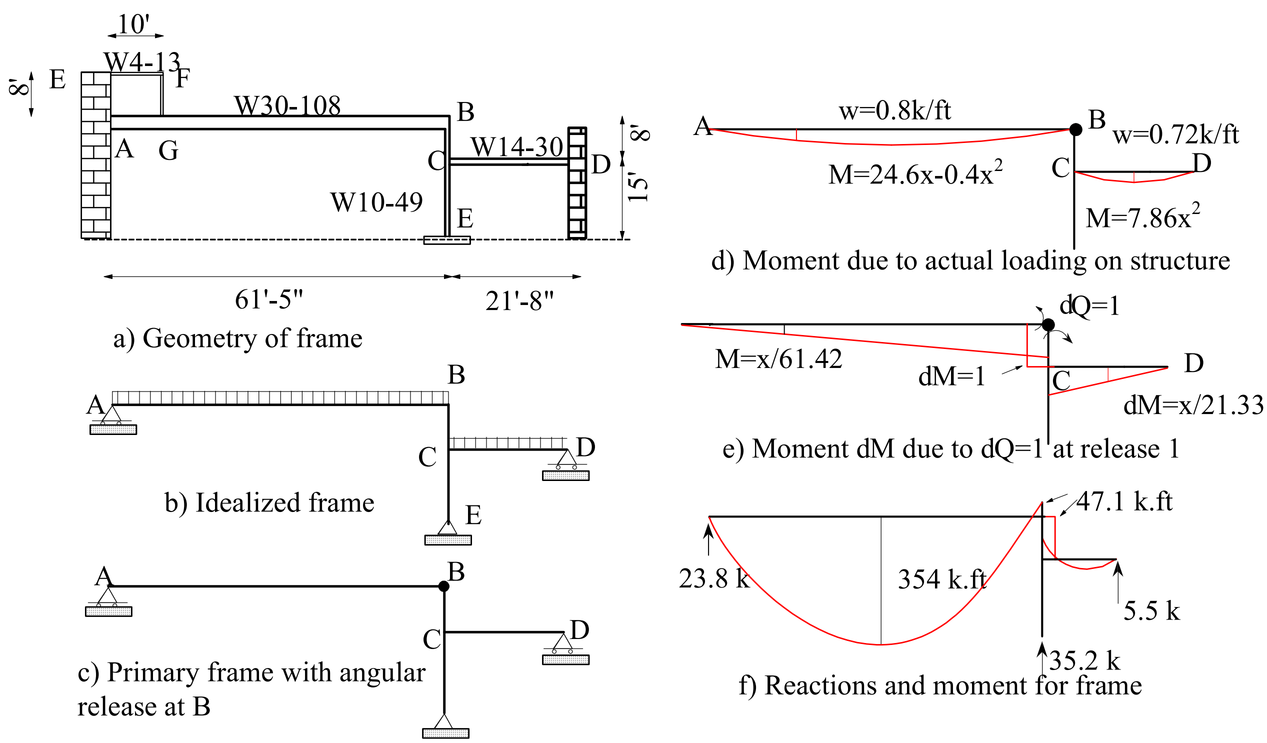
$$\boxed{R_1 := \frac{-\Delta_1}{f_{11}} = -6.764\cdot\text{kip}}$$

7. The two displacement terms in the equation must carry opposite signs to account for their differences in direction.

Comments The bending moment in the frame differes only slightly from that of structure (b). In other words, the tie member has such high axial stiffness that it provides nearly as much restraint as the foundation of structure (b). Frames with tie members are used widely in industrial buildings. A lesson to be learned here is that it is easy to provide high stiffness through an axially loaded member.
The maximum moment in frames (b) and (c) is about 55% of the maximum moment in frame (a). This effect of continuity and redundancy is typical - the positive bending moments in the members are lowered while the joint moments are increased and a more economical design can be realized. Finally, we should notice that the vertical reactions at the bases of the columns do not change with the degree of horizontal restraint at the bases.

14.2 Analysis of Irregular Building Frame

The structural steel frame for the Church of the Holy Spirit, Penfield, New York is shown below. In this example we will discuss the idealization of the structure and then determine the forces and bending moments acting on the frame.



Solution:

1. The two main horizontal members of the frame are supported at points A and D by masonry walls.
2. The connection used at these points is not intended to transmit axial forces from the frame to the wall; accordingly, the axial forces in the horizontal members are assumed to be zero and the joints at A and D are idealized as rollers that transmit vertical forces only.
3. The base joint E is designed to resist both horizontal and vertical loads, but not moment, and is assumed to be a hinge.
4. Finally, joints B and C are designed to provide continuity and will be taken as rigid; that is, the angles of intersection of the members at the joint do not change with applied loading.
5. The frame is simplified for analysis by removing the small 4 in. wide flange members EF and FG and replacing their load effect by applying the roof load with acts on EF directly to the segment AG.
6. the idealized frame is shown above.
7. The dead load on the higher portion of the frame is $p_{AB} := 25\text{psf}$ times the frame spacing of $\text{spacing} := 13.33\text{ft}$, or

$w_{AB} := p_{AB} \cdot \text{spacing} = 333 \cdot \frac{\text{lbf}}{\text{ft}}$ along the frame.

8. The dead load on CD is less beacuse the weight of the frame member is substantially smaller, and the dead load is about $p_{CD} := 19\text{psf}$,

or $w_{CD} := p_{CD} \cdot \text{spacing} = 253 \cdot \frac{\text{lbf}}{\text{ft}}$.

9. Snow load is $S := 35\text{psf}$ over both areas, or $w_{Snow} := S \cdot \text{spacing} = 467 \cdot \frac{\text{lbf}}{\text{ft}}$.

10. The total loads are then

Member AB: $w_{totAB} := w_{AB} + w_{Snow} = 0.8 \cdot \frac{\text{kip}}{\text{ft}}$

Member CD: $w_{totCd} := w_{CD} + w_{Snow} = 0.72 \cdot \frac{\text{kip}}{\text{ft}}$

11. the frame has four unknown reaction components and therefore has one redundant. Although several different releases are possible, we choose an angular (bending) release at point B.
12. The resulting primary structure is shown as (c) in the figure above, where the redundant quantitity R1 is the bending moment at point B.
13. The equation of compatability is

$$\theta_1 + f_{11}\cdot M_1 = 0$$

where θ_1 is the relative angular rotation corresponding to release 1 as produced by the actual loading, and f_{11} is the flexibility coefficient for a unit moment acting at the release.

14. From virtual work we have

$$1\cdot\theta_1 = \int \delta\cdot\overline{M}\cdot\frac{M}{EI}\,dx$$

and

$$1\cdot f_{11} = \int \delta\cdot\overline{M}\cdot\frac{M}{EI}\,dx$$

where $\delta\cdot\overline{M}$ and M are defined in (e) in the figure above.

15. Then

$$\theta_1(EI_{AB},EI_{CD}) := \left[\frac{1}{EI_{AB}}\cdot\int_0^{61.42}\frac{x}{61.42}\cdot\left(24.6x - 0.4x^2\right)dx + \frac{1}{EI_{CD}}\cdot\int_0^{21.33}\frac{x}{21.33}\cdot\left(7.68x - 0.36x^2\right)dx\right]$$

$$\theta_1(EI_{AB},EI_{CD}) \rightarrow \frac{291.31658667}{EI_{CD}} + \frac{7763.6329512000000002}{EI_{AB}}$$

16. With $I_{AB} := 4470in^4$ and $I_{CD} := 290in^4$

$$\theta_1(E) := \left(\frac{291}{E\cdot I_{CD}} + \frac{7764}{E\cdot I_{AB}}\right)\cdot in^4 \rightarrow \frac{118411}{43210\cdot E}$$

17. Similarly

$$f_{11}(EI_{AB},EI_{CD},EI_{BC}) := \frac{1}{EI_{AB}}\cdot\int_0^{61.42}\left(\frac{x}{61.42}\right)^2 dx + \frac{1}{EI_{CD}}\cdot\int_0^{21.33}\left(\frac{x}{21.33}\right)^2 dx + \frac{1}{EI_{BC}}\cdot\int_0^{8}(1)^2\,dx$$

$$f_{11}(EI_{AB},EI_{CD},EI_{BC}) \rightarrow \frac{8}{EI_{BC}} + \frac{20.473}{EI_{AB}} + \frac{7.11}{EI_{CD}}$$

18. With $I_{BC} := 273in^4$

$$f_{11}(E) := \left(\frac{8}{E\cdot I_{BC}} + \frac{20.473}{E\cdot I_{AB}} + \frac{7.11}{E\cdot I_{CD}}\right)\cdot in^4 \text{ simplify } \rightarrow \frac{0.058}{E}$$

Note that the numerators of θ_1 and f_{11} have units of $\frac{kip\cdot ft^2}{in^4}$

19. Applying the compatability equation,

$$\frac{118411}{43210\cdot E} + \frac{0.058}{E}\cdot M_1 = 0$$

and the bending moment at point B is

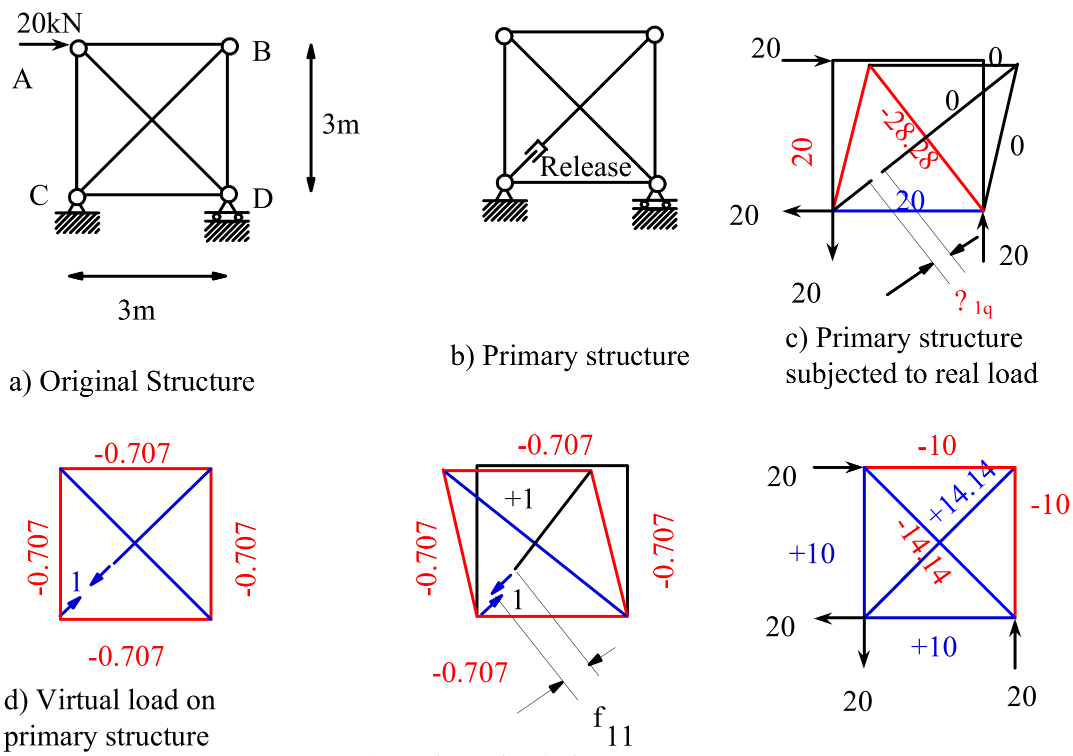
$$M_1 := \frac{-\frac{118411}{43210 \cdot E}}{\frac{0.058}{E}} \cdot kip \cdot ft = -47.248 \cdot kip \cdot ft$$

The reactions and moments in the structure are given in (f) in the figure above.

20. Once we have M_1, the structure is by now statically determinate and from statics we can complete the shear and moment diagrams.

14.3 Redundant Truss Analysis

Determine the bar forces in the steel truss shown below using the flexibility method. The truss is part of a supporting tower for a tank, and the 20 kN horizontal load is produced by wind loading on the tank.



a) Original Structure
b) Primary structure
c) Primary structure subjected to real load
d) Virtual load on primary structure
e) Action of unit force corresponding to redundant R_1
f) Final results

Solution:

1. Applying the criteria for indeterminancy $2 \cdot 4 = 8$ equations, 6 members + 3 reactions => one degree of indeterminancy. A logitudnial release in any of the six bars may be chosen.
2. Because the truss members carry only axial load, the longitudinal release is identical to actually cutting the member and removing its axial force capability from the truss.
3. In analyzing the trusses with double diaganols it is both convenient and customary to select the release in one of the diganol members because the resulting primary structure will be the conventional truss to which we are accustomed.
4. Choosing the diagonal member BC for the release, we cut it and remove its axial stiffness from the structure. The primary structure is shown in (b) in the figure above.
5. The analysis problem reduces to applying an equation of compatability to the changes in length of the release member. The relative displacement Δ_1 of the two cut ends of member BC, as producued by the real loading, is shown in (d) in the figure above.
6. The displacement is always measured along the length of the redundant member, and since the redundant is unstressed at this stage of the analysis, the displacement Δ_1 is equal to the relative displacement of joint B with respect to joint C.
7. This displacement must be eliminated by the relative displacements of the cut ends of member BC when the redundant force is acting on the member. The latter displacement is written in terms of the axial flexibility coefficient f_{11}, and the desired equation of consistent deformation is

$$\Delta_1 + f_{11} \cdot R_1 = 0$$

8. The quantity Δ_1 is given by

$$1 \cdot \Delta_1 = \sum \delta \cdot \overline{P} \left(\frac{PL}{AE} \right)$$

where $\delta \cdot \overline{P}$ and P are given in (d) and (e) in the figure above, respectively

9. Similarly,

$$f_{11} = \sum \delta \cdot \bar{P} \left(\frac{\bar{P} \cdot L}{AE} \right)$$

10. Evaluating these summations in tabular form:

Member	P kN	δP kN	L m	δPPL	δPPL
AB	0	-0.707	3	0	1.5
BD	0	-0.707	3	0	1.5
CD	20	-0.707	3	-42.42	1.5
AC	20	-0.707	3	-42.42	1.5
AD	-28.28	1	4.242	-119.96	4.242
BC	0	1	4.242	0	4.242
Total				-204.8	14.484

11. Since A = constant for each member

$$\Delta_1 = \sum \delta \cdot \bar{P} \left(\frac{PL}{AE} \right) = \frac{-204.8}{AE} \cdot kN^2 \cdot m \qquad \text{and} \qquad f_{11} = \frac{14.484}{AE} \cdot kN^2 \cdot m$$

then

$$\frac{1}{AE} \cdot \left(-204.8 + 14.484 \cdot R_1 \right) = 0$$

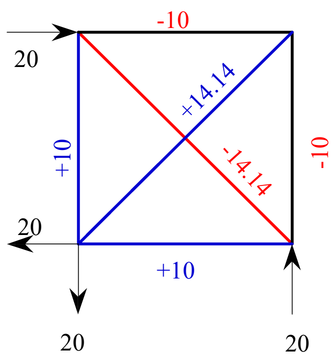
12. the solution for the redundant force value is $R_1 := \frac{204.8}{14.484} \cdot kN = 14.14 \cdot kN$

13. The final values for forces in each of the truss members are given by superimposing the forces due to the redundant and the forces due to the real loading.

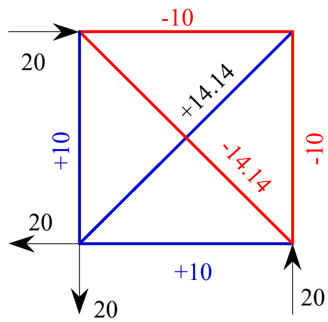
14. The real loading forces are shown in (c) in the figure above, while the redundant force effect is computed by multiplying the member forces in (d) in the figure above by 2.83, the value of the redundant.

Member	δP kN	$R_1\delta P$ kN	P kN	P_{total} kN
AB	-0.707	-10	0	-10
BD	-0.707	-10	0	-10
CD	-0.707	-10	20	10
AC	-0.707	-10	20	10
AD	1	14.14	-28.28	-14.14
BC	1	14.14	0	14.14

15. It is informative to compare the member forces from this solution to the approximate analysis obtained by assuming that the double diagonals each carry half the total shear in the panel. The comparison is given in the figure below



a) Exact analysis



b) Approximate analysis

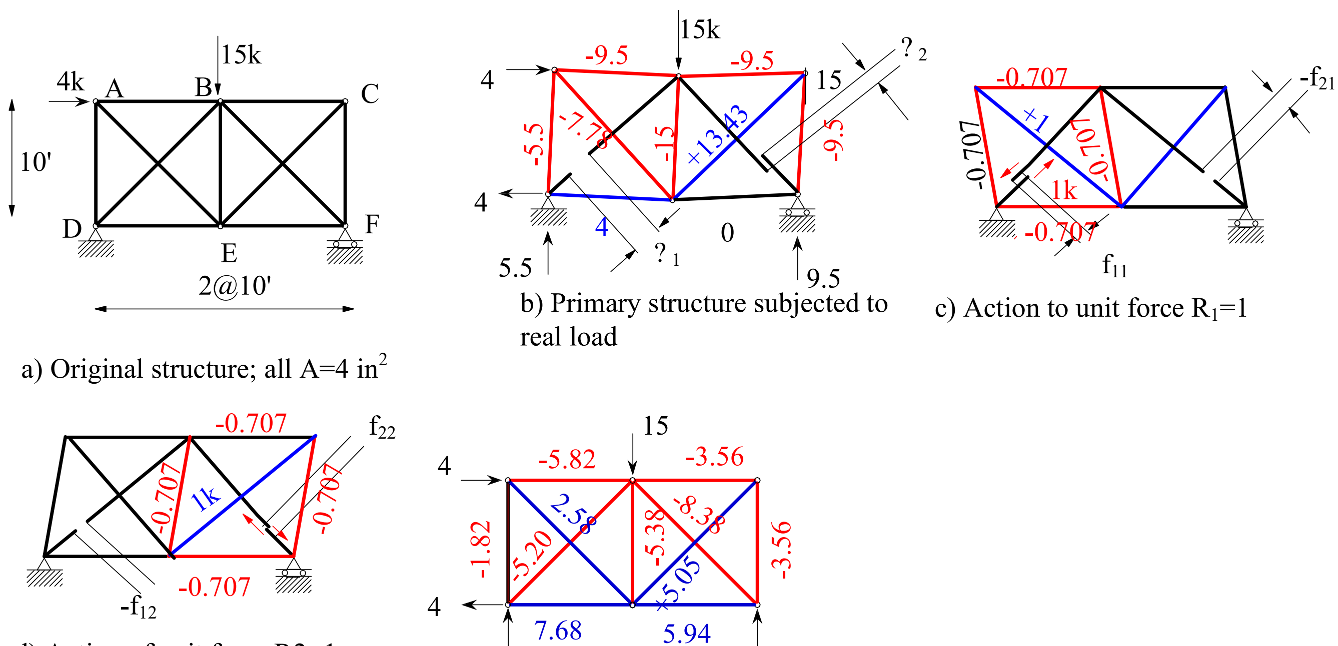
It reveals that the approximate analysis is the same as the exact analysis for this particular truss. The reason for this is that the stiffness provided by each of the diagonal members (against "shear" deformation of the rectangular panel) is the same, and therefore they each carry an equal portion of the total shear accross the panel.

14.4 Truss with Two Redundants

Another panel with a second redundant member is added to the truss of the preceeding example

Solution:

1. The twice redundant truss is converted to a determinate primary structure by releasing two members of the truss; we choose two diagonals (DB and BF).
2. Releasing both diagonals in a single panel, such as members AE and DB, is inadmissible since it leads to an unstable truss form.
3. The member forces and required displacements for the real loading and for the two redundant forces in members DB and BF are given in the figure below.



a) Original structure; all A=4 in^2

b) Primary structure subjected to real load

c) Action to unit force R_1=1

d) Action of unit force R2=1

e) Final results

4. Although the rela loading ordinarily stresses all members of the entire truss, we see that the unit forces corresponding to the redundant stress only those members in the panel that contain the redundant; all other bar forces are zero.
5. Recognizing this fact enables us to solve the double diagonal truss problem more rapidly than a frame with multiple redundants.
6. The virtual work equations for computing the six required displacements (two due to load and four flexibilities) are

$$1\cdot\Delta_1 = \sum \delta\cdot\overline{P_1}\left(\frac{PL}{AE}\right)$$

$$1\cdot\Delta_2 = \sum \delta\cdot\overline{P_2}\left(\frac{PL}{AE}\right)$$

$$1\cdot f_{11} = \sum \delta\cdot\overline{P_1}\left(\frac{\overline{P_1}\cdot L}{AE}\right)$$

$$1\cdot f_{21} = \sum \delta\cdot\overline{P_2}\left(\frac{\overline{P_1}\cdot L}{AE}\right)$$

$f_{12} = f_{21}$ by reciporocal theorem

$$1\cdot f_{22} = \sum \delta\cdot\overline{P_2}\left(\frac{\overline{P_2}\cdot L}{AE}\right)$$

7. If we assume tension in a truss member as positive, use tensile unit loads when computing the flexibility coeeficients corresponding to the redundants, and let all displacement terms carry their own signs, then in the for the redundants a positive value of force indicates tension while a negative value menas the member is in compression.
8. The calculation of f_{22} involves only the six members in the left panel of the truss; f_{21} invloves only member BE.
9. The simple used for performing the displacement analyses, as summarized in tabular form, leads one quickly to the compatability equations which state that the cut ends of both redundant members must match (there can be no gaps or overlaps of members in the actual structure).

Member	P	P_1	P_2	Δ_1	Δ_1	f_{11}	f_{21}	f_{22}
				$\delta P_1 PL$	$\delta P_2 PL$	$\delta P_1 P_1 L$	$\delta P_2 P_1 L$	$\delta P_2 P_2 L$
AB	-9.5	-0.707	0	806	0	60	0	0
BC	-9.5	0	-0.707	0	806	0	0	60
CF	-9.5	0	-0.707	0	806	0	0	60
EF	0	0	-0.707	0	0	0	0	60
DE	4	-0.707	0	-340	0	60	0	0
AD	-5.5	-0.707	0	466	0	60	0	0
AE	7.78	1	0	1322	0	170	0	0
BE	-15	-0.707	-0.707	1272	1272	60	60	60
CE	13.43	0	1	0	2280	0	0	170
BD	0	1	0	0	0	170	0	0
BF	0	0	1	0	0	0	0	170
Total				3528	5164	580	60	580

10. The equations are

$$\Delta_1 + f_{11} \cdot R_1 + f_{12} \cdot R_2 = 0$$
$$\Delta_2 + f_{21} \cdot R_1 + f_{22} \cdot R_2 = 0$$

or

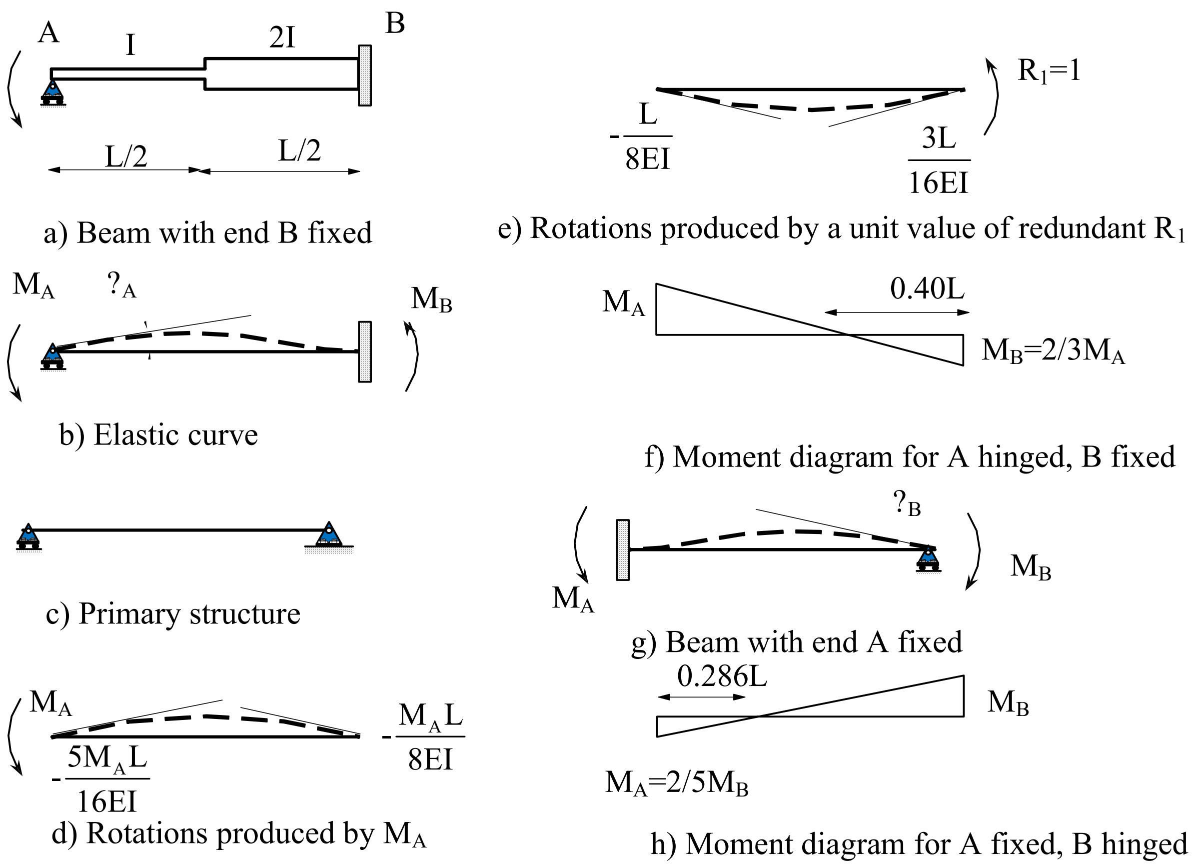
$$\frac{1}{AE} \cdot \begin{pmatrix} 580 & 60 \\ 60 & 580 \end{pmatrix} \cdot \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \frac{-1.}{AE} \cdot \begin{pmatrix} 3528 \\ 5164 \end{pmatrix}$$

11. The final set of forces in the truss is obtained by adding up, for each member, the three separate effects. In terms of the forces shown in the figure above, the force in any member is given by $F = P + R_1 \cdot \overline{P_1} + R_2 \cdot \overline{P_2}$

Member	P	P_1	P_2	$R_1 P_1$	$R_2 P_2$	P_{total}
AB	-9.5	-0.707	0	3.676	0	-5.82
BC	-9.5	0	-0.707	0	5.925	-3.56
CF	-9.5	0	-0.707	0	5.925	-3.56
EF	0	0	-0.707	0	5.925	5.94
DE	4	-0.707	0	3.676	0	7.68
AD	-5.5	-0.707	0	3.676	0	-1.82
AE	7.78	1	0	-5.2	0	2.58
BE	-15	-0.707	-0.707	3.676	5.925	-5.38
CE	13.43	0	1	0	-8.38	5.05
BD	0	1	0	-5.2	0	-5.2
BF	0	0	1	0	-8.38	-8.38

14.5 Analysis of Nonprismatic Members

The nonprismatic beam is loaded with an end moment M_A at its hinged end A. Determine the moment induced at the fixed end B by this loading.



Solution:

1. The beam has one redundant force; we select M_B as the redundant R_1, and obtain the primary structure shown in the figure above. It can be shown that the flexibility coefficients for unit moments applied at each end are those in (d) and (e) in the figure above, with a sing convention of counterclockwise as positive.
2. The equation of consistent displacement at B is

$$\frac{-M_A\cdot L}{8E\cdot I}+\frac{3}{16}\cdot\frac{L}{E\cdot I}\cdot R_1 = 0$$

and the value of M_B is

$$M_B = R_1 = \frac{2}{3}\cdot M_A$$

3. The resulting moment diagram is given in (f). We note that the inflection point is 0.4L from the fixed end. If the beam had a uniform value of I across its span, the inflection point would be L/3 from the fixed end. Thus the inflection point shifts toward the section of reduced stiffness.
4. The end rotation θ_A is given by

$$\theta_A = \frac{5}{16}\cdot\frac{M_A\cdot L}{E\cdot I} - \frac{1}{8}\cdot\left(\frac{2}{3}\cdot M_A\right)\cdot\frac{L}{E\cdot I} = \frac{11}{48}\cdot\frac{M_A\cdot L}{E\cdot I}$$

5. The ratio of the applied end moment to rotation $\frac{M_A}{\theta_A}$ is called the rotational stiffness and is

$$\frac{M_A}{\theta_A} = \frac{48}{11} \cdot \frac{E \cdot I}{L}$$

6. If we now reverse the boundary conditions, making A fixed and B hinged, and repeat the analysis for an applied moment M_B, the resulting moment diagram will be as given in (h) in the above figure. The moment induced at end A is only 40% of the applied end moment M_B. The inflection point is 0.286L from the fixed end A. The corresponding end rotation θ_B in (g) in the figure above is

$$\theta_B = \frac{11}{80} \cdot \frac{M_B \cdot L}{E \cdot I}$$

7. The rotational stiffness $\frac{M_B}{\theta_B}$ is

$$\frac{M_A}{\theta_A} = \frac{80}{11} \cdot \frac{E \cdot I}{L}$$

8. A careful comparison of the rotational stiffnesses, and of the moment diagrams in (f) and (h) in the figure above, illustrate the fact that flexural sections of increased stiffness attract more moment, and that inflection points always shift in the direction of decreased stiffness.
9. The approach illustrated here may be used to determine moments and end rotations in any type of non-prismatic member. The end rotations need in the force analysis may be calculated by either virtual work or moment area (or by other methods). Complex variations in EI are handles by numerical integration of the virtual work equation or by approximating the resultant M/EI area and their locations in the moment area method.

14.6 Fixed End Moments for Nonprismatic Members

The beam of the previous example

Solution:

1. The beam has two redundant forces and we select M_A and M_B. Releasing these redundants, R_1 and R_2, we obtain the primary sturcture.
2. The equations of consistent deformations are

$$\Delta_1 + f_{11} \cdot R_1 + f_{12} \cdot R_2 = 0$$

$$\Delta_2 + f_{21} \cdot R_1 + f_{22} \cdot R_2 = 0$$

where R_1 is M_A and R_2 is M_B.

3. The values of Δ_1 and Δ_2, the end rotations produced by the real loading on the primary structure, can be computed by the virtual work method.
4. The flexibility coefficients are also separately derived (not yet in these notes)
5. We define counterclockwise end moments and rotations as positive and obtain

$$\frac{L}{E \cdot I} \cdot \begin{pmatrix} \frac{5}{16} & \frac{-1}{8} \\ \frac{-1}{8} & \frac{3}{16} \end{pmatrix} \cdot \begin{pmatrix} R_1 \\ R_1 \end{pmatrix} = \frac{w \cdot L^3}{E \cdot I} \cdot \begin{pmatrix} -0.352 \\ 0.0273 \end{pmatrix}$$

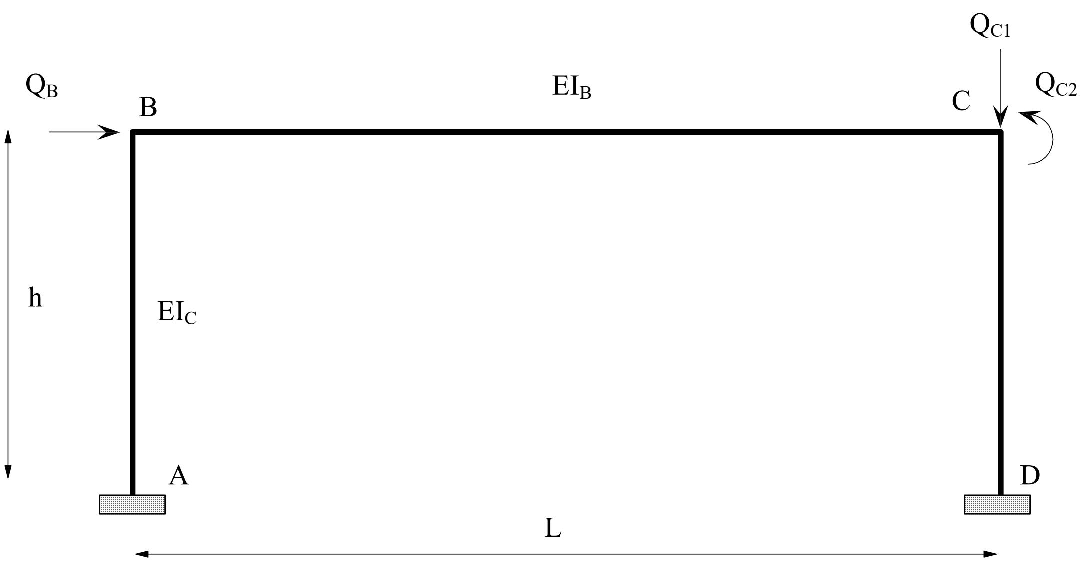
from which

$$\boxed{R_1 = M_A = 0.0742 w \cdot L^2}$$

$$\boxed{R_2 = M_B = -0.0961 w \cdot L^2}$$

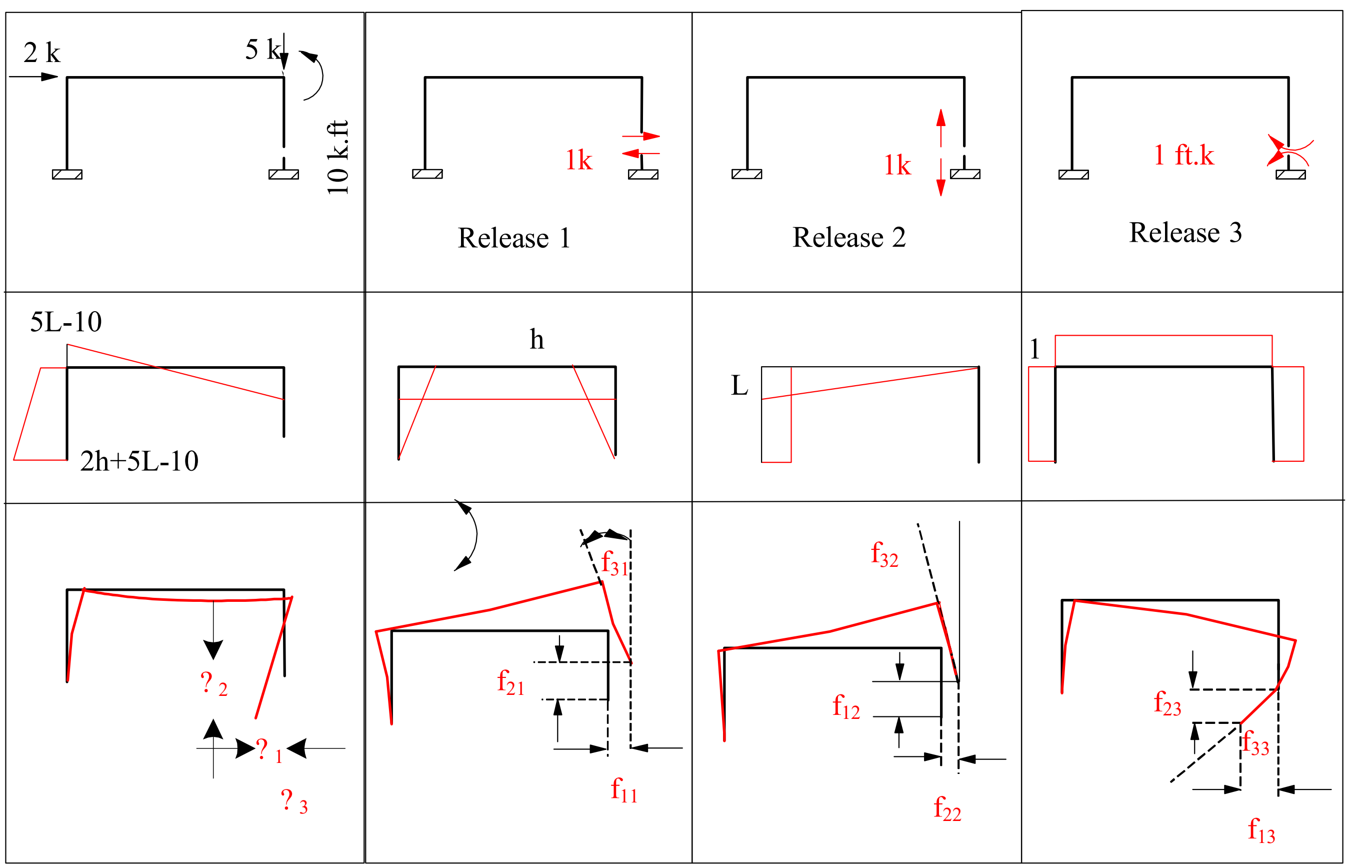
6. The stiffer end of the beam attracts 30% more than the flexible end.
7. For a prismatic beam with constant I, the fixed end moments are equal in magnitude ($M_A = -M_B = \frac{w \cdot L^2}{12}$) and intermediate in value between the two end moments determiend above.
8. Fixed end moments are an essential part of indeterminate analysis based on the displacement (stiffness) method and will be used exclusively in the Moment Distribution method.

14.7 Rectangular Frame; External Load



Solution:

1. The structure is statically indeterminate to the third degree, and the displacements (flexibility terms) are shown in the figure above.
2. In order to evaluate the 9 flexibility terms, we refer tot he table.



3. Substituting $h := 10ft$, $L := 20ft$, and $EI_B = EI_C = EI$, the flexibility matrix then becomes

$$f = \frac{1}{EI} \cdot \begin{pmatrix} 2667 & 3000 & -300 \\ 3000 & 6667 & -400 \\ -300 & -400 & 40 \end{pmatrix}$$

and the vector of displacements for the primary structure is

$$\Delta = \frac{1}{EI} \cdot \begin{pmatrix} -12833 \\ -31333 \\ 1800 \end{pmatrix}$$

where the units are kips and feet.

4. The inverse of the flexibility matrix is

$$f^{-1} = 10^{-3} \cdot EI \cdot \begin{pmatrix} 2.4 & 0 & 18 \\ 0 & 0.375 & 3.75 \\ 18 & 3.75 & 197.5 \end{pmatrix}$$

5. Hence the reactions are determined from

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} := 10^{-3} \cdot EI \cdot \begin{pmatrix} 2.4 & 0 & 18 \\ 0 & 0.375 & 3.75 \\ 18 & 3.75 & 197.5 \end{pmatrix} \cdot \left[\frac{1}{EI} \cdot \begin{pmatrix} -12833 \\ -31333 \\ 1800 \end{pmatrix} \right] \rightarrow \begin{pmatrix} 1.601 \\ -5 \\ 7.007 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} 1.6 \\ -5 \\ 7 \end{pmatrix}}$$

14.8 Frame with Temperature and Support Displacements

The single bay frame, of the previous example, has a height $h := 10ft$ and span $L := 20ft$ and its two supportsrigidly connected and is constructed of reinforced concrete. It supports a roof and wall partitions in such a manner that linear temperature variation occurs across the depth of the frame members when inside and outside temperatures differ. Assume the member depth is constant at 1 ft, and that the structure was built with fixed bases A and D at a temperature of 85F. The temperature is now 70F inside and 20F outside. We wish to determine the reactions at D under these conditions. Assume that the coefficient of linear expansion of reinforced concrete is $\alpha := 0.0000055\cdot\frac{1}{\Delta°F}$.

Solution:

1. Our analysis proceeds as before, with the [D] vector interpreted appropriately. The three releases shown in the previous example, will be used.
2. The first stage in the analysis is the computation of the relative displacements $\Delta_{1\Delta}, \Delta_{2\Delta}, \Delta_{3\Delta}$, of the primary structure caused by temperature effects. These diaplcements are caused by two effects: axial shortening of the members because of the drop in average temperature (a middepth of the members), and curvature of the members because of the temperature gradient.
3. In the following discussion the contributions to the displacements due to the axial strain are denoted with a single prime (') and those due to curvature by a double prime (").
4. Consider the axial strain first. A unit length of frame member shortens as a result of the temperature decrease from 85F to 45F at the midepth of the member. the strain is therefore

$$\alpha\Delta T := \alpha\cdot 40\cdot\Delta°F = 0.00022$$

5. The effect of axial strain on the relative displacements needs little analysis. The horizontal member shortens by an amount $\alpha\Delta T\cdot 20ft = 0.0044\cdot ft$. The shortening of the vertical members results in no relative displacement in the vertical direction. No rotation occurs.
6. We therefore have $\Delta'_{1\Delta} := -\alpha\Delta T\cdot 20ft = -0.0044\cdot ft$, $\Delta'_{2\Delta} := 0ft$, and $\Delta'_{3\Delta} := 0rad$
7. The effect of curvature must also be considered. A frame element of length dx undergoes an angular strain as a result of the temperature gradient. The change in length at an extreme fiber is

$$\varepsilon(dx) := \alpha\cdot 25\cdot\Delta°F\cdot dx \rightarrow 0.0001375\cdot dx$$

8. With the resulting real rotation of the cross section

$$d\phi(dx) := \frac{\varepsilon(dx)}{0.5}\cdot rad \rightarrow 0.000275\cdot dx\cdot rad$$

9. The relative displacements of the primary structure at D are found by the virtual force method.
10. A virtual force $\delta\cdot\overline{Q}$ is applied in the direction of the desired displacement and the resulting moment diagram $\delta\cdot\overline{M}$ determined.
11. The virtual work equation

$$\delta\cdot\overline{Q}\cdot\Delta = \int \delta\cdot\overline{M}\, d\phi$$

is used to obtain each of the desired displacements at D.

12. The results, which you should verify, are

$$\Delta''_1 := 0.0828ft$$
$$\Delta''_2 := 0.1104ft$$
$$\Delta''_3 := -0.01104rad$$

13. Combining the effects of axial and rotational strain, we have

$$\Delta_1 := \Delta'_{1\Delta} + \Delta''_1 = 0.0784\cdot ft$$
$$\Delta_2 := \Delta'_{2\Delta} + \Delta''_2 = 0.1104\cdot ft$$
$$\Delta_3 := \Delta'_{3\Delta} + \Delta''_3 = -0.01104\cdot rad$$

14. We now compute the reduntants caused by temperature effects

$$R = f^{-1}\cdot(-\Delta)$$

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = 10^{-3} \cdot EI \cdot (18 \quad 3.75 \quad 197.5) \cdot \begin{pmatrix} -0.0784 \\ -0.1104 \\ 0.01104 \end{pmatrix} = \begin{pmatrix} 0.0106 \\ 0.355 \end{pmatrix} \cdot 10^{-3} \cdot EI$$

where the units are kips and feet.

15. You should construct the moment diagram for the structure using the values of the redundant found in the analysis.
16. Notice the stiffness term EI does not cancel out in this case. Internal forces and reactions in a statically indeterminate subject to effects other than loads (such as temperature) are dependent on the actual stiffnesses of the structure.
17. The effects of axial strain caused by forces in the members have been neglected in this analsis. This is usual for low frames where bending strain dominates behavior. To illustrate the significance of this assumption, consider member BC. We have found $R_1 = 10.6 \cdot 10^{-6} \cdot \frac{EI}{EA}$. The tension in BC has this same value, resulting in a strain for the member of $10.6 \cdot 10^{-6} \cdot \frac{EI}{EA}$.For a rectangular member, $\frac{I}{A} = \frac{b \cdot d^3}{12 \cdot bd} = \frac{d^2}{12}$. In our case d=1ft, therefore the axial strain is $10.6 \cdot 10^{-6} \cdot (0.0833) = 8.83 \times 10^{-7}$, which is several orders of magnitude smaller than the temperature strain computed for the same member. We may therefore rest assured that neglecting axial strain caused by forces does not affect the values of the redundants in a significant manner for this structure.
18. Now considert the effects of foundation movement on the same structure. The intermediate frame behavior depends on a structure that we did not design: the earth. The earth is an essential part of nearly all structures, and we must understand the effects of foundation behavior on structural behavior. for the purposes of this example, assume that a foundation has revealed the possibility of a clockwise rotation of the support at D of 0.001 radians and a downward movement of the support at D of 0.12 ft. We wish to evaluate the redundants R_1, R_2, and R_3 caused by this foundation movement.
19. No analysis is needed to determine the values of $\Delta_{1\Delta}$, $\Delta_{2\Delta}$, and $\Delta_{3\Delta}$ for the solution of the redundants. These displacements are found directly from the support movements, with proper consideration, of the originally chosen sign convention which defined the positive direction of the relative displacements. From the given support displacements, we find $\Delta_{1\Delta} := 0$, $\Delta_{2\Delta} := 0.12\text{ft}$, and $\Delta_{3\Delta} := 0.001\text{rad}$. Can you evaluate these quantities for a casr in which the displacements occured at A instead of D?
20. The values of the redundants is given by

$$R = f^{-1} \cdot (-\Delta)$$

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = 10^{-3} \cdot EI \cdot (18 \quad 3.75 \quad 197.5) \cdot \begin{pmatrix} -0.12 \\ 0.001 \end{pmatrix} = \begin{pmatrix} 18 \\ -252.5 \end{pmatrix} \cdot 10^{-3} \cdot EI$$

with units in kips and feet.

21. A moment diagram may now be constructed, and other internal force quantities computed from the now known values of the redundants. The redundants have been evaluated separately for effects of temperature and foundation settlement. These effects may be combined with those due to loading using the principle of superposition.

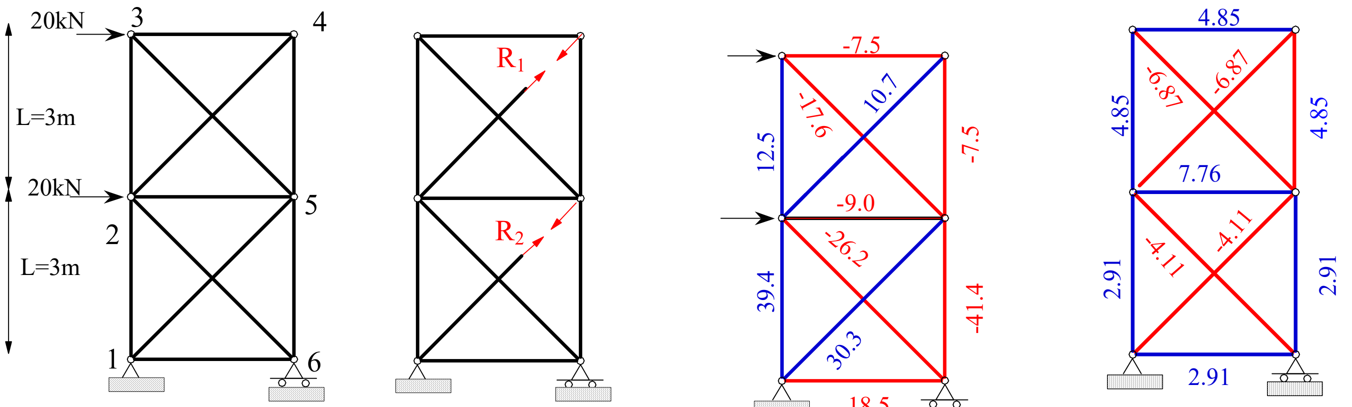
14.9 Braced Bent with Loads and Temperature Change

The struss shown below represents an internal braced bent in an enclosed shed, with lateral loads of 20 kN at the panel points. A temperature drop of 30C may occur on the other members (members 1-2, 2-3, 3-4, 4-5, and 5-6). We wish to analyze the truss for loading and for temperature effects.

Solution:

1. The first step in the analysis is the definition of the two redundants. The choice of forces in diagonals 2-4 and 1-5 as redundants facilitates the computations because some of the load effects are easy to analyze.
2. The computations are organized in tabular form in Table ??. The first column gives the force in each bar caused by a unit load (1 kN) corresponding to release 1. These are denoted p_i and also represent the bar force $\frac{\overline{q_1}}{\delta \cdot \overline{Q_1}}$ caused by a virtual force $\delta \cdot \overline{Q_1}$ applied at the same location. Column 3 lists the same quantitiy for a unit load and for a virutal force $\delta \cdot \overline{Q_2}$ applied at release 2. These three columns constitute a record of the truss analysis needed for this problem.

Verticals: 500 mm^2 ; Webs: 250 mm^2; E=200,000 N/mm^2; a=1.0x10^{-5}/ °C



a) Structure b) Redundant forces c) Forces due to lateral load d) Forces due to temperature drop

3. Column 4 gives the value of L/EA for each bar in terms of L_c/EA_c of the vertical members. This is useful because the term L/EA cancels out in some of the calculations.
4. The method of virtual work is applied directly to compute the displacements Δ_{1Q} and Δ_{2Q} corresponding to the releases and caused by the actual loads. Apply a virtual force $\delta \cdot \overline{Q_1}$ at release 1. The internal virtual forces $\overline{q_1}$ are found in column 2. The internal virtual work $\overline{q_1} \cdot \Delta l$ is found in column 5 as the product of columns 1, 2 and 4. The summation of column 5 is $\Delta_{1q} = -122.42 \cdot \frac{L_c}{EA_c}$. Similarly, column 6 is the product of columns 1, 3, and 4, giving $\Delta_{2q} = -273.12 \cdot \frac{L_c}{EA_c}$.
5. The same method is used to compute the flexibilities. In this case, the real loading is a unit load corresponding to release 1 leading to f_{11} and f_{21}, and then to release 2 leading to f_{12} and f_{22}. Column 7 shows the computation for f_{11}. It is the product of column 2, representing force due to the real unit load with column 2 representing force due to a virtual force $\delta \cdot \overline{Q_1}$ at the same location (release 1) multiplied by column 4 to include the L_c/EA_c term. Column 8 derives from columns 2, 3, and 4 and leads to f_{21}. Columns 9 and 10 are the computations for the remaining flexibilities.

6. We have assumed that a temperature drop of 30C occurs in the outer members. The corresponding length changes are found in column 11. Again using the virtual work method, column 12 tabulates the internal virtual work of th virtual forces through displacements Δl where for each bar, $\Delta l = \alpha l \Delta T$. Column 12 is therefore the product of columsn 2 and 11. The summation of the elements of column 12 is the displacement Δl corresponding to release 1. Column 13 repeats this process for Δ_2 corresponding to release 2.

7. The tabulated information provides the necessary terms for a matrix solution of the problem. We have

$$f = \begin{pmatrix} 8.66 & 1 \\ 1 & 8.66 \end{pmatrix} \cdot \frac{L_c}{EA_c}$$

$$\Delta q = \begin{pmatrix} -122.42 \\ -273.12 \end{pmatrix} \cdot \frac{L_c}{EA_c}$$

$$\Delta_\Delta = \begin{pmatrix} 6.36 \\ 4.24 \end{pmatrix} \cdot 10^{-4} \cdot L_c$$

therefore

$$f^{-1} = \begin{pmatrix} 0.117 & -0.0134 \\ -0.0134 & 0.117 \end{pmatrix} \cdot \frac{EA_c}{L_c}$$

8. The redundant forces due to the applied load are

$$R = f^{-1} \cdot (-\Delta_Q)$$

$$R = (-0.0134 \;\; 0.117) \cdot \frac{EA_c}{L_c} \cdot \begin{pmatrix} 122.42 \\ 273.12 \end{pmatrix} \cdot \frac{L_c}{EA_c} = \begin{pmatrix} 10.66 \\ 30.32 \end{pmatrix}$$

9. Thus $R_1 = 10.66\text{kN}$, $R_2 = 30.32\text{kN}$

10. The redundant forces due to the temperature drop are $R = f^{-1} \cdot (-\Delta_\Delta)$

$$R = (-0.0134 \;\; 0.117) \cdot \frac{EA_c}{L_c} \cdot \begin{pmatrix} -6.36 \\ -4.24 \end{pmatrix} \cdot 10^{-4} \cdot L_c = \begin{pmatrix} -6.87 \\ -4.11 \end{pmatrix} \cdot 10^{-5} \cdot EA_c$$

11. Thus with $E := 200 \frac{\text{kN}}{\text{mm}^2}$, $A_c := 500\text{mm}^2$

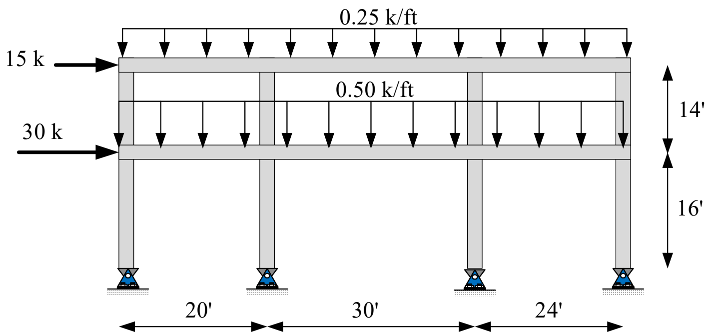
$$R_1 := -6.87 \cdot 10^{-5} \cdot E \cdot A_c = -6.87 \cdot \text{kN}$$

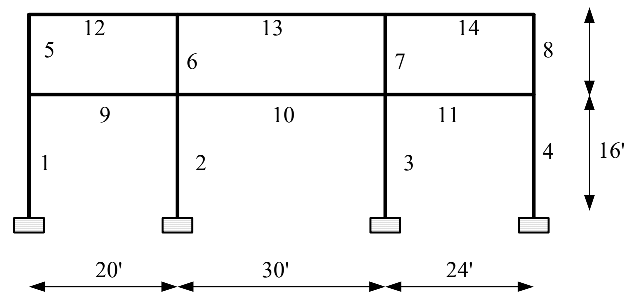
$$R_2 := -4.11 \cdot 10^{-5} \cdot E \cdot A_c = -4.11 \cdot \text{kN}$$

12. Using the redundant forces from each of these analyses, the remainder of the bar forces are computed by simple equilibrium. Table ?? contains such computations. The bar forces in any bar is the force in column 1 added to that in column 2 multiplied by R_1 plus that is column 3 multiplied by R_2. This follows from the fact that columns 2 and 3 are bar forces caused by a force of unitiy corresponding to each of the redundants. The results of the calcultations are shown for the applied loading and for the temperature drop. The forces caused by the temperature drop are similar in magnitude to those caused by wind load in this example. Temperature differences, shrinkage, support settlement, or tolerance errors can cause important effects in statically indeterminate structures.

15.1 Approximate Analysis of a Frame Subjected to Vertical and Horizontal Loads

Draw the shear and moment diagrams for the following frame.

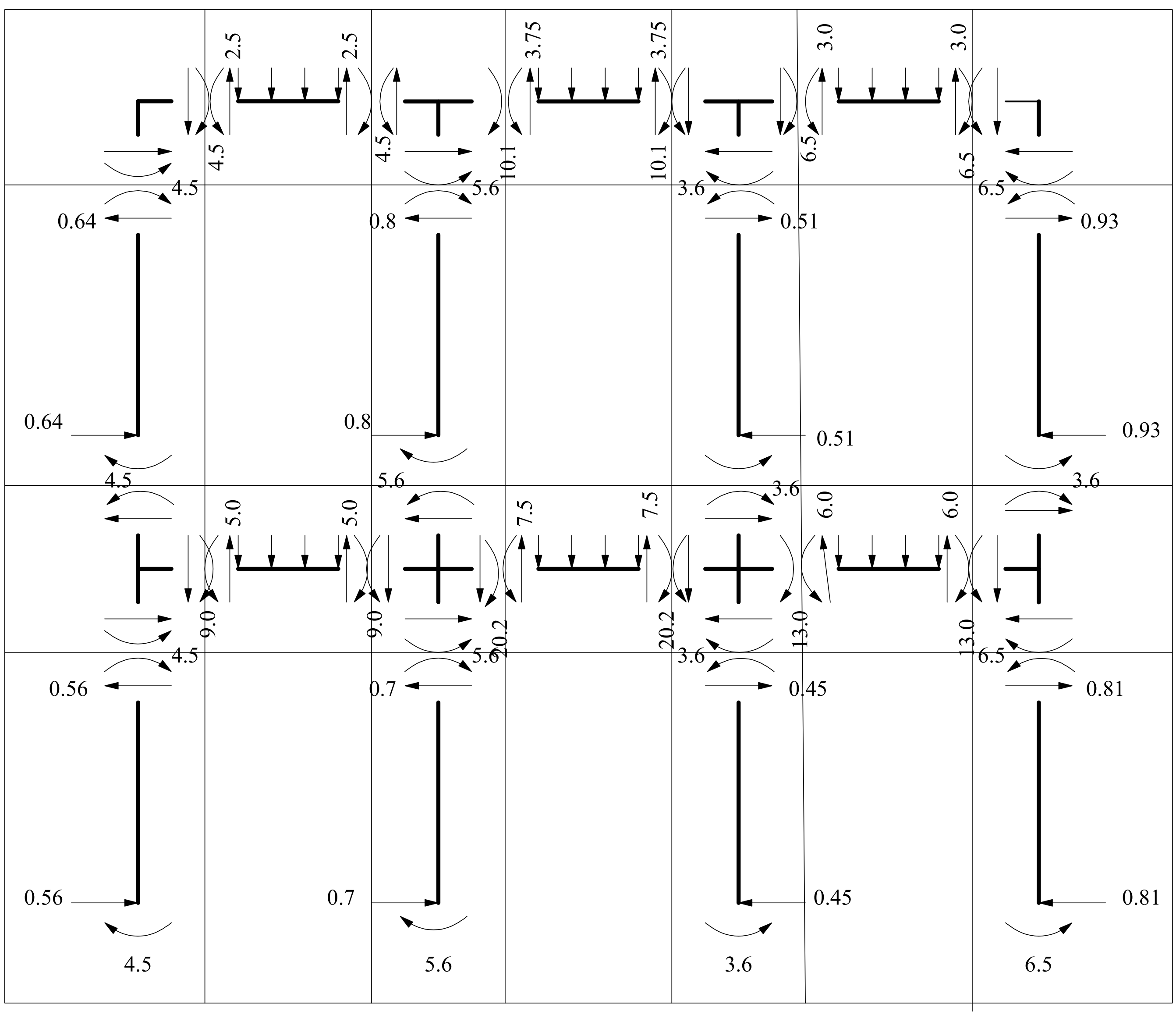




Solution:

The analysis should be conducted in conjunction with the free body diagram shown below.

Vertical Loads



Gravity Loads

$w_{12} := 0.25\frac{kip}{ft}$ $\quad w_{13} := w_{12} = 0.25\cdot\frac{kip}{ft}$ $\quad w_{14} := w_{12} = 0.25\cdot\frac{kip}{ft}$

$w_{9} := 0.5\frac{kip}{ft}$ $\quad w_{10} := w_{9} = 0.5\cdot\frac{kip}{ft}$ $\quad w_{11} := w_{9} = 0.5\cdot\frac{kip}{ft}$

Approximate Equations

$M_{left}(w,L) := -0.045\cdot w\cdot L^{2}$ Maximum negative moment at girder end

$M_{cent}(w,L) := 0.08\cdot w\cdot L^{2}$ Maximum positive moment

1. Top Girder Moments

$M_{12lft} := M_{left}(w_{12}, 20ft) = -4.5\cdot kip\cdot ft$

$M_{12cnt} := M_{cent}(w_{12}, 20ft) = 8\cdot kip\cdot ft$

$M_{12rgt} := M_{12lft} = -4.5\cdot kip\cdot ft$

$M_{13lft} := M_{left}(w_{13}, 30ft) = -10.125\cdot kip\cdot ft$

$M_{13cnt} := M_{cent}(w_{13}, 30ft) = 18\cdot kip\cdot ft$

$M_{13rgt} := M_{13lft} = -10.125\cdot kip\cdot ft$

$M_{14lft} := M_{left}(w_{14}, 24ft) = -6.48\cdot kip\cdot ft$

$M_{14cnt} := M_{cent}(w_{14}, 24ft) = 11.52\cdot kip\cdot ft$

$M_{14rgt} := M_{14lft} = -6.48\cdot kip\cdot ft$

2. Bottom Girder Moments

$M_{9lft} := M_{left}(w_{9}, 20ft) = -9\cdot kip\cdot ft$

$M_{9cnt} := M_{cent}(w_{9}, 20ft) = 16\cdot kip\cdot ft$

$M_{9rgt} := M_{9lft} = -9\cdot kip\cdot ft$

$M_{10lft} := M_{left}(w_{10}, 30ft) = -20.25\cdot kip\cdot ft$

$M_{10cnt} := M_{cent}(w_{10}, 30ft) = 36\cdot kip\cdot ft$

$M_{10rgt} := M_{10lft} = -20.25\cdot kip\cdot ft$

$M_{11lft} := M_{left}(w_{11}, 24ft) = -12.96\cdot kip\cdot ft$

$M_{11cnt} := M_{cent}(w_{11}, 24ft) = 23.04\cdot kip\cdot ft$

$M_{11rgt} := M_{11lft} = -12.96\cdot kip\cdot ft$

3. Top Column Moments

$M_{5top} := M_{12lft} = -4.5\cdot kip\cdot ft$

$M_{5bot} := -M_{5top} = 4.5\cdot kip\cdot ft$

$M_{6top} := -M_{12rgt} + M_{13lft} = -5.625\cdot kip\cdot ft$

$M_{6bot} := -M_{6top} = 5.625\cdot kip\cdot ft$

$M_{7top} := -M_{13rgt} + M_{14lft} = 3.645\cdot kip\cdot ft$

$M_{7bot} := -M_{7top} = -3.645\cdot kip\cdot ft$

$M_{8top} := -M_{14rgt} = 6.48\cdot kip\cdot ft$

$M_{8bot} := -M_{8top} = -6.48\cdot kip\cdot ft$

4. Bottom Column Moments

$M_{1top} := M_{5bot} + M_{9lft} = -4.5 \cdot kip \cdot ft$

$M_{1bot} := -M_{1top} = 4.5 \cdot kip \cdot ft$

$M_{2top} := M_{6bot} - M_{9rgt} + M_{10lft} = -5.625 \cdot kip \cdot ft$

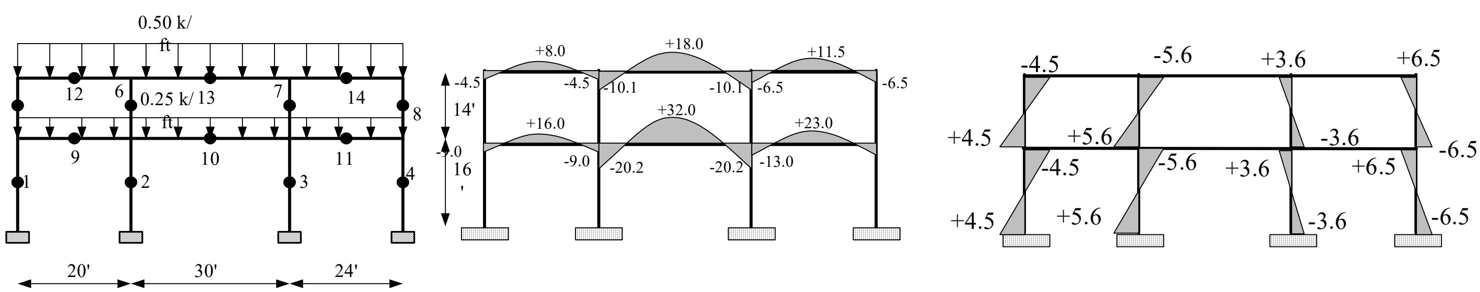
$M_{2bot} := -M_{2top} = 5.625 \cdot kip \cdot ft$

$M_{3top} := M_{7bot} - M_{10rgt} + M_{11lft} = 3.645 \cdot kip \cdot ft$

$M_{3bot} := -M_{3top} = -3.645 \cdot kip \cdot ft$

$M_{4top} := M_{8bot} - M_{11rgt} = 6.48 \cdot kip \cdot ft$

$M_{4bot} := -M_{4top} = -6.48 \cdot kip \cdot ft$



5. Top Girder Shear

$$V_{12lft} := \frac{w_{12} \cdot 20ft}{2} = 2.5 \cdot kip$$

$V_{12rgt} := -V_{12lft} = -2.5 \cdot kip$

$$V_{13lft} := \frac{w_{13} \cdot 30ft}{2} = 3.75 \cdot kip$$

$V_{13rgt} := -V_{13lft} = -3.75 \cdot kip$

$$V_{14lft} := \frac{w_{14} \cdot 24ft}{2} = 3 \cdot kip$$

$V_{14rgt} := -V_{14lft} = -3 \cdot kip$

6. Bottom Girder Shear

$$V_{9lft} := \frac{w_{9} \cdot 20ft}{2} = 5 \cdot kip$$

$V_{9rgt} := -V_{9lft} = -5 \cdot kip$

$$V_{10lft} := \frac{w_{10} \cdot 30ft}{2} = 7.5 \cdot kip$$

$V_{10rgt} := -V_{10lft} = -7.5 \cdot kip$

$$V_{11lft} := \frac{w_{11} \cdot 24ft}{2} = 6 \cdot kip$$

$V_{11rgt} := -V_{11lft} = -6 \cdot kip$

7. Column Shears

$h_5 := 14ft \quad h_6 := h_5 = 14 \cdot ft \quad h_7 := h_5 = 14 \cdot ft \quad h_8 := h_5 = 14 \cdot ft$

$h_1 := 16ft \quad h_2 := h_1 = 16 \cdot ft \quad h_3 := h_1 = 16 \cdot ft \quad h_4 := h_1 = 16 \cdot ft$

$$V_5 := \frac{M_{5top}}{\frac{h_5}{2}} = -0.643 \cdot kip$$

$$V_6 := \frac{M_{6top}}{\frac{h_6}{2}} = -0.804 \cdot kip$$

$$V_7 := \frac{M_{7top}}{\frac{h_7}{2}} = 0.521 \cdot kip$$

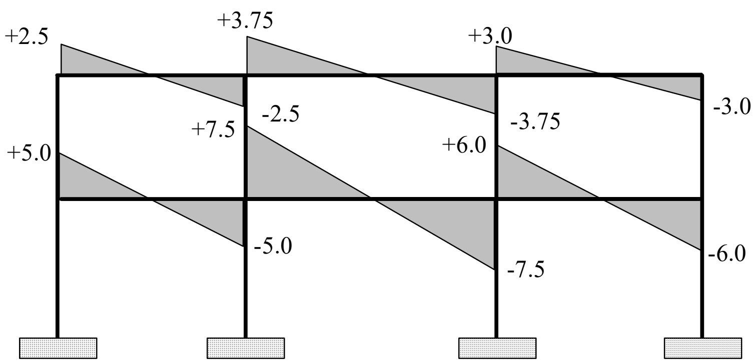
$$V_8 := \frac{M_{8top}}{\frac{h_8}{2}} = 0.926 \cdot kip$$

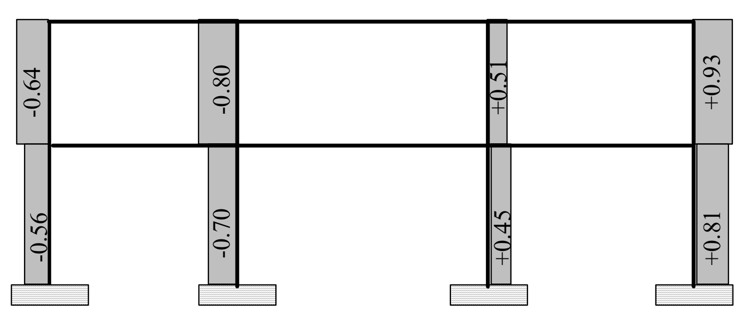
$$V_1 := \frac{M_{1top}}{\frac{h_1}{2}} = -0.563 \cdot kip$$

$$V_2 := \frac{M_{2top}}{\frac{h_2}{2}} = -0.703 \cdot kip$$

$$V_3 := \frac{M_{3top}}{\frac{h_3}{2}} = 0.456 \cdot kip$$

$$V_4 := \frac{M_{4top}}{\frac{h_4}{2}} = 0.81 \cdot kip$$





8. Top Column Axial Forces

$P_5 := V_{12lft} = 2.5 \cdot kip$

$P_6 := -V_{12rgt} + V_{13lft} = 6.25 \cdot kip$

$P_7 := -V_{13rgt} + V_{14lft} = 6.75 \cdot kip$

$P_8 := -V_{14rgt} = 3 \cdot kip$

9. Bottom Chord Axial Forces

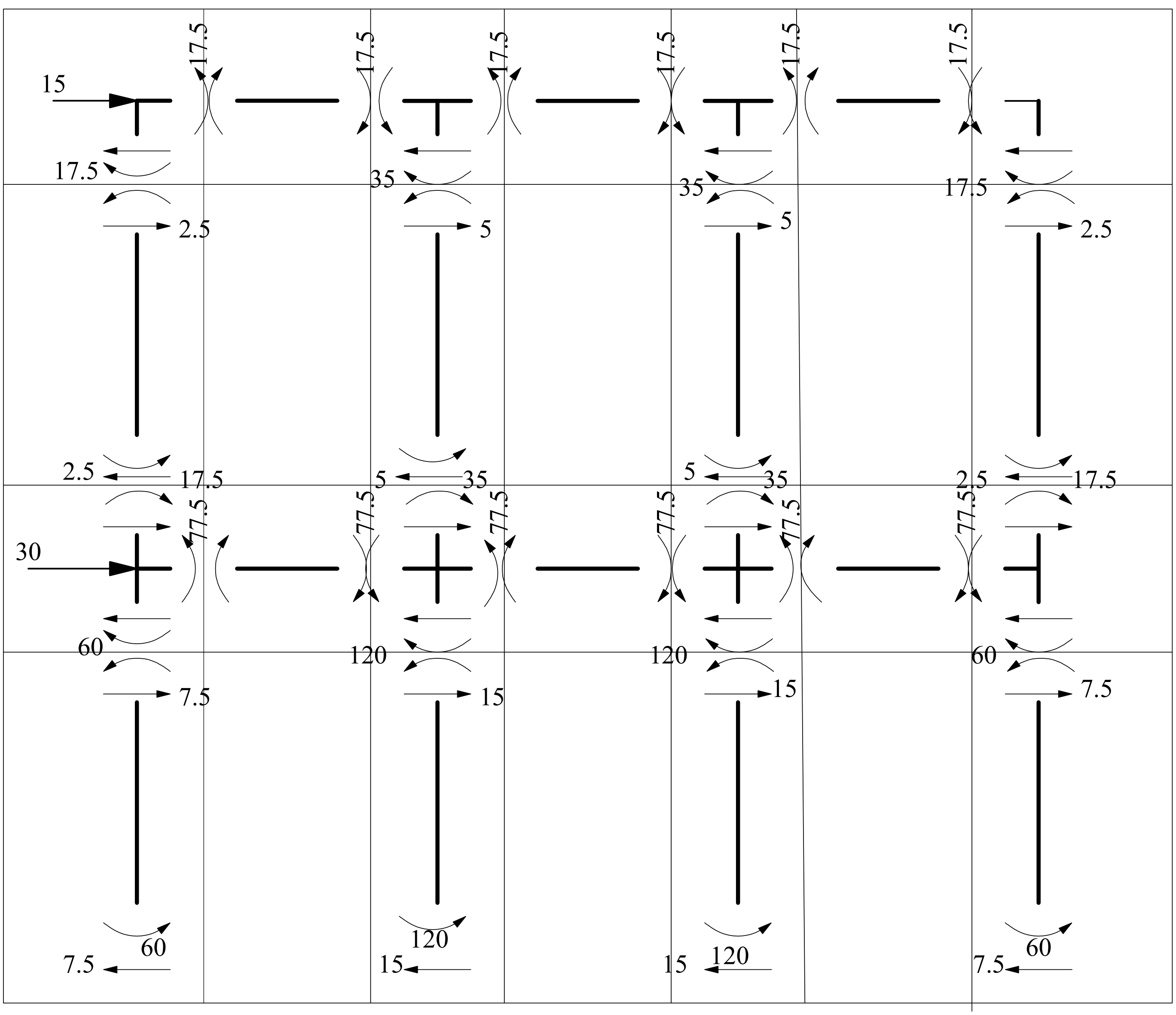
$P_1 := P_5 + V_{9lft} = 7.5 \cdot kip$

$P_2 := P_6 - V_{10rgt} + V_{9lft} = 18.75 \cdot kip$

$P_3 := P_7 - V_{11rgt} + V_{10lft} = 20.25 \cdot kip$

$P_4 := P_8 - V_{11rgt} = 9 \cdot kip$

Horizontal Loads



Horizontal Loads

$F_5 := 15kip$

$F_1 := 30kip$

Approximate Equations

$V_{ext}(F, n) := \frac{F}{2 \cdot n}$

$n_{bays} := 3$

1. Column Shears

$V_5 := V_{ext}(F_5, n_{bays}) = 2.5 \cdot kip$

$V_6 := 2 \cdot V_5 = 5 \cdot kip$

$V_7 := 2 \cdot V_5 = 5 \cdot kip$

$V_8 := V_5 = 2.5 \cdot kip$

$V_1 := V_{ext}(F_1 + F_5, n_{bays}) = 7.5 \cdot kip$

$V_2 := 2 \cdot V_1 = 15 \cdot kip$

$V_3 := 2 \cdot V_1 = 15 \cdot kip$

$V_4 := V_1 = 7.5 \cdot kip$

2. Top Column Moments

$M_{5top} := \frac{V_5 \cdot h_5}{2} = 17.5 \cdot kip \cdot ft$

$M_{5bot} := -M_{5top} = -17.5 \cdot kip \cdot ft$

$M_{6top} := \frac{V_6 \cdot h_6}{2} = 35 \cdot kip \cdot ft$

$M_{6bot} := -M_{6top} = -35 \cdot kip \cdot ft$

$M_{7top} := \frac{V_7 \cdot h_7}{2} = 35 \cdot kip \cdot ft$

$M_{7bot} := -M_{7top} = -35 \cdot kip \cdot ft$

$M_{8top} := \frac{V_8 \cdot h_8}{2} = 17.5 \cdot kip \cdot ft$

$M_{8bot} := -M_{8top} = -17.5 \cdot kip \cdot ft$

3. Bottom Column Moments

$M_{1top} := \frac{V_1 \cdot h_1}{2} = 60 \cdot kip \cdot ft$

$M_{1bot} := -M_{1top} = -60 \cdot kip \cdot ft$

$M_{2top} := \frac{V_2 \cdot h_2}{2} = 120 \cdot kip \cdot ft$

$M_{2bot} := -M_{2top} = -120 \cdot kip \cdot ft$

$M_{3top} := \frac{V_3 \cdot h_3}{2} = 120 \cdot kip \cdot ft$

$M_{3bot} := -M_{3top} = -120 \cdot kip \cdot ft$

$M_{4top} := \frac{V_4 \cdot h_4}{2} = 60 \cdot kip \cdot ft$

$M_{4bot} := -M_{4top} = -60 \cdot kip \cdot ft$

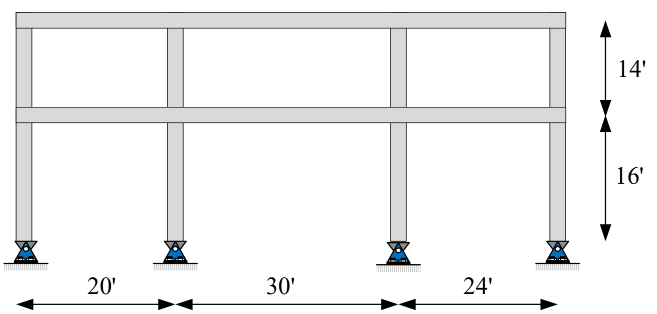
$$M_{4bot} := -M_{4top} = -60 \cdot kip \cdot ft$$

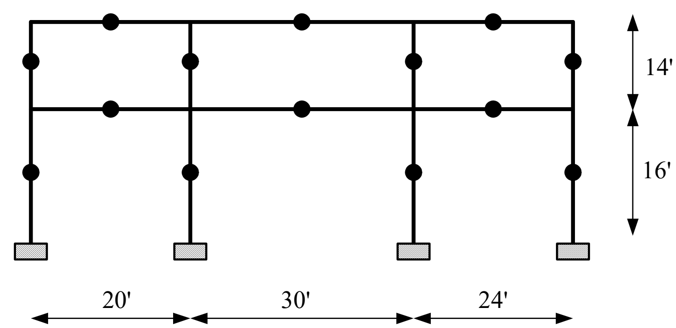
4. Top Girder Moments

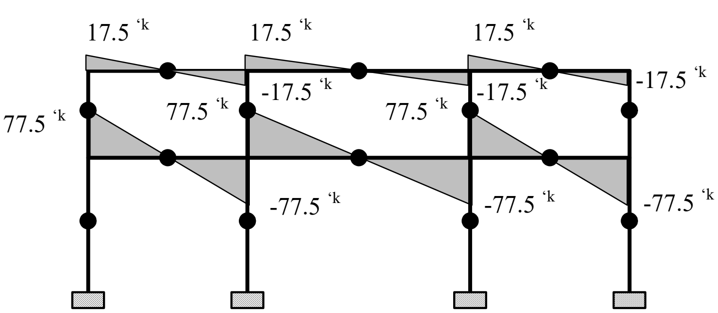
$$M_{12lft} := M_{5top} = 17.5 \cdot kip \cdot ft$$
$$M_{12rgt} := -M_{12lft} = -17.5 \cdot kip \cdot ft$$
$$M_{13lft} := M_{12rgt} + M_{6top} = 17.5 \cdot kip \cdot ft$$
$$M_{13rgt} := -M_{13lft} = -17.5 \cdot kip \cdot ft$$
$$M_{14lft} := M_{13rgt} + M_{7top} = 17.5 \cdot kip \cdot ft$$
$$M_{14rgt} := -M_{14lft} = -17.5 \cdot kip \cdot ft$$

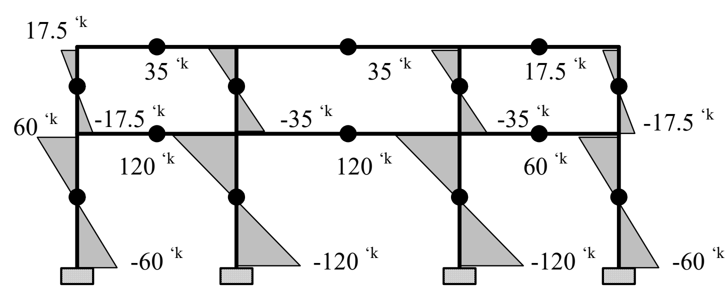
5. Bottom Girder Moments

$$M_{9lft} := M_{1top} - M_{5bot} = 77.5 \cdot kip \cdot ft$$
$$M_{9rgt} := -M_{9lft} = -77.5 \cdot kip \cdot ft$$
$$M_{10lft} := M_{9rgt} + M_{2top} - M_{6bot} = 77.5 \cdot kip \cdot ft$$
$$M_{10rgt} := -M_{10lft} = -77.5 \cdot kip \cdot ft$$
$$M_{11lft} := M_{10rgt} + M_{3top} - M_{7bot} = 77.5 \cdot kip \cdot ft$$
$$M_{11rgt} := -M_{11lft} = -77.5 \cdot kip \cdot ft$$









6. Top Girder Shear

$$V_{12lft} := \frac{-2 \cdot M_{12lft}}{20ft} = -1.75 \cdot kip$$
$$V_{12rgt} := V_{12lft} = -1.75 \cdot kip$$
$$V_{13lft} := \frac{-2 \cdot M_{13lft}}{30ft} = -1.167 \cdot kip$$
$$V_{13rgt} := V_{13lft} = -1.167 \cdot kip$$
$$V_{14lft} := \frac{-2 \cdot M_{14lft}}{24ft} = -1.458 \cdot kip$$
$$V_{14rgt} := V_{14lft} = -1.458 \cdot kip$$

7. Bottom Girder Shear

$$V_{9lft} := \frac{-2 \cdot M_{9lft}}{20ft} = -7.75 \cdot kip$$

$$V_{9rgt} := V_{9lft} = -7.75 \cdot kip$$

$$V_{10lft} := \frac{-2 \cdot M_{10lft}}{30ft} = -5.167 \cdot kip$$

$$V_{10rgt} := V_{10lft} = -5.167 \cdot kip$$

$$V_{11lft} := \frac{-2 \cdot M_{11lft}}{24ft} = -6.458 \cdot kip$$

$$V_{11rgt} := V_{11lft} = -6.458 \cdot kip$$

8. Top Column Axial Forces (+ve tension, -ve compression)

$$P_5 := -V_{12lft} = 1.75 \cdot kip$$

$$P_6 := V_{12rgt} - V_{13lft} = -0.583 \cdot kip$$

$$P_7 := V_{13rgt} - V_{14lft} = 0.292 \cdot kip$$

$$P_8 := V_{14rgt} = -1.458 \cdot kip$$

8. Bottom Column Axial Forces (+ve tension, -ve compression)

$$P_1 := P_5 - V_{9lft} = 9.5 \cdot kip$$

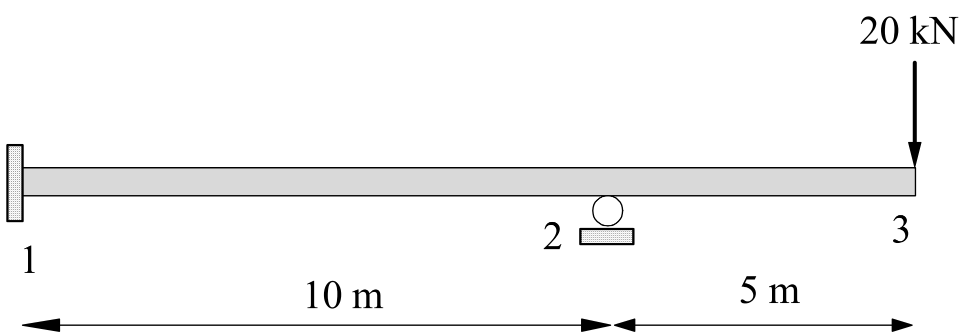
$$P_2 := P_6 - V_{10rgt} + V_{9lft} = -3.167 \cdot kip$$

$$P_3 := P_7 - V_{11rgt} + V_{10lft} = 1.583 \cdot kip$$

$$P_4 := P_8 + V_{11rgt} = -7.917 \cdot kip$$

16.1 Propped Cantilever Beam

Find the end moments for the beam



Solution:

1. The beam is kinematicall indeterminate to the third degreee ($\theta_2, \Delta_3, \theta_3$), however by replacing the overhang by a fixed end moment equal to 100 kN-m at support 2, we reduce the degreee of kinematic indeterminancy to one (θ_2).
2. The equilibrium relation is

$$M_{21} - 100kN\cdot m = 0$$

3. The member end moments in terms of the rotations are

$$M_{12} = 2\cdot E\cdot K_{12}\cdot(2\cdot\theta_1 + \theta_2) = \frac{2}{10}\cdot E\cdot I\cdot\theta_2$$

$$M_{21} = 2\cdot E\cdot K_{12}\cdot(\theta_1 + 2\cdot\theta_2) = \frac{4}{10}\cdot E\cdot I\cdot\theta_2$$

4. Substituting into the equilibrium equations

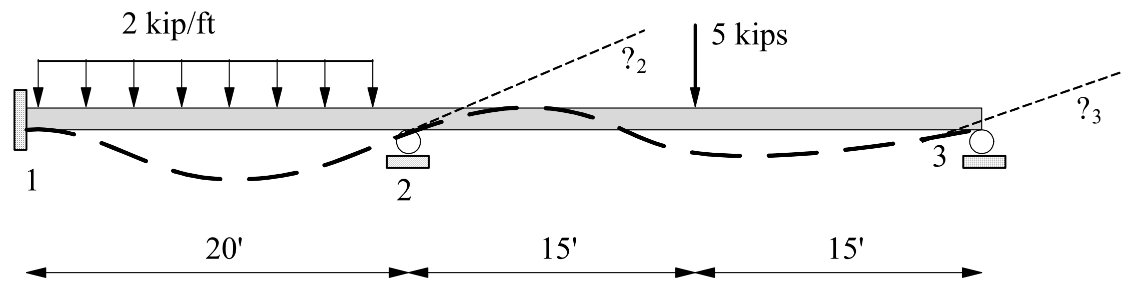
$$\theta_2 = \frac{10}{4E\cdot I}\cdot M_{21} = \frac{250}{EI}$$

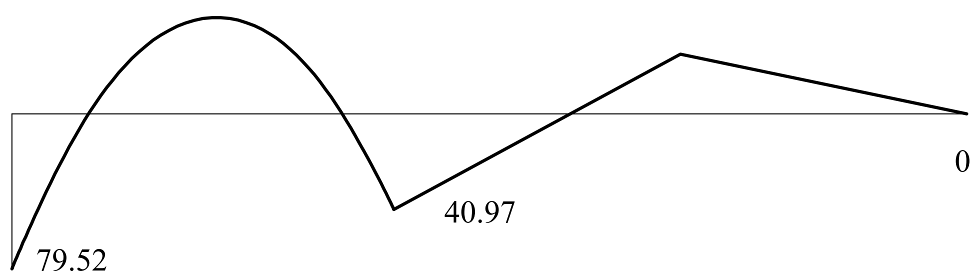
or

$$\boxed{M_{12} = \frac{2}{10}\cdot EI\cdot\theta_2 = \frac{2\cdot EI}{10}\cdot\frac{250}{EI} = 50kN\cdot m}$$

16.2 Two-Span Beam, Slope Deflection

Draw the moment diagram for the two span beam





Solution:

1. The unknows are θ_2 and θ_3
2. The equilibrium relations are

$$M_{21} + M_{23} = 0$$

$$M_{32} = 0$$

3. The fixed end moments are given by

$$M_{12F} = -M_{21F} = \frac{-w \cdot L^2}{12} \qquad M_{12F} := \frac{-\left(-2\,\frac{kip}{ft}\right)\cdot(20ft)^2}{12} = 66.667 \cdot kip \cdot ft$$

$$M_{23F} = -M_{32F} = \frac{-PL}{8} \qquad M_{23F} := \frac{-(-5kip)(30ft)}{8} = 18.75 \cdot kip \cdot ft$$

4. The member end moments in terms of the rotations are

$$M_{12} = 2 \cdot E \cdot K_{12} \cdot \theta_2 + M_{12F} = \frac{2EI}{L_1} \cdot \theta_2 + M_{12F} = \frac{EI}{10} \cdot \theta_2 + M_{12F}$$

$$M_{21} = 2 \cdot E \cdot K_{12} \cdot \left(2\theta_2\right) + M_{21F} = \frac{4EI}{L_1} \cdot \theta_2 + M_{21F} = \frac{EI}{5} \cdot \theta_2 + M_{21F}$$

$$M_{23} = 2 \cdot E \cdot K_{23} \cdot \left(2\theta_2 + \theta_3\right) + M_{23F} = \frac{2EI}{L_2} \cdot \left(2\theta_2 + \theta_3\right) + M_{23F} = \frac{EI}{7.5} \cdot \theta_2 + \frac{EI}{15} \cdot \theta_3 + M_{23F}$$

$$M_{32} = 2 \cdot E \cdot K_{23} \cdot \left(\theta_2 + 2\theta_3\right) + M_{32F} = \frac{2EI}{L_2} \cdot \left(\theta_2 + 2\theta_3\right) + M_{32F} = \frac{EI}{15} \cdot \theta_2 + \frac{EI}{7.5} \cdot \theta_3 + M_{32F}$$

5. Substituting in the equilibrium equations

5. Substituting in the equilibrium equations

$$\frac{EI}{5}\cdot\theta_2 + M_{21F} + \frac{EI}{7.5}\cdot\theta_2 + \frac{EI}{15}\cdot\theta_3 + M_{23F} = 0 \quad => \quad \frac{EI}{5}\cdot\theta_2 + \frac{EI}{7.5}\cdot\theta_2 + \frac{EI}{15}\cdot\theta_3 = 47.917\ \text{kip}\cdot\text{ft}$$

$$\frac{EI}{15}\cdot\theta_2 + \frac{EI}{7.5}\cdot\theta_3 + M_{32F} = 0 \quad => \quad \frac{EI}{15}\cdot\theta_2 + \frac{EI}{7.5}\cdot\theta_3 = 18.75\ \text{kip}\cdot\text{ft}$$

or

$$EI\cdot\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 718.755 \\ 281.25 \end{pmatrix}$$

$$\begin{pmatrix} EI\theta_2 \\ EI\theta_3 \end{pmatrix} := \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}^{-1}\cdot\begin{pmatrix} 718.755 \\ 281.25 \end{pmatrix} = \begin{pmatrix} 128.473 \\ 76.388 \end{pmatrix}$$

6. Substituting for the moments

$$M_{12} := \frac{EI\theta_2\cdot\text{kip}\cdot\text{ft}}{10} + M_{12F} = 79.514\cdot\text{kip}\cdot\text{ft}$$

$$M_{21} := \frac{EI\theta_2\cdot\text{kip}\cdot\text{ft}}{5} - M_{12F} = -40.972\cdot\text{kip}\cdot\text{ft}$$

$$M_{23} := \frac{EI\theta_2\cdot\text{kip}\cdot\text{ft}}{7.5} + \frac{EI\theta_3\cdot\text{kip}\cdot\text{ft}}{15} + M_{23F} = 40.972\cdot\text{kip}\cdot\text{ft}$$

$$M_{32} := \frac{EI\theta_2\cdot\text{kip}\cdot\text{ft}}{15} + \frac{EI\theta_3\cdot\text{kip}\cdot\text{ft}}{7.5} - M_{23F} = 0\cdot\text{kip}\cdot\text{ft}$$

16.3 Two-Span Beam, Slope Deflection, Initial Deflection

Determine the end moments for the previous problem if the middle support settles by 6 in.

Solution:

1. Since we are performing a linear elastic analysis, we can separately analyze the beam for support settlement, and then add the moments to those due to the applied loads.
2. The unknowns are θ_2 and θ_3
3. The equilibrium relations are

$$M_{21} + M_{23} = 0$$

$$M_{32} = 0$$

4. The member end moments in terms of the rotations are

$$M_{12} = 2\cdot E\cdot K_{12}\cdot\left(\theta_2 - 3\frac{\Delta}{L_{12}}\right) = \frac{2EI}{20}\cdot\left(\theta_2 + 3\cdot\frac{0.5}{20}\right) = \frac{EI}{10}\cdot\theta_2 + \frac{3EI}{400}$$

$$M_{21} = 2\cdot E\cdot K_{12}\cdot\left(2\theta_2 - 3\frac{\Delta}{L_{12}}\right) = \frac{2EI}{20}\cdot\left(2\theta_2 + 3\cdot\frac{0.5}{20}\right) = \frac{EI}{5}\cdot\theta_2 + \frac{3EI}{400}$$

$$M_{23} = 2\cdot E\cdot K_{23}\cdot\left(2\theta_2 + \theta_3 - 3\frac{\Delta}{L_{23}}\right) = \frac{2EI}{30}\cdot\left(2\theta_2 + \theta_3 + 3\cdot\frac{0.5}{30}\right) = \frac{EI}{7.5}\cdot\theta_2 + \frac{EI}{15}\cdot\theta_3 + \frac{EI}{300}$$

$$M_{32} = 2\cdot E\cdot K_{23}\cdot\left(\theta_2 + 2\theta_3 - 3\frac{\Delta}{L_{23}}\right) = \frac{2EI}{30}\cdot\left(\theta_2 + 2\theta_3 + 3\cdot\frac{0.5}{30}\right) = \frac{EI}{15}\cdot\theta_2 + \frac{EI}{7.5}\cdot\theta_3 + \frac{EI}{300}$$

5. Substituting into the equilibrium equations

5. Substituting into the equilibrium equations

$$\frac{EI}{5}\cdot\theta_2 + \frac{3EI}{400} + \frac{EI}{7.5}\cdot\theta_2 + \frac{EI}{15}\cdot\theta_3 + \frac{EI}{300} = 0 \qquad \frac{EI}{3}\cdot\theta_2 + \frac{EI}{15}\cdot\theta_3 = \frac{-13EI}{1200}$$

$$\frac{EI}{15}\cdot\theta_2 + \frac{EI}{7.5}\cdot\theta_3 + \frac{EI}{300} = 0 \qquad \frac{EI}{15}\cdot\theta_2 + \frac{EI}{7.5}\cdot\theta_3 = \frac{-EI}{300}$$

or

$$EI\cdot\begin{pmatrix}100 & 20\\ 20 & 40\end{pmatrix}\cdot\begin{pmatrix}\theta_2\\ \theta_3\end{pmatrix} = EI\cdot\begin{pmatrix}\frac{-13}{4}\\ -1\end{pmatrix}$$

which will give

$$\begin{pmatrix}\theta_2\\ \theta_3\end{pmatrix} := \begin{pmatrix}100 & 20\\ 20 & 40\end{pmatrix}^{-1}\cdot\begin{pmatrix}\frac{-13}{4}\\ -1\end{pmatrix} \rightarrow \begin{pmatrix}-\frac{11}{360}\\ -\frac{7}{720}\end{pmatrix}$$

6. Thus the additional moments due to the settlement are

$$M_{12}(EI) := \frac{EI}{10}\cdot\theta_2 + \frac{3EI}{400} \text{ simplify } \rightarrow \frac{EI}{225}$$

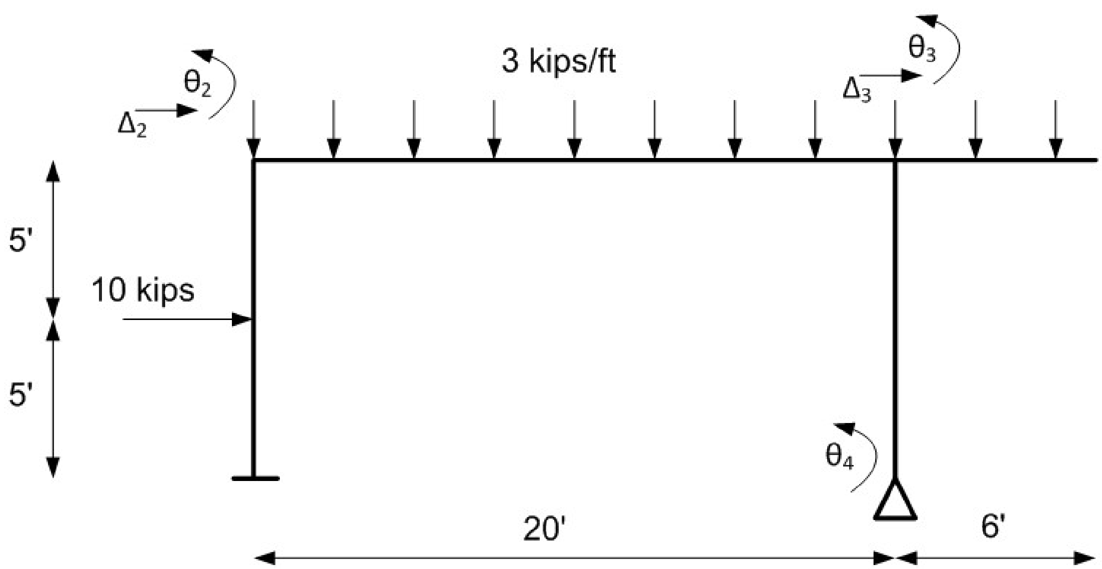
$$M_{21}(EI) := \frac{EI}{5}\cdot\theta_2 + \frac{3EI}{400} \text{ simplify } \rightarrow \frac{EI}{720}$$

$$M_{23}(EI) := \frac{EI}{7.5}\cdot\theta_2 + \frac{EI}{15}\cdot\theta_3 + \frac{EI}{300} \text{ simplify } \rightarrow -0.0013888888888888888889\cdot EI$$

$$M_{32}(EI) := \frac{EI}{15}\cdot\theta_2 + \frac{EI}{7.5}\cdot\theta_3 + \frac{EI}{300} \text{ simplify } \rightarrow -3.7037037037037037037e\text{-}24\cdot EI$$

16.4 Frame, Slope Deflection

Determine the end moments for the frame



Solution:

1. The effect of the 35 cantilever can be included by replacing it with its end moment.

$$M_3 = -w \cdot L \cdot \frac{L}{2} \qquad M_3 := 3\,\frac{kip}{ft} \cdot 6ft \cdot 6\,\frac{ft}{2} = 54 \cdot kip \cdot ft$$

2. The unknown displacements and rotations are

Δ_2 and θ_2 at joint 2

θ_3 and θ_4 at joints 3 and 4

We observe that due to the lack of symmetry, there will be a lateral displacement in the frame, and neglecting axial deformations, $\Delta_2 = \Delta_3$

3. The equilibrium relations are

$$M_{23} + M_{32} = 0$$

$$M_{32} + M_{34} = -54 kip \cdot ft$$

$$M_{43} = 0$$

$$V_{12} + V_{43} - 10 kip = 0$$

Thus we have four unknown displacements and four equations. However, the last two equations are in terms of of the shear forces, and we need to have them in terms of the end moments. This can be achieved through the following equilibrium relations

$$V_{12} = \frac{M_{12} + M_{21} + 50}{L_{12}}$$

$$V_{43} = \frac{M_{34} + M_{43}}{L_{34}}$$

Hence, all four equations are now in terms of the moments.

4. The fixed end moments for member 23 are

4. The fixed end moments for member 23 are

$$M_{21F} = \frac{-PL}{8} \qquad M_{21F} := \frac{-10 kip \cdot 10 ft}{8} = -12.5 \cdot kip \cdot ft$$

$$M_{23F} = \frac{-w \cdot L^2}{12} \qquad M_{21F} := \frac{-3\,\frac{kip}{ft} \cdot (20ft)^2}{12} = -100 \cdot kip \cdot ft$$

5. The member end moments in terms of the rotations are

$$M_{12} = 2 \cdot E \cdot K_{12} \cdot \left(\theta_2 - 3\,\frac{\Delta_2}{L_{12}}\right) + M_{12F} = 0.2EI \cdot \left(\theta_2 - 0.3\Delta_2\right) + 12.5$$

$$M_{21} = 2 \cdot E \cdot K_{12} \cdot \left(2\theta_2 - 3\,\frac{\Delta_2}{L_{12}}\right) + M_{21F} = 0.2EI \cdot \left(2\theta_2 - 0.3\Delta_2\right) - 12.5$$

$$M_{23} = 2 \cdot E \cdot K_{23} \cdot \left(2\theta_2 + \theta_3\right) + M_{23F} = 0.1EI \cdot \left(2\theta_2 + \theta_3\right) + 100$$

$$M_{32} = 2 \cdot E \cdot K_{32} \cdot \left(\theta_2 + 2\theta_3\right) + M_{32F} = 0.1EI \cdot \left(\theta_2 + 2\theta_3\right) - 100$$

$$M_{34} = 2 \cdot E \cdot K_{34} \cdot \left(2\theta_3 + \theta_4 - 3\,\frac{\Delta_2}{L_{34}}\right) = 0.2EI \cdot \left(2\theta_3 + \theta_4 - 0.3\Delta_2\right) + 100$$

$$M_{43} = 2 \cdot E \cdot K_{34} \cdot \left(\theta_3 + 2\theta_4 - 3\,\frac{\Delta_2}{L_{34}}\right) = 0.2EI \cdot \left(\theta_3 + 2\theta_4 - 0.3\Delta_2\right) + 100$$

6. Substituting into the equilibrium equations and dividing by EI

$$6\theta_2 + \theta_3 - 0.6\Delta_2 = \frac{-875}{EI}$$

$$\theta_2 + 6\theta_3 + 2\theta_4 - 0.6\Delta_2 = \frac{460}{EI}$$

$$\theta_3 + 2\theta_4 - 0.3\Delta_2 = 0$$

and the last equilibrium equation is obtained by substituting V_{12} and V_{43} and multiplying by 10/EI

$$\theta_2 + \theta_3 + \theta_4 - 0.4\Delta_2 = \frac{-83.3}{EI}$$

or

$$EI\cdot\begin{pmatrix} 6 & 1 & 0 & -0.6 \\ 1 & 6 & 2 & -0.6 \\ 0 & 1 & 2 & -0.3 \\ 1 & 1 & 1 & -0.4 \end{pmatrix}\cdot\begin{pmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \\ \Delta_2 \end{pmatrix} = \frac{1}{EI}\cdot\begin{pmatrix} -875 \\ 460 \\ 0 \\ -83.3 \end{pmatrix}$$

which will give

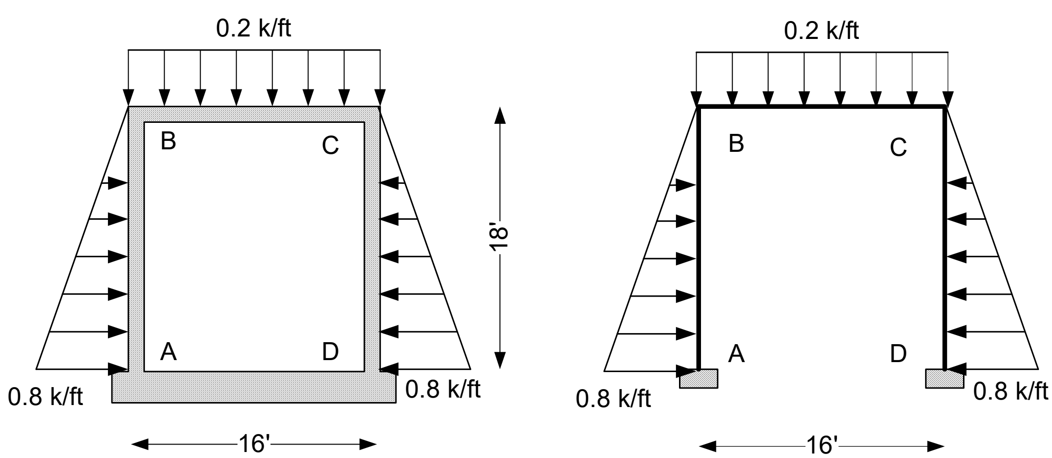
$$\begin{pmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \\ \Delta_2 \end{pmatrix} = \frac{1}{EI}\cdot\begin{pmatrix} -294.8 \\ 68.4 \\ -240.6 \\ -1375.7 \end{pmatrix}$$

7. Substituting into the slope deflection equations gives the end moments

$$\begin{pmatrix} M_{12} \\ M_{21} \\ M_{23} \\ M_{32} \\ M_{34} \\ M_{43} \end{pmatrix} = \begin{pmatrix} 36 \\ -47.88 \\ 47.88 \\ -115.8 \\ 61.78 \\ 0 \end{pmatrix}$$

16.5 Box Culvert Slope Deflection

Draw the shear and moment diagram for the following box girder



Solution:

1. From symmetry $\theta_B = -\theta_C$, and at the base $\theta_A = \theta_D = 0$.
2. The fixed end moments are given by

$$FEM_{BC} = \frac{w \cdot L^2}{12} \qquad FEM_{BC} := \frac{0.2\frac{kip}{ft}\cdot(16ft)^2}{12} = 4.267 \cdot kip \cdot ft$$

$$FEM_{CB} = \frac{-w \cdot L^2}{12} \qquad FEM_{CB} := -\frac{0.2\frac{kip}{ft}\cdot(16ft)^2}{12} = -4.267 \cdot kip \cdot ft$$

$$FEM_{AB} = \frac{w \cdot L^2}{20} \qquad FEM_{AB} := \frac{0.8\frac{kip}{ft}\cdot(18ft)^2}{20} = 12.96 \cdot kip \cdot ft$$

$$FEM_{BA} = \frac{-\left(w \cdot L^2\right)}{30} \qquad FEM_{BA} := -\frac{0.8\frac{kip}{ft}\cdot(18ft)^2}{30} = -8.64 \cdot kip \cdot ft$$

3. The moments are given by

$$M_{BC} = \frac{2EI}{16}\cdot\left(2\theta_B + \theta_C\right) + FEM_{BC} = \frac{EI}{8}\cdot\theta_B + 4.267$$

$$M_{BA} = \frac{2EI}{18}\cdot\left(2\theta_B + 0\right) + FEM_{BA} = \frac{2EI}{9}\cdot\theta_B - 8.64$$

$$M_{AB} = \frac{2EI}{18}\cdot\left(2\theta_B\right) + FEM_{AB} = \frac{EI}{9}\cdot\theta_B + 12.96$$

4. Equilibrium at joint B

$$M_{BA} + M_{BC} = 0$$

$$\frac{2EI}{9}\cdot\theta_B - 8.64 + \frac{EI}{8}\cdot\theta_B + 4.267 = 0$$

$$\theta_B(EI) := \frac{8.64 - 4.267}{\frac{2EI}{9} + \frac{EI}{8}} \rightarrow \frac{12.594}{EI}$$

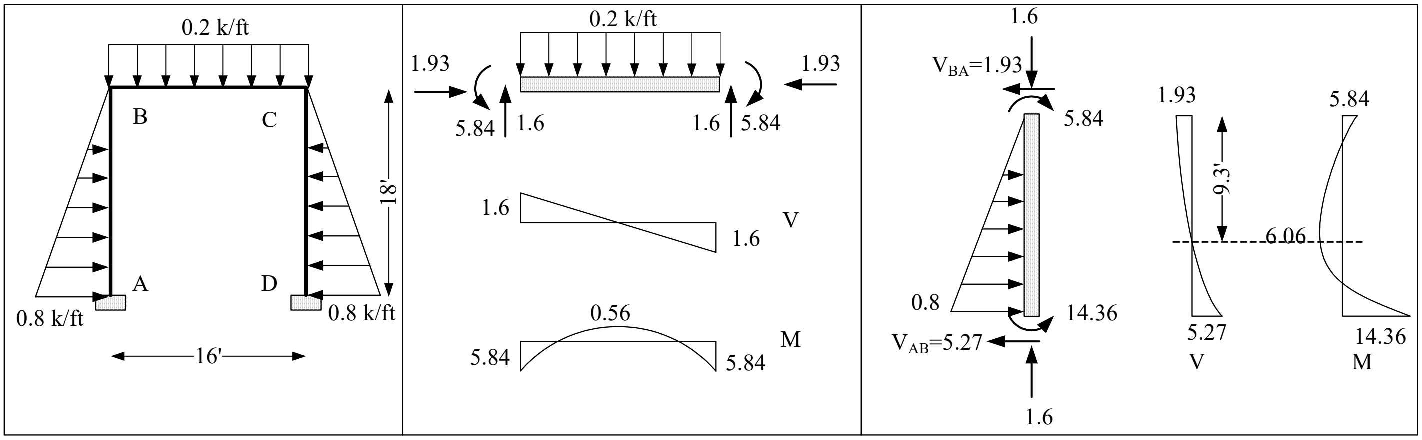
5. Substitute θ_B to get the moments

$$M_{BC} := \left[\frac{EI}{8}\cdot\left(\frac{12.594}{EI}\right) + 4.267\right]\cdot kip \cdot ft \rightarrow 5.841 \cdot ft \cdot kip$$

$$M_{AB} := \left[\frac{EI}{9}\cdot\left(\frac{12.594}{EI}\right) + 12.96\right]\cdot kip \cdot ft \rightarrow 14.359 \cdot ft \cdot kip$$

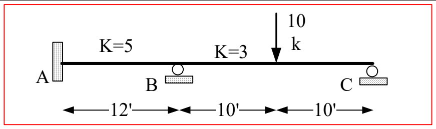
$$M_{BA} := \left[\frac{2EI}{9}\cdot\left(\frac{12.594}{EI}\right) - 8.64\right]\cdot kip \cdot ft \rightarrow -5.841 \cdot ft \cdot kip$$

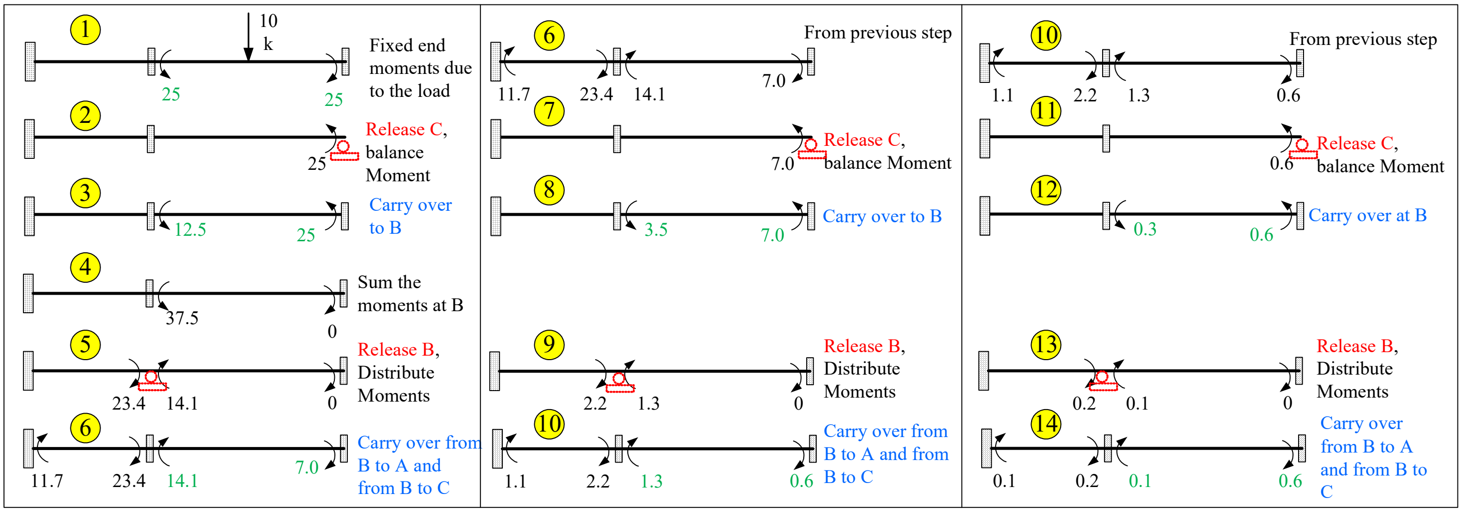
6. Member forces are determined from statics. Careful, the moment diagram is now based on the so-called "design" sign convention.

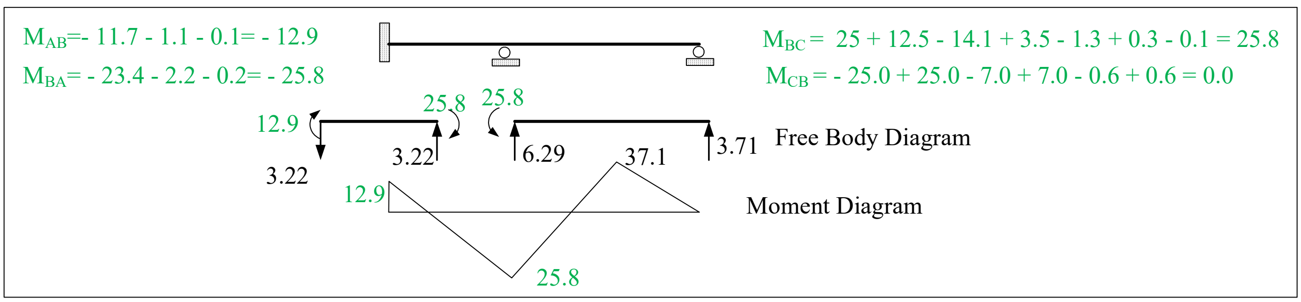


16.6 Continuous Beam

Solve for the moments at A and B by moment distribution, using (a) the ordinary method, and (b) the simplified method.







Solution:

For this example the fixed-end moments are computed as follows:

$$M_{BCF} = \frac{PL}{8} \qquad M_{BCF} := \frac{10kip \cdot 20ft}{8} = 25 \cdot kip \cdot ft$$

$$M_{CBF} := -M_{BCF} = -25 \cdot kip \cdot ft$$

2. Since the relative stiffness is given in each span, the distribution factors are

$$DF_{AB} = \frac{K_{AB}}{\Sigma K} = \frac{5}{\infty + 5} = 0$$

$$DF_{BA} = \frac{K_{BA}}{\Sigma K} = \frac{5}{5 + 3} = 0.625$$

$$DF_{BC} = \frac{K_{BC}}{\Sigma K} = \frac{3}{5 + 3} = 0.375$$

$$DF_{CB} = \frac{K_{CB}}{\Sigma K} = \frac{3}{3} = 1$$

3. The balancing computations are shown below.

Joint	A	B		C
Member	AB	BA	BC	CB
K	5	5	3	3

Joint	A	B		C
Member	AB	BA	BC	CB
K	5	5	3	3
DF	0	0.625	0.375	1
FEM			25	-25
			12.5	25
	-11.7	-23.4	-14.1	-7
			3.5	7
	-1.1	-2.2	-1.3	-0.6
			0.3	0.6
	-0.1	-0.2	-0.1	
Total	-12.9	-25.8	25.8	0

4. The above solution is that referred to as the ordinary method, so named to designate the manner of handling the balancing at the simple support at C. It is known, of course, that the final moment must be zero at this support because it is simple
5. Consequently, the first step is to balance the fixed-end moment at C to zero. The carry-over is then amde immediately to B. When B is balanced, however, a carry-over must be made back to C simply because the relative stiffness of BC is based on end C of this span being fixed. It is apparent, however, that the moment carried back to C (in this case -7.0) cannot exist at this joint. Accordingly, it is immediately balanced out, and a carry-over is again made to B, this carry-over being considerably smaller than the first. Now B is again balanced, and the process continues until the numbers involved become too small to have any practical value.
6. Alternatively, we can use the simplified method. It was previously shown that if the support at C is simple and a moment is applied at B, then the resistance of the span BC to this moment is reduced to three-fourths the value it would have had wit C fixed. Consequently, the relative stiffness of span BC is reduced to three-fourths of the value given, it will not be necessary to carry over to C.

Joint	A	B		C
Member	AB	BA	BC	CB
K	5	5	2.25	3
DF	0	0.69	0.31	1
FEM			25	-25
			12.5	25
	-12.9	-25.8	-11.7	
Total	-12.9	-25.8	25.8	0

7. From the standpoint of work involved, the advantage of the simplified method is obvious. It should always be used when the external (terminal) end of a member rests on a simple support, but it does not apply when a structure is continuous at a simple support. Attention is called to the fact that when the opposite end of the member is simply supported, the reduction factor for the stiffness is always 3/4 for a prismatic member but a variable quantity for a nonprismatic member.
8. One valuable feature of the tabular arrangement is that of dropping down one line for each balancing operation and making the carry-over on the same line. The practice clearly indicates the order of balancing the joints, which in turn makes it possible to check back in the event of an error. Moreover, the placing of the carry-over on the same line with the balancing moments definitely decreases the chances of omitting a carry-over.
9. The correctness of the answers may ina sense be checked by verifying $\Sigma M = 0$ at each joint. However, even though the final answers satisfy this equation at every join, this is in no way a check on the inital fixed end moments. These fixed end moments, therefore, should be checked with great care before beginning the balancing operation. Moreover, it occasionally happens that conpensating errors are made in the balancing, and these errors will not be apparent when cheking $\Sigma M = 0$ at each joint.

10. To draw the final shear and moment diagram, we start by drawing the free body diagram of each beam segment with the computed

moments, and then solve from statics for the reactions

$12.9\text{kip}\cdot\text{ft} + 25.8\text{kip}\cdot\text{ft} - 12\text{ft}\,V_A = 0$ $\qquad V_A := \dfrac{12.9\text{kip}\cdot\text{ft} + 25.8\text{kip}\cdot\text{ft}}{12\text{ft}} = 3.225\cdot\text{kip}$

$V_A + V_{BL} = 0$ $\qquad V_{BL} := -V_A = -3.225\cdot\text{kip}$

$25.8\text{kip}\cdot\text{ft} + 10\text{kip}\cdot 10\text{ft} - 20\text{ft}\cdot V_{BR} = 0$ $\qquad V_{BR} := \dfrac{25.8\text{kip}\cdot\text{ft} + 10\text{kip}\cdot 10\text{ft}}{20\text{ft}} = 6.29\cdot\text{kip}$

$6.29\text{kip} + V_C - 10\text{kip} = 0$ $\qquad V_C := 10\text{kip} - 6.29\text{kip} = 3.71\cdot\text{kip}$

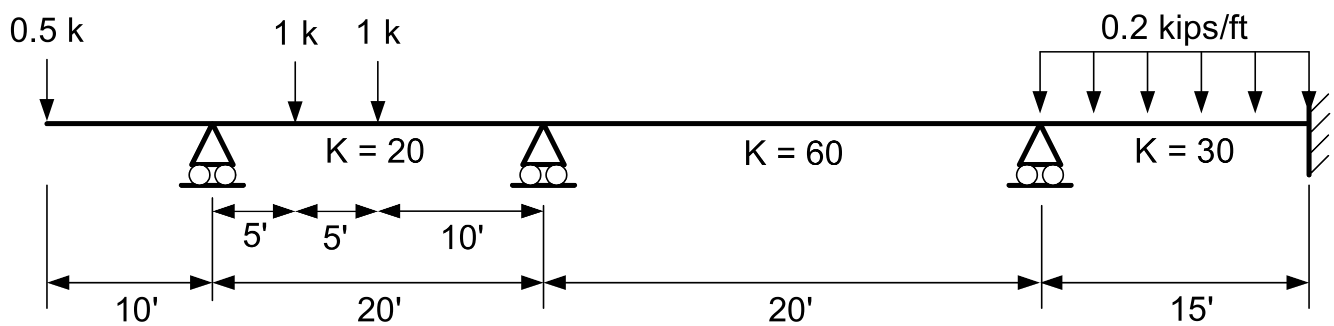
$-V_{BL} - V_{BR} + R_B = 0$ $\qquad R_B := V_{BL} + V_{BR} = 3.065\cdot\text{kip}$

Check: $R_A + R_B + R_C - 10\text{kip} = 0$ $\qquad V_A + R_B + V_C - 10\text{kip} = 0\cdot\text{kip}$

$M_{BC} := V_C\cdot 10\text{ft} = 37.1\cdot\text{kip}\cdot\text{ft}$

16.7 Continuous Beam

Using the simplified method of moment distribution, find the moments in the following continuous beam. The values of I as indicated by the various values of K, are different for the various spans. Determine the values of the reactions, draw the shear and bending moment diagrams, and sketch the deflected structure.



Solution:

1. Fixed-end moments

$M_{AOF} := -0.5\text{kip}\cdot 10\text{ft} = -5\cdot\text{kip}\cdot\text{ft}$

For the 1 kip load:

$$M_{ABF} = \frac{P\cdot a\cdot b^2}{L^2} \qquad M_{ABF1} := \frac{1\text{kip}\cdot 5\text{ft}\cdot(15\text{ft})^2}{(20\text{ft})^2} = 2.812\cdot\text{kip}\cdot\text{ft}$$

$$M_{BAF} = \frac{P\cdot a^2\cdot b}{L^2} \qquad M_{BAF1} := \frac{1\text{kip}\cdot(5\text{ft})^2\cdot 15\text{ft}}{(20\text{ft})^2} = 0.938\cdot\text{kip}\cdot\text{ft}$$

For the 4 kip load:

$$M_{ABF} = \frac{P\cdot L}{8} \qquad M_{ABF4} := \frac{4\text{kip}\cdot 20\text{ft}}{8} = 10\cdot\text{kip}\cdot\text{ft}$$

$$M_{BAF4} := -M_{ABF4} = -10\cdot\text{kip}\cdot\text{ft}$$

For the uniform load:

$$M_{CDF} = \frac{w\cdot L^2}{12} \qquad M_{CDF} := \frac{0.2\frac{\text{kip}}{\text{ft}}\cdot(15\text{ft})^2}{12} = 3.75\cdot\text{kip}\cdot\text{ft}$$

$$M_{DCF} := -M_{CDF} = -3.75\cdot\text{kip}\cdot\text{ft}$$

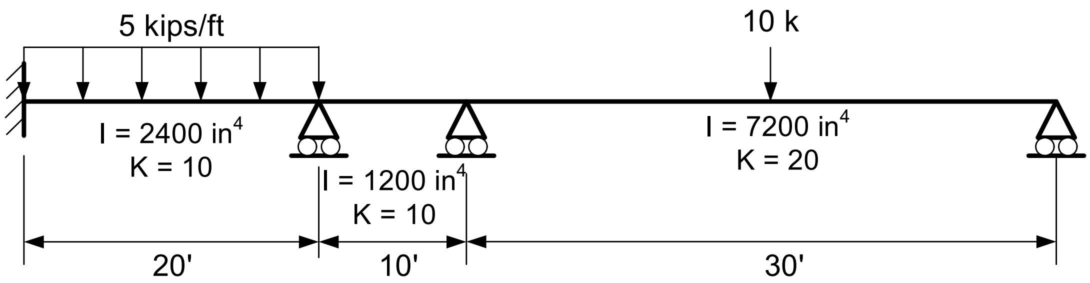
2. The balancing operation is shown below

Joint	A		B		C		D
Member	AO	AB	BA	BC	CB	CD	DC
K	0	20	15	60	60	40	40
DF	0	1	0.2	0.8	0.6	0.4	0
FEM	-5	12.8	-10.9			3.8	-3.8
		-7.8	-3.9				
			2.9	11.9	5.9		
				-2.9	-5.8	-3.9	-1.9
			0.6	2.3	1.1		
				-0.3	-0.7	-0.4	-0.2
			0.1	0.2			
Total	-5	5	-11.2	11.2	0.5	-0.5	-5.9

3. The only new point in this example is the method of handling the overhanging end. It is obvious that the final internal moment at A must be 5 kip-ft and, accordingly, the first step is to balance out 7.8 kip-ft of the fixed end moment at AB, leaving the required 5 kip-ft for the internal moment at AB. Since the relative stiffness of BA has been reduced to three-fourths of its original value, to permit considering the support at A as simple in balancing, no carry-over from B to A is required.
4. The easiest way to determine the reactions is to consider each span as a free body. End shears are first determined as caused by the loads alone on each span and, following this, the end shears caused by the end moments are computed. These two shears are added algebraically to obtain the net end shear for each span. An algebraic summation of the end shears at any support will give the total reaction.

16.8 Continuous Beam, Initial Settlement

For the following beam find the moments at A, B, and C by moment distribution. The support at C settles by 0.1 in. Use $E := 30000\text{ksi}$



Solution:

1. Fixed-end moments

Uniform load:

$$M_{ABF} = \frac{w \cdot L^2}{12} \qquad M_{ABF} := \frac{5\frac{kip}{ft}\cdot(20ft)^2}{12} = 166.667\cdot kip\cdot ft$$

$$M_{BAF} := -M_{ABF} = -166.667\cdot kip\cdot ft$$

Concentrated load:

$$M_{CDF} = \frac{P \cdot L}{8} \qquad M_{CDF} := \frac{10kip\cdot 30ft}{8} = 37.5\cdot kip\cdot ft$$

$$M_{DCF} := -M_{CDF} = -37.5\cdot kip\cdot ft$$

Moments caused by deflection:

$$M_{BCF} = \frac{6EI\cdot\Delta}{L^2} \qquad M_{BCF} := \frac{6\cdot E\cdot 1200in^4\cdot 0.1in}{(10ft)^2} = 125\cdot kip\cdot ft$$

$$M_{CBF} := M_{BCF} = 125\cdot kip\cdot ft$$

$$M_{CDF} = -\frac{6EI\cdot\Delta}{L^2} \qquad M_{CDF} := -\frac{6\cdot E\cdot 7200in^4\cdot 0.1in}{(30ft)^2} = -83.333\cdot kip\cdot ft$$

$$M_{DCF} := M_{CDF} = -83.333\cdot kip\cdot ft$$

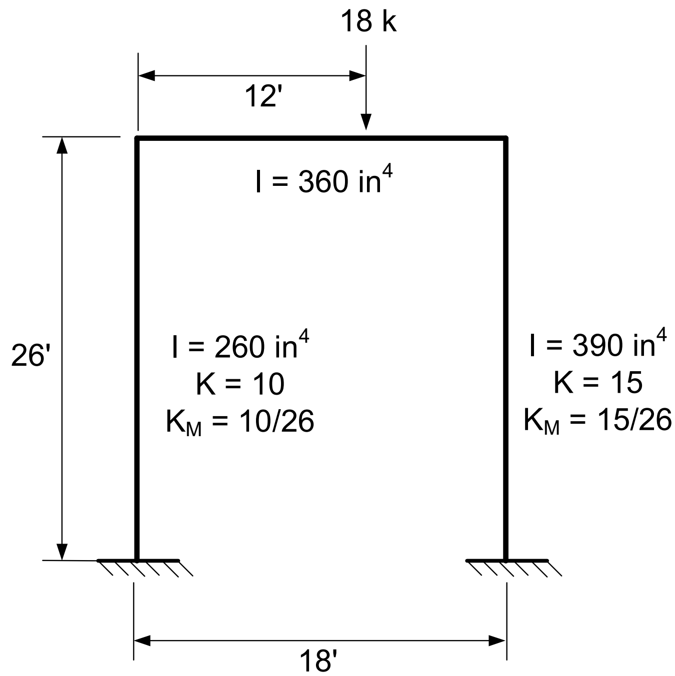
2. Moment distribution

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
K	10	10	10	10	15	20
DF	0	0.5	0.5	0.4	0.6	1
FEM	167	-167			38	-38
			125	125	-83	-83
					60	121
			-28	-56	-84	
	17	35	35	17		
			-3	-7	-10	
	1	2	1			
Total	185	-130	130	79	-79	0

The fixed-end moments caused by a settlement of supports have the same sign at both ends of each span adjacent to the settling support. The above computations have been carried to the nearest kip-ft, which for the moments of the magnitudes involved, would be significantly close for purposes of design.

16.9 Frame

Find all moments by moment distribution for the following frame. Draw the bending moment diagram and the deflected structure.



Solution:

1. The first step is to perform the usual moment distribution. The reader should fully understand that this balancing operation adjusts the internal moments at the ends of the members by a series of corrections as the joints are considered to rotate, until $\Sigma M = 0$ at each joint. The reader should also realize that during this balancing operation no translation of any joint is permitted.
2. The fixed-end moments are

$$M_{BCF} := \frac{18\text{kip}\cdot 12\text{ft}\cdot(6\text{ft})^2}{(18\text{ft})^2} = 24\cdot\text{kip}\cdot\text{ft}$$

$$M_{CBF} := -\frac{18\text{kip}\cdot 6\text{ft}\cdot(12\text{ft})^2}{(18\text{ft})^2} = -48\cdot\text{kip}\cdot\text{ft}$$

3. Moment distribution

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
K	10	10	20	20	15	15
DF	0	0.333	0.667	0.571	0.429	0
			24	-48		
			13.7	27.4	20.6	10.3
	-6.3	-12.6	-25.1	-12.5		
FEM			3.6	7.1	5.4	2.7
	-0.6	-1.2	-2.4	-1.2		
			0.3	0.7	0.5	0.2
		-0.1	-0.2			
Total	-6.9	-13.9	13.9	-26.5	26.5	13.2

4. The final moments listed in the table are correct only if there is no translation of any joint. It is therefore necessary to determine whether or not, with the above moments existing, there is any tendency for side lurch of the top of the frame.
5. If the frame is divided into three free bodies, the result will be as shown below.

Inspection of this sketch indicates that if the moments of the first balance exist in the frame, there is a net force of $1.53\text{kip} - 0.8\text{kip} = 0.73\cdot\text{kip}$ tending to sway the frame to the left. In order to prevent side-sway, and thus allow these moments to exist (temporarily for the purpose of the analysis), it is necessary that an imaginary horizontal force ce considered to act to the right at B or C. This force is designated as the artificial joint restraint (abbreviated as AJR) and is shown below.

6. This illustration shows the complete load system which would have to act on the structure if the final moments of the first balance are to be correct. The AJR, however, cannot be permitted to remain, and thus its effect must be cancelled. This may be accomplished by finding the moments in the frame resulting from a force equal but opposite to the AJR and applied at the top.
7. Although it is not possible to make a direct solution for the moments resulting from this force, they may be determined inddirectly. Assume some unknown force P acts on the frame, as shown below, and causes it to deflect laterally to the left, without joint rotation, through some distance Δ. Now, regardless of the value of P and the value of the resulting Δ, the fixed-end moments induced in the ends of the columns must be proprtional to the respective values of K_M.

Recalling that the fixed end moment is $M_F = 6EI\cdot\frac{\Delta}{L^2} = 6EK_M\cdot\Delta$, where $K_M = \frac{1}{L^2} = \frac{K}{L}$ we can write

$$\Delta = \frac{M_{ABF}}{6E\cdot K_M} = \frac{M_{DCF}}{6E\cdot K_M}$$

$$\frac{M_{ABF}}{M_{DCF}} = \frac{K_{MAB}}{K_{MDC}} = \frac{10}{15}$$

These fixed-end moments could, for example, have values of -10 and -15 kip-ft, or -20 and -30, or -30 and -45, or any other combination so long as the above ratio is maintained. The proper procedure is to choose values for the fixed-end moments of approximately the same order of magnitude as the original fixed-end moments due to the real loads. This will result in the same accuracy for the results of the balance for the side-sway correction that was realized in the first balance for the real loads. Accordingly, it will be assumed that P, and the resulting Δ, are of such magnitude as to result in fixed-end moments shown below.
8. Obviously $\Sigma M = 0$ is not satisfied for joints B and C in this deflected frame. Therefore, these joints must rotate until equilibrium is reached. The effect of this rotation is determined in the distribution below.

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
K	10	10	20	20	15	15
DF	0	0.333	0.667	0.571	0.429	0
FEM	-30	-30			-45	-45
			12.9	25.8	19.2	9.6
	2.8	5.7	11.4	5.7		
			-1.6	-3.3	-2.4	-1.2
	0.2	0.5	1.1	0.5		
				-0.3	-0.2	-0.1
Total	-27	-23.8	23.8	28.4	-28.4	-36.7

9. During the rotation of joints B and C, as represented by the above distribution, the value of Δ has remained constant, with P varying in magnitude as required to maintain Δ.
10. It is now possible to determine the final value of P simply by adding the shears in the columns. The shear in any member, without external loads applied along its length, is obtained by adding the end moments algebraically and dividing by the length of the member. The final value of P is the force necessary to maintain the deflection of the frame after the joints have rotated. In other words, it is the force which will be consistent with the displacement and internal moments of the structure as determined by the second balancing operation. Hence this final value of P will be called the consistent joint force (abbreviated as CJF).
11. The consistent joint force is given by

$$CJF := \frac{27kip\cdot ft + 23.8kip\cdot ft}{26ft} + \frac{28.4kip\cdot ft + 36.7kip\cdot ft}{26ft} = 4.458\cdot kip$$

and inspection clearly indicates that the CJF must act to the left.
12. Obviously, then, the results of the last balance above are moments which will exist in the frame when a force of 4.45 kip acts to the left at the top level. It is necessary, however, to determine the moments resulting from a force of 0.73 kip acting to the left at the top level, and some as yet unknown factor "z" times 4.45 kip will be used to represent this force acting to the left.
13. The free body diagram for the member BC is shown above. $\Sigma H = 0$ must be satisfied for this figure, and if forces to the left are considered as positive, the result is $4.45z - 0.73 = 0$, and

$$z := \frac{0.73}{4.45} = 0.164$$

If this factor is applied to the moments obtained from the second balance, the result will be the moments caused by a force of 0.73 kip acting to the left at the top level. If these moments are then added to the moments obtained from the first balance, the result will be the final moments for the frame, the effect of the AJR having been cancelled. The combintaion of moments is shown below.

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
M from 1st balance	-6.9	-13.9	13.9	-26.5	26.5	13.2
z x M from 2nd balance	-4.4	-3.9	3.9	4.7	-4.7	-6
Final Moments	-11.3	-17.8	17.8	-21.8	21.8	7.2

14. If the moments are correct, the shears in the two columns of the frame shoul dbe equal and opposite to satisfy $\Sigma H = 0$ for the entire frame. This check is expressed

$$\frac{11.3kip\cdot ft + 17.8kip\cdot ft}{26ft} + \frac{-21.8kip\cdot ft + -7.2kip\cdot ft}{26ft} = 0\cdot kip$$

and

$1.12kip - 1.11kip = 0.01\cdot kip$ (nearly)

The signs of all moments taken from the previous table have been reversed to give the correct signs for the end moments external to the columns. It will be remembered that the moments considered in moment distribution are always internal for each member. However, the above check actually considers each column as a free body and so external moments must be used.
15. The moment under the 18 kip load is obtained by treating BC as a free body:

$$M_{18} := 5.77kip\cdot 12ft - 17.8kip\cdot ft = 51.44\cdot kip\cdot ft$$

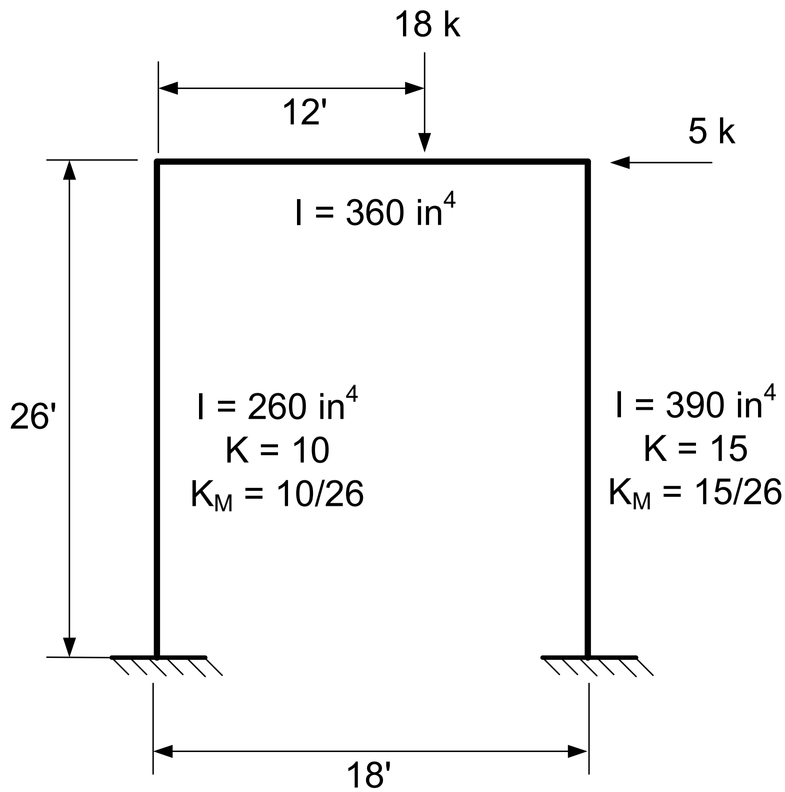
16. The direction of side lurch may be determined from the obvious fact that the frame will always lurch in a direction opposite to the AJR. If required, the magnitude of this side lurch may be found. The procedure which follows will apply.

A force P of sufficient magnitude to result in the indicated column moments and the lurch Δ was applied to the frame. During the second balance this value of Δ was held constant as the joints B and C rotated, and the value of P was considered to vary as necessary. The final value of P was found to be 4.45 kip. Since Δ was held constant, however, its magnitude may be determined from the equation $M = 6EI\cdot\frac{\Delta}{L^2}$,

where M is the fixed-end moment for either column, I is the moment of inertia for that column, and L is the length. This Δ will be the lurch for 4.45 kip acting at the top level. For any other force acting horizontally, Δ would vary proportially and thus the final lurch of the frame would be the factor z multiplied by Δ determined above.

16.10 Frame with Side Load

Find by moment distribution the moments in the following frame



Solution:

The first balance will give the results shown

AB	BA	BC	CB	CD	DC
-7.2	-14.6	14.6	-22.5	22.5	0

A check of the member BC as a free body for $\Sigma H = 0$ will indicate that an AJR is necessary as follows:

$$AJR + 0.84kip - 0.87kip - 5kip = 0$$

from which

$$AJR := 5kip + 0.87kip - 0.84kip = 5.03 \cdot kip$$ in the direction assumed

The values of K_M for the two columns are shown, with K_M for column CD being K/2L because of the pin at the bottom. The horizontal displacement Δ of the top of the frame is assumed to cause the fixed-end moments shown there. These moments are proportional to the values of K_M and of approximately the same order of magnitude as the original fixed-end moments due to the real loads. The results of balancing out these moments are

AB	BA	BC	CB	CD	DC
-34.4	-28.4	28.4	23.6	-23.6	0

$$CJF := \frac{34.4kip \cdot ft + 28.4kip \cdot ft + 23.6kip \cdot ft}{26ft} = 3.323 \cdot kip$$

and

$$5.03 - 3.32z = 0$$

from which

$$z := \frac{5.03kip}{CJF} = 1.514$$

The final results are

	AB	BA	BC	CB	CD	DC
M from 1st balance	-7.2	-14.6	14.6	-22.5	22.5	0
z x M 2nd balance	-52.1	-43	43	35.8	-35.8	0
Final moments	-59.3	-57.6	57.6	13.3	-13.3	0

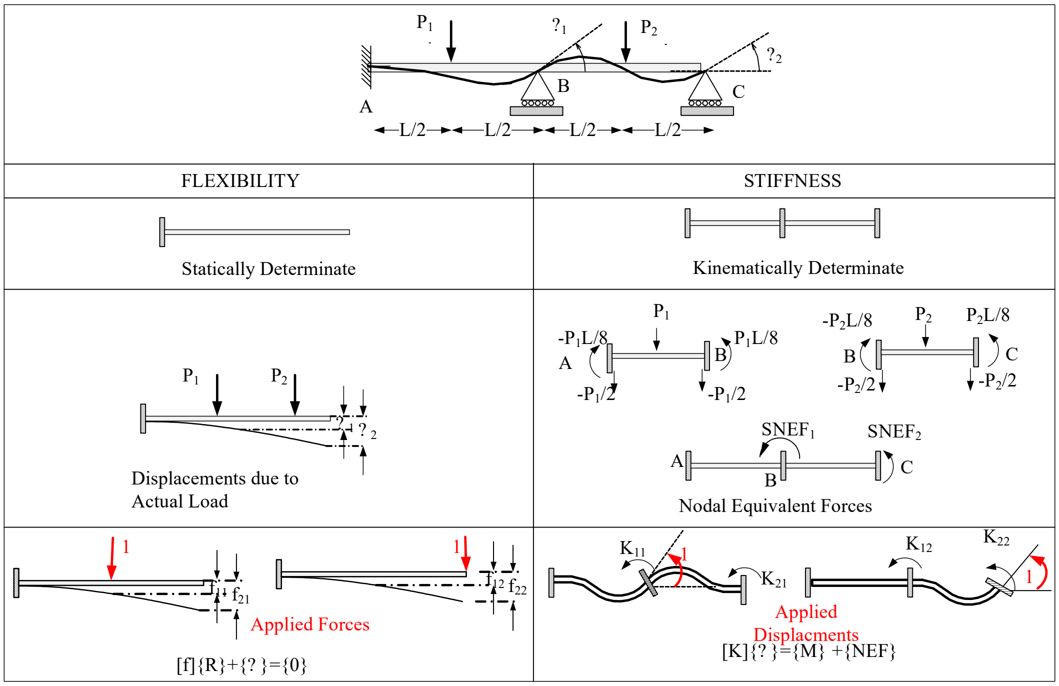
If these final moments are correct, the sum of the column shears will be 5 kip

$$\Sigma V := \frac{59.3 \cdot kip \cdot ft + 57.6kip \cdot ft + 13.3kip \cdot ft}{26ft} = 5.01 \cdot kip$$

The 5 kip horizontal load acting at C enters into the problem only in connection withthe determination of the AJR. If this load had been applied to the column CD between the ends, it would have resulted in intial fixed-end moments in CD and these would be computed in the usual way. In addition, such a load would have enetered into the determination of the AJR, since the horizontal reaction of CD against the right end of BC would have been computed by treating CD as a free body.

17.1 Beam

Considering the figure, let $P_1 = 2P$, $M = PL$, $P_2 = P$, and $P_2 = P$. Solve for the displacements.



Solution:

1. Using the previously defined sign convention (counterclockwise positive)

$$\Sigma NEF_1 = \frac{P_1 \cdot L}{8} - \frac{P_2 \cdot L}{8} = \frac{2P \cdot L}{8} - \frac{P \cdot L}{8} = \frac{P \cdot L}{8}$$

$$\Sigma NEF_2 = \frac{P \cdot L}{8}$$

2. If it takes 4EI/L (k_{44AB}) to rotate AB and 4EI/L (k_{22BC}) to rotate BC, it will take a total force of 8EI/L to simultaneously rotate AB and BC (Note that a rigid joint is assumed)

3. Hence, K_{11} which is the sum of the rotational stiffnesses at global d.o.f. 1, will be equal to $K_{11} = \frac{8EI}{L}$; similarly, $K_{21} = \frac{2EI}{L}$ (k_{42BC})

4. If we rotate dof 2 by a unit angle, then we will have $K_{22} = \frac{4EI}{L}$ and $K_{12} = \frac{2EI}{L}$

5. The equilibrium relation can be written as

$$\begin{pmatrix} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} PL \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{PL}{8} \\ \frac{PL}{8} \end{pmatrix}$$

or

$$\begin{pmatrix} PL + \frac{PL}{8} \\ \frac{PL}{8} \end{pmatrix} = \begin{pmatrix} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

We note that the matrix corresponds to the structure's stiffness matrix, and not the augmented one.

6. The two by two matrix is next inverted

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{pmatrix}^{-1} \cdot \begin{pmatrix} PL + \frac{PL}{8} \\ \frac{PL}{8} \end{pmatrix} = \begin{pmatrix} \frac{17}{112} \cdot \frac{P \cdot L^2}{EI} \\ \frac{-5}{112} \cdot \frac{P \cdot L^2}{EI} \end{pmatrix}$$

7. Next we need to determine both the reactions and the internal forces.

8. Recall that for each element $\mathbf{p} = \mathbf{k} \cdot \delta$, and in this case $\mathbf{p} = \mathbf{P}$ and $\delta = \Delta$ for element AB. The element stiffness matrix has been previously derived, and in the case of the global and local d.o.f. are the same.

9. hence, the equilibrium equation for element AB, at the element level, can be written as

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{17}{112} \cdot \frac{P \cdot L^2}{EI} \end{pmatrix} + \begin{pmatrix} \frac{2P}{2} \\ \frac{2PL}{8} \\ \frac{2P}{2} \\ \frac{-2PL}{8} \end{pmatrix}$$

Solving

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix} = \begin{pmatrix} \frac{107}{56} \cdot P & \frac{31}{56} \cdot PL & \frac{5}{56} \cdot P & \frac{5}{14} \cdot PL \end{pmatrix}$$

10. Similarly, for element BC:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \frac{17}{112} \cdot \frac{P \cdot L^2}{EI} \\ 0 \\ \frac{-5}{112} \cdot \frac{P \cdot L^2}{EI} \end{pmatrix} + \begin{pmatrix} \frac{P}{2} \\ \frac{PL}{8} \\ \frac{P}{2} \\ \frac{-PL}{8} \end{pmatrix}$$

or

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix} = \begin{pmatrix} \frac{7}{8} \cdot P & \frac{9}{14} \cdot PL & \frac{-1}{7} \cdot P & 0 \end{pmatrix}$$

11. This simple example calls for the following observations:

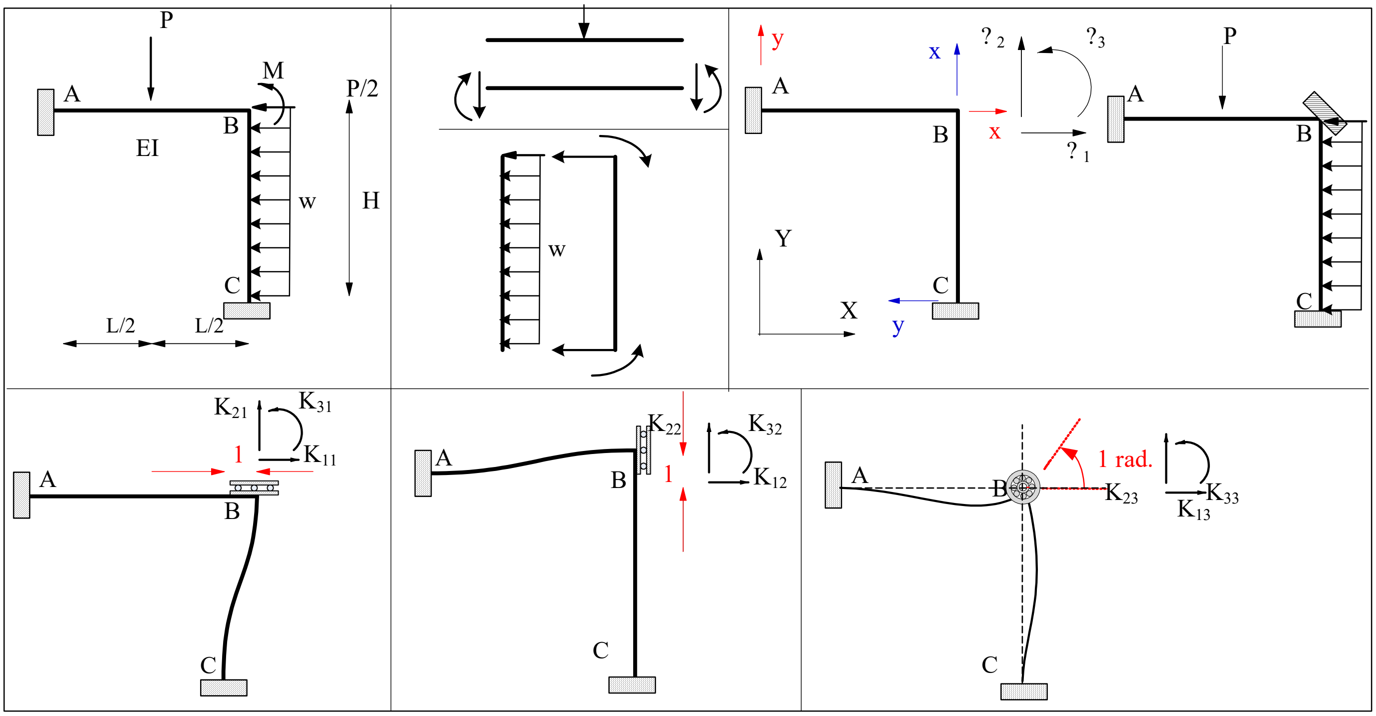
a) Node A has contributions from element AB only, while node B has contributions from both AB and BC

b) We observe that $p_{3AB} \neq p_{1BC}$ even though they both correspond to a shear force at node B, the difference between them is equal to the reaction at B. Similarly, $p_{4AB} \neq p_{2BC}$ due to the externally applied moment at B

c) From this analysis, we can draw the complete free body diagram and then the shear and moment diagrams which is what the engineer is most interested in for design purposes.

17.2 Frame

Whereas in the first example all local coordinate systems were identical to the global one, in this example we consider the orthogonal frame shown below.



Solution:

1. Assuming axial deformations, we do have three global degress of freedom Δ_1, Δ_2, and θ_3.
2. Constrain all degrees of freedom, and thus make the structure kineamatically determinate.
3. Determine the nodal equivalent forces for each element in its own local coordinate system (the first three values are associated with the first node, and the last three with the second node):

$$\begin{pmatrix} p_1 & v_1 & m_1 & p_2 & v_2 & m_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{-P}{2} & \frac{-PL}{8} & 0 & \frac{-P}{2} & \frac{PL}{8} \end{pmatrix} \quad \text{Member AB}$$

$$\begin{pmatrix} p_1 & v_1 & m_1 & p_2 & v_2 & m_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{-wH}{2} & \frac{-w\cdot H^2}{12} & 0 & \frac{-wH}{2} & \frac{w\cdot H^2}{12} \end{pmatrix} \quad \text{Member BC}$$

4. Summing the nodal equivalent forces at node B in global coordinates we have

$$\begin{pmatrix} P_1 & P_2 & P_3 \end{pmatrix} = \begin{pmatrix} \frac{-wH}{2} & \frac{-P}{2} & \frac{PL}{8} - \frac{w\cdot H^2}{12} \end{pmatrix}$$

5. Next, we apply a unit displacement in each of the three global degrees of freedom, and we seek to determine the structure global stiffness matrix. Each entry K_{ij} of the global stiffness matrix will correspond to the internal force in degreee of freedome i, due to a unit displacement in degree of freedom j.
6. Recalling the force displacement relations derived earlier, we can assemble the global stiffness matrix in terms of contributions from both AB and BC.

		K_{i1} Δ_1	K_{i2} Δ_2	K_{i3} Δ_3
K_{1j}	AB	EA/L	0	0
	BC	$12EI/H^3$	0	$6EI/H^2$
K_{2j}	AB	0	$12EI/L^3$	$-6EI/L^2$
	BC	0	EA/H	0
K_{3j}	AB	0	$-6EI/L^2$	4EI/L
	BC	$6EI/H^2$	0	4EI/H

7. Summing up, the structure global stiffness matrix [**K**] is

$$K = \begin{pmatrix} \frac{EA}{L}+\frac{12EI}{H^3} & 0 & \frac{6EI}{H^2} \\ 0 & \frac{12EI}{L^3}+\frac{EA}{H} & \frac{-6EI}{L^2} \\ \frac{6EI}{H^2} & \frac{-6EI}{L^2} & \frac{4EI}{L}+\frac{4EI}{H} \end{pmatrix}$$

8. The global equation of equilibrium can now be written

$$\begin{pmatrix} \frac{EA}{L}+\frac{12EI}{H^3} & 0 & \frac{6EI}{H^2} \\ 0 & \frac{12EI}{L^3}+\frac{EA}{H} & \frac{-6EI}{L^2} \\ \frac{6EI}{H^2} & \frac{-6EI}{L^2} & \frac{4EI}{L}+\frac{4EI}{H} \end{pmatrix} \cdot \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} \frac{-P}{2} \\ 0 \\ M \end{pmatrix} - \begin{pmatrix} \frac{wH}{2} \\ \frac{P}{2} \\ \frac{-PL}{8}+\frac{w\cdot H^2}{12} \end{pmatrix}$$

9. Solve for the displacements

$$\begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} \frac{EA}{L}+\frac{12EI}{H^3} & 0 & \frac{6EI}{H^2} \\ 0 & \frac{12EI}{L^3}+\frac{EA}{H} & \frac{-6EI}{L^2} \\ \frac{6EI}{H^2} & \frac{-6EI}{L^2} & \frac{4EI}{L}+\frac{4EI}{H} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{-P}{2}-\frac{wH}{2} \\ \frac{-P}{2} \\ M+\frac{PL}{8}-\frac{w\cdot H^2}{12} \end{pmatrix}$$

10. To obtain the element internal forces, we will multiply each element stiffness matrix by the local displacements. For element AB, the local and global coordinates match, thus

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix} = \begin{pmatrix} \frac{EA}{L} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ \frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Delta_1 \\ \Delta_2 \\ \theta_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{P}{2} \\ \frac{PL}{8} \\ 0 \\ \frac{P}{2} \\ \frac{-PL}{8} \end{pmatrix}$$

11. For element BC, the local and global coordinates do not match, hence we will need to transform the dispalcements from their global to their local components. But since vector (displacement and load) and matrix transformation have not yet been covered, we not by inspection that the relationship between global and local coordinates for element BC is

Local	δ_1	δ_2	θ_3	δ_4	δ_5	θ_6
Global	0	0	0	Δ_2	$-\Delta_1$	θ_3

and we observe that there are no local or global displacements associated with dof 1-3; Hence, the internal forces for element BC are given by:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix} = \begin{pmatrix} \frac{EA}{L} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ \frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Delta_2 \\ -\Delta_1 \\ \theta_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{wH}{2} \\ -\frac{w \cdot H^2}{12} \\ 0 \\ -\frac{wH}{2} \\ \frac{w \cdot H^2}{12} \end{pmatrix}$$

Note that the element is defined as going from C to B hence x,y,z correspond to Y, -X, Z.