

## Von Mises (revisited)

*Note to Sam Fares:* What you asked would have been simpler two years ago; I was still working and had ready access to Mathcad 15. Fortunately for me but unfortunately for you I've retired, and all I have left is Prime Express and a very nice Mathcad Viewer from Luc that allows me to see the older sheets. What you posted had things overlapping some regions, so the program couldn't be seen completely. Rather than abandon you I've undertaken a Von Mises tutorial; ALL of this information you could harvest from the internet.

Reference: <https://www.simscale.com/docs/simwiki/fea-finite-element-analysis/what-is-von-mises-stress/#:~:text=Von%20Mises%20stress%20is%20a,ductile%20materials%2C%20such%20as%20metals.>

The von Mises stress is a criterion for yielding, widely used for metals and other ductile materials. It states that yielding will occur in a body if the components of stress acting on it are greater than the criterion:

$$\frac{(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 + 6 \cdot (\tau_{12}^2 + \tau_{23}^2 + \tau_{13}^2)}{6} = k^2$$

*Cauchy stress tensor*

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

The constant  $k$  is defined through experiment and  $\tau$  is the stress tensor. Common experiments for defining  $k$  are made from uniaxial stress, where the above expression reduces to:  $\frac{\tau_y^2}{3} = k^2$ .

If  $\tau_y$  reaches the simple tension elastic limit,  $S_y$ , then the above expression becomes:  $\frac{S_y^2}{3} = k^2$

Substituting:

$$\sqrt{\frac{(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 + 6 \cdot (\tau_{12}^2 + \tau_{23}^2 + \tau_{13}^2)}{2}} = S_y$$

The von Mises stress,  $\tau_y$ , is defined as:  $\tau_y^2 = 3 \cdot k^2$ . Therefore, the von Mises yield criterion is also commonly rewritten as:  $\tau_y \geq S_y$

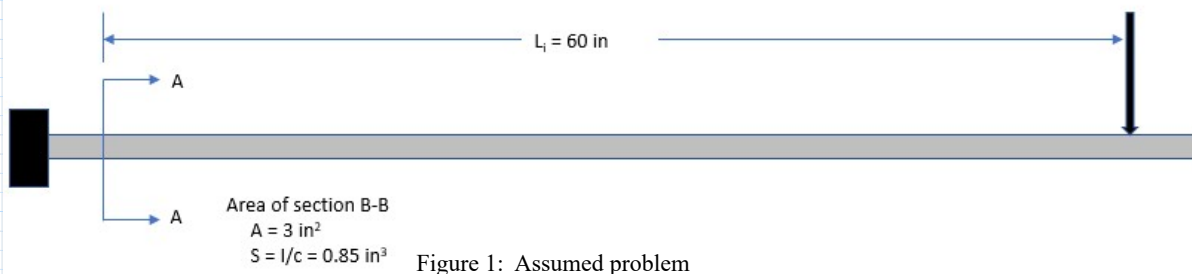
### Application to the problem in the sheet:

The problem was never defined, but we can make a reasonable guess:

$$A := 6 \cdot \text{in}^2$$

$$S := 2 \cdot \text{in}^3$$

$$L_1 := 60.0 \text{ in}$$



We'll assume a cantilever beam with a concentrated load a fixed distance from the vertical plane of interest.

taking the 11 direction as along the beam, and looking at an element on the top surface, the stress tensor becomes:

$$\tau(p) := \begin{bmatrix} \frac{p \cdot L_1}{S} & \frac{-p}{A} & 0 \\ \frac{p}{A} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \tau(1 \text{ kip}) = \begin{bmatrix} 30 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ ksi} \quad \tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

The Von Mises stress formula. To make the subscript numbers consistent: **ORIGIN**  $\equiv$  1

$$\sigma_v(\tau) := \sqrt{\frac{(\tau_{1,1} - \tau_{2,2})^2 + (\tau_{2,2} - \tau_{3,3})^2 + (\tau_{3,3} - \tau_{1,1})^2 + 6 \cdot (\tau_{1,2}^2 + \tau_{2,3}^2 + \tau_{1,3}^2)}{2}}$$

$\sigma_v(\tau(1 \text{ kip})) = 30 \text{ ksi}$  If the bar has a yield strength less than 70 ksi, it is yielding.

Okay, what happens if we rotate the tensor plane from vertical?

$$R_z(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider now a second coordinate system, with base vectors  $\mathbf{e}'_1, \mathbf{e}'_2$ , obtained from the first by a rotation  $\theta$ . The components of the transformation matrix are

$$Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}'_1 & \mathbf{e}_1 \cdot \mathbf{e}'_2 \\ \mathbf{e}_2 \cdot \mathbf{e}'_1 & \mathbf{e}_2 \cdot \mathbf{e}'_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90+\theta) \\ \cos(90-\theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$T_{ij} = Q_{ip} Q_{jq} T'_{pq}, \quad T'_{ij} = Q_{pi} Q_{qj} T_{pq} \quad \text{Tensor Transformation Formulae (1.13.5)}$$

the components of **S** in the second coordinate system are  $[\mathbf{S}'] = [\mathbf{Q}^T] [\mathbf{S}] [\mathbf{Q}]$ , so

or, in matrix form,

$$[\mathbf{T}] = [\mathbf{Q}] [\mathbf{T}'] [\mathbf{Q}^T] \quad [\mathbf{T}'] = [\mathbf{Q}^T] [\mathbf{T}] [\mathbf{Q}] \quad (1.13.6) \quad \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

So the rotated stress tensor becomes:  $T(p, \theta) := R_z(\theta)^T \cdot \tau(p) \cdot R_z(\theta)$

$$\tau(1 \text{ kip}) = \begin{bmatrix} 30 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ ksi} \quad T(1 \text{ kip}, 0 \text{ deg}) = \begin{bmatrix} 30 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ ksi}$$

$$T(1 \text{ kip}, 90 \text{ deg}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ ksi} \quad T(1 \text{ kip}, 45 \text{ deg}) = \begin{bmatrix} 15 & -15 & 0 \\ -15 & 15 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ ksi}$$

$$f := 0, 0.01 \dots \frac{\pi}{2} \quad \sigma_v \left( T \left( 1 \text{ kip}, \begin{bmatrix} 0 \\ 90 \\ 45 \end{bmatrix} \text{ deg} \right) \right) = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix} \text{ ksi}$$

Looks HORRIBLE but look at the vertical scale!

Figure 2: Von Mises stress VS reference plane angle



Just for fun, let's rotate the load vector. That's going to change a bunch of stuff

$$P(p, \theta) := p \cdot \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix} \quad \theta \text{ is the angle of the vector from vertical (measured clockwise)}$$

$$R := \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \quad P(1 \text{ kip}, 45 \text{ deg}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ kip}$$

$$M(p, \theta) := (R \times P(p, \theta))_3 \quad M(1 \text{ kip}, 0 \text{ deg}) = 5000 \text{ ft} \cdot \text{lb} \quad 1 \text{ kip} \cdot L_1 = 5000 \text{ ft} \cdot \text{lb}$$



$$M(1 \text{ kip}, f) \text{ (ft} \cdot \text{lb)}$$

$$TT(t, \theta) := R_z(\theta)^T \cdot t \cdot R_z(\theta)$$

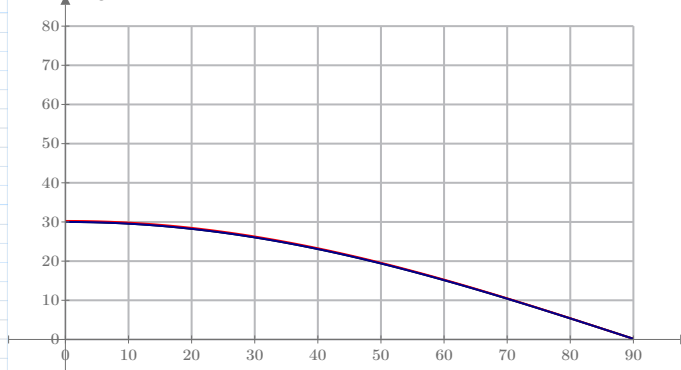
$$\tau_F(p, \theta) := \begin{bmatrix} \frac{M(p, \theta)}{S} + \frac{P(p, \theta)_1}{A} & -\frac{P(p, \theta)_2}{A} & 0 \\ \frac{P(p, \theta)_2}{A} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tau_F(1 \text{ kip}, 90 \text{ deg}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ ksi}$$

$G := \text{Grid}$

We see that the reference frame angle (0, 45, 90 degrees) doesn't change the Von Mises stress but the angle of the applied load does! Q E D

Figure 3: Von Mises Stress variation with force rotation



$$G^{(2)}$$

$$\sigma_v(TT(\tau_F(1 \text{ kip}, f), 0 \text{ deg})) \text{ (ksi)}$$

$$\sigma_v(TT(\tau_F(1 \text{ kip}, f), 45 \text{ deg})) \text{ (ksi)}$$

$$\sigma_v(TT(\tau_F(1 \text{ kip}, f), 90 \text{ deg})) \text{ (ksi)}$$

$$G^{(1)}$$

$$f \text{ (deg)}$$

Okay, how can we solve for the maximum load?

I put a limit on von Mises stress to be less or equal 50 ksi, but the resulting von Mises stress is much larger. i am not sure why.

(Like I said, that program was flawed!

We've seen that the Von Mises stress is (essentially) unchanged regardless of the reference frame angle (Figure 2.) So what would the allowable load be for a given yield strength material?

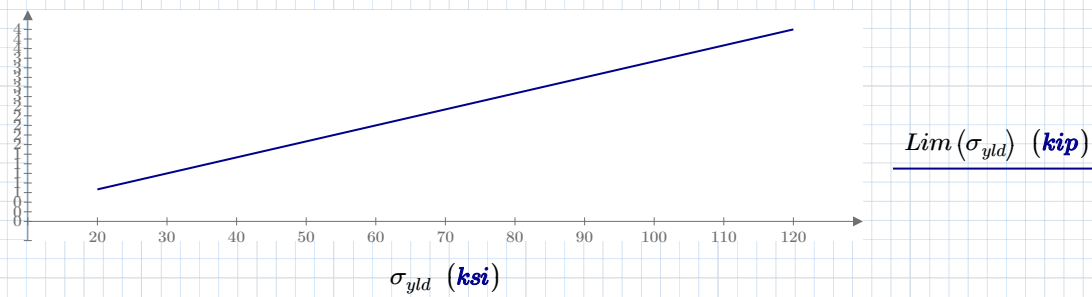
We have an equation for the Von Mises stress as a function of stress tensor, and an equation for the stress tensor as a function of applied load.

$$Fn(p, yield) := \sigma_v(\tau(p)) - yield \quad p_g := 0.5 \text{ kip}$$

$$Lim(yield) := \text{root}(Fn(p_g, yield), p_g) \quad \text{Find where Von Mises equals the yield stress.}$$

$$Lim(70 \text{ ksi}) = 2 \text{ kip} \quad \sigma_v(\tau(0.992 \text{ kip})) = 30 \text{ ksi}$$

$$\sigma_{yld} := 20 \text{ ksi}, 21 \text{ ksi}.. 120 \text{ ksi}$$



Sam's new problem:

For each load value for P, i'd like to compute von Mises stress when the load is applied at different angles provided the von Mises stress doesn't exceed the limit (50 ksi). it should report each load, the angles, and von Mises. if the stress goes over the limit then not to report the angles or the stress because it exceeds the limit.

$$\tau_F(p, \theta) := \begin{bmatrix} \frac{M(p, \theta)}{S} + \frac{P(p, \theta)_1}{A} & -\frac{P(p, \theta)_2}{A} & 0 \\ \frac{P(p, \theta)_2}{A} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$i := 1..5 \quad P_i := i \cdot \text{kip}$$

$$j := 1..10 \quad \phi_j := (j-1) \cdot 10 \text{ deg}$$

$$\sigma_{mises}(p, \theta) := \sigma_v(TT(\tau_F(p, \theta), 0 \text{ deg}))$$

$$\begin{cases} P_{y_i} \leftarrow P_i \cdot \sin(\phi_j) \\ P_{x_i} \leftarrow P_i \cdot \cos(\phi_j) \\ M_i \leftarrow P_{y_i} \cdot L_1 \\ \tau_i \leftarrow \frac{P_{y_i}}{A} \\ \sigma_{normal_i} \leftarrow \frac{P_{x_i}}{A} + \frac{M_i}{S} \end{cases}$$

$$\sigma_{VM_{i,j}} := \sigma_{mises}(P_i, \phi_j)$$

$$\sigma_{VM} = \begin{bmatrix} 30 & 30 & 28 & 26 & 23 & 19 & 15 & 10 & 5 & 0 \\ 60 & 59 & 56 & 52 & 46 & 39 & 30 & 21 & 11 & 0 \\ 90 & 89 & 85 & 78 & 69 & 58 & 45 & 31 & 16 & 1 \\ 120 & 118 & 113 & 104 & 92 & 78 & 61 & 42 & 21 & 1 \\ 150 & 148 & 141 & 130 & 115 & 97 & 76 & 52 & 27 & 1 \end{bmatrix} \text{ ksi}$$

$$Title := \text{augment}\left(\left[ \text{"Load (kip)"} \right], \frac{\phi^T}{deg}\right) \quad Body := \text{augment}\left(\frac{P}{kip}, \frac{\sigma_{VM}}{ksi}\right)$$

$$Table := \text{stack}(Title, Body)$$

Table =	["Load (kip)"]	0	10	20	30	40	50	60	70	80	90
	1	30	30	28	26	23	19	15	10	5	0
	2	60	59	56	52	46	39	30	21	11	0
	3	90	89	85	78	69	58	45	31	16	1
	4	120	118	113	104	92	78	61	42	21	1
	5	150	148	141	130	115	97	76	52	27	1

$$\sigma_{mises}(1 \text{ kip}, 0 \text{ deg}) = 30 \text{ ksi}$$

$$\sigma_{mises}(2 \text{ kip}, 34 \text{ deg}) = 50 \text{ ksi}$$

$$\sigma_{mises}(3 \text{ kip}, 56.5 \text{ deg}) = 50 \text{ ksi}$$

$$\sigma_{mises}(4 \text{ kip}, 65.5 \text{ deg}) = 50 \text{ ksi}$$

$$\sigma_{mises}(5 \text{ kip}, 71 \text{ deg}) = 50 \text{ ksi}$$

$$A = 6 \text{ in}^2 \quad S = 2 \text{ in}^3 \quad L_1 = 60 \text{ in}$$