Table of Contents

Include << D:\edt3Engineer\PDM Vault\edt3\EDT3\Functions.mcdx

Kingston, WA 98346
(360) 981-2720
david@edt3.com
www.edt3.com

Fibonacci Numbers:

A series of numbers in which each number (Fibonacci number) is the sum of the two preceding numbers.

$F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$ Resulting the series 1, 1, 2, 3, 5, 8, 13, 21, 34...

John

$$Fibon(n) := \begin{array}{l} F_1 \leftarrow 1 \\ \text{if } n = 1 \\ \quad \text{return submatrix}(F,1,1,0,0) \\ F_2 \leftarrow 1 \\ \text{if } n = 2 \\ \quad \text{return submatrix}(F,1,2,0,0) \\ \text{for } i \in 3..n \\ \quad F_i \leftarrow F_{i-1} + F_{i-2} \\ row_F \leftarrow rows(F) - 1 \\ \text{submatrix}(F,1,row_F,0,0) \end{array}$$

$$Fibon(100) = \begin{bmatrix} 1.000 \\ 1.000 \\ 2.000 \\ 3.000 \\ 5.000 \\ 8.000 \\ 13.000 \\ 21.000 \\ \vdots \end{bmatrix}$$

$F_2 := 1$

$$Fibon(2) = \begin{bmatrix} 1.000 \\ 1.000 \end{bmatrix}$$

$$Fibon(100)_{99} = 354224848179262000000$$

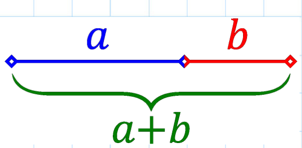
$$\frac{Fibon(100)_5}{Fibon(100)_4} = 1.600 \qquad \frac{Fibon(100)_{99}}{Fibon(100)_{98}} = 1.618$$

$$i := 0,1..12 \qquad Fibon_{ratio}(i) := \frac{Fibon(100)_{i+1}}{Fibon(100)_i} \qquad V_i := Fibon_{ratio}(i)$$

$$V^T = [1.00000 \;\; 2.00000 \;\; 1.50000 \;\; 1.66667 \;\; 1.60000 \;\; 1.62500 \;\; 1.61538 \;\; 1.61905 \;\; 1.61765 \;\; 1.61818 \;\; 1.61798 \;\; 1.61806 \;\; 1.61803]$$

Golden Ratio (Mean)

The golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. The figure on the right illustrates the geometric relationship.



By Traced by Stannered - File:Golden ratio line.png, Public Domain, https://commons.wikimedia.org/w/index.php?curid=1830029

For $a>b$ The Golden Rule φ is defined as $\frac{a+b}{a}=\frac{a}{b}$

Determine the value of the Golden Rule

$\frac{a+b}{a}=\frac{a}{b} \Rightarrow \frac{a+b}{a}=\frac{a}{a}+\frac{b}{a} \Rightarrow 1+\frac{b}{a}=\frac{a}{b}$ **clear** (φ) Let $\varphi=\frac{a}{b}$ φ represents the Golden Rule

$$1+\frac{1}{\varphi}=\varphi \Rightarrow \varphi^2-\varphi-1=0 \xrightarrow{solve, assume, \varphi>0} \frac{\sqrt{5}+1}{2} \xrightarrow{float, 6} 1.61803$$

$$F_n=F_{n-1}+F_{n-2} \quad F_{n+1}=F_n+F_{n-1} \Rightarrow \frac{F_{n+1}}{F_n}=\frac{F_n+F_{n-1}}{F_{n-1}+F_{n-2}}=\frac{F_n+F_{n-1}}{F_n}=\varphi$$

Continued fraction can be used to solve for φ.

$$\varphi=1+\frac{1}{\varphi} \quad \varphi=1+\frac{1}{1+\frac{1}{\varphi}} \quad \varphi=1+\frac{1}{1+\frac{1}{1+\frac{1}{\varphi}}} \quad \text{etc.}$$

Therefore $\frac{F_{n+1}}{F_n}=1+\frac{1}{1+\frac{1}{1+\frac{1}{\varphi}}}$ etc.

$$\varphi:=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}}}}}}}}}}}} \rightarrow \frac{987}{610} \xrightarrow{float, 20} 1.6180327868852459016$$

Taking the ratios if the successive Fibonacci numbers $i:=0,1..12$ and $Fibon_{ratio}(i):=\frac{Fibon(14)_{i+1}}{Fibon(14)_i}$

($V_i:=Fibon_{ratio}(i)$) leads to the following series:

$V^T = [1.00000 \;\; 2.00000 \;\; 1.50000 \;\; 1.66667 \;\; 1.60000 \;\; 1.62500 \;\; 1.61538 \;\; 1.61905 \;\; 1.61765 \;\; 1.61818 \;\; 1.61798 \;\; 1.61806 \;\; 1.61803]$ This converses to the golden ratio.

Closed Form - Fibonacci Numbers

Binet's Fibonacci number formula $F_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n \cdot \sqrt{5}}$

or $F_n = \dfrac{\varphi^n - \psi^n}{\varphi - \psi}$ and

$\varphi := \dfrac{1+\sqrt{5}}{2} \qquad \psi := \dfrac{1-\sqrt{5}}{2} \xrightarrow{float,5} -0.61803 \quad \Rightarrow \quad \psi := 1 - \varphi = -0.61803$

$\psi = \dfrac{-1}{\varphi}$ and $\varphi - \psi = \sqrt{5}$

Substitute for ψ

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \xrightarrow{substitute, \psi = \frac{-1}{\varphi}, explicit} F_n = \frac{\sqrt{5} \cdot \left(\varphi^n - \left(-\frac{1}{\varphi}\right)^n\right)}{5}$$

$$F_{closed}(n) := \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \qquad F_{closed}(-6) = -8.000$$

Closed form for Fibonacci Numbers. Reference: Wikipedia - Fibonacci Numbers

Kingston, WA 98346
(360) 981-2720
david@edt3.com
www.edt3.com

John Conway Series 1, 1, 2, 2, 3, 4, 4, 4, 5 ...

Reference: Schroeder, Manfred. Fractals, Chaos, Power Laws. page 59

$$F(n) = F(F(n-1)) + F(n - F(n-1))$$

$F_0 := 1$ $F_1 := F_0 = 1$

```
Conway(n) := | J_1 ← 1
             | if n = 1
             |   return submatrix(J, 1, 1, 0, 0)
             | J_2 ← 1
             | if n = 2
             |   return submatrix(J, 1, 2, 0, 0)
             | for i ∈ 3..n
             |   J_i ← J_(J_(i-1)) + J_(i - J_(i-1))
             | row_J ← rows(J) - 1
             | submatrix(J, 1, row_J, 0, 0)
```

$$Conway(100) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 4 \\ 4 \\ 4 \\ 5 \\ \vdots \end{bmatrix}$$

```
J_n(n) := | J_1 ← 1
          | if n = 1
          |   return J_n
          | J_2 ← 1
          | if n = 2
          |   return J_n
          | for i ∈ 3..n
          |   J_i ← J_(J_(i-1)) + J_(i - J_(i-1))
          | J_n
```

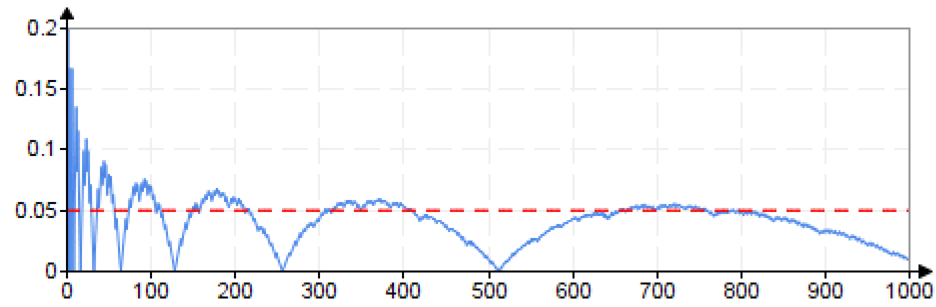
$$J_n(10000) = 5373$$

Kingston, WA 98346
(360) 981-2720
david@edt3.com
www.edt3.com

Conway challenge

Find n_0 such that for all $n > n_0$ $\left| \frac{J_n}{n} - \frac{1}{2} \right| \le 0.05$

$n_0 := 1000$ $i := 1, 2 .. n_0$



Kingston, WA 98346
(360) 981-2720
david@edt3.com
www.edt3.com

Euler's continued fractions (accessed November 9, 2020)

Euler derived the formula as connecting a finite sum of products with a finite continued fraction (accessed November 9, 2020).

$$a_0 + a_0 \cdot a_1 + a_0 \cdot a_1 \cdot a_2 + \blacksquare..\blacksquare \text{ to } a_0 \cdot a_1 \cdot a_2 \cdot \blacksquare..a_n = \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{a_2}{1 + a_2 - \cfrac{a_3}{\blacksquare}}}} \text{etc.}$$

or

$$\sum_{i=\blacksquare}^{\infty} \left(\prod_{j=1}^{i} r_j \right) = \cfrac{1}{1 - \cfrac{r_1}{1 + r_1 - \cfrac{r_2}{1 + r_2 - \cfrac{r_3}{\blacksquare}}}}$$

The continued fraction method can be extended to functions that can be written as a series. Example $e^z = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{z}{j}$

$$e^z \xrightarrow{confrac} \begin{bmatrix} 1 & z \\ 1 & z \\ -2 & z \\ -3 & z \\ 2 & z \\ 5 & 0 \end{bmatrix} \qquad e^z \xrightarrow{confrac, fraction} 1 + \cfrac{z}{1 + \cfrac{z}{-2 + \cfrac{z}{-3 + \cfrac{z}{2 + \cfrac{z}{5}}}}}$$

$$x\pi := \pi \xrightarrow{confrac} \begin{bmatrix} 3 \\ 7 \\ 15 \\ 1 \\ 292 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad Confrac(x\pi) \rightarrow [3 \text{ ";" } 7 \;\; 15 \;\; 1 \;\; 292 \;\; 1 \;\; 1 \;\; 1]$$