Problem 40 in Shklarsky, Chentzov and Yaglom

A hiker is lost in a forest. He knows the shape and the size of the forest, but doesn’t know where he is. Does he have a better choice of escape route in the following cases than the route described below?

a) If the shape of the forest is a circle of diameter *d*, then a straight-line walk of length less than *d* in an arbitrary direction leads him out of the forest (see Fig. 9/a).

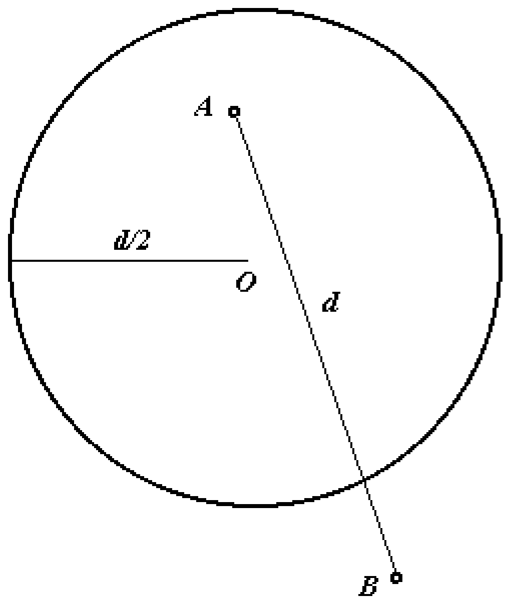

Fig. 9a

b) If the shape of the forest is a half-plane and the hiker knows that he is not farther than *d* from the edge, he will escape if he follows a straight-line path of length *d* in an arbitrary direction, then follows a circular path of radius *d* and centered at his starting point *A*. This route of length *d+2pd (» 7.28d)* will surely lead him out of the forest (see Fig. 9/b).

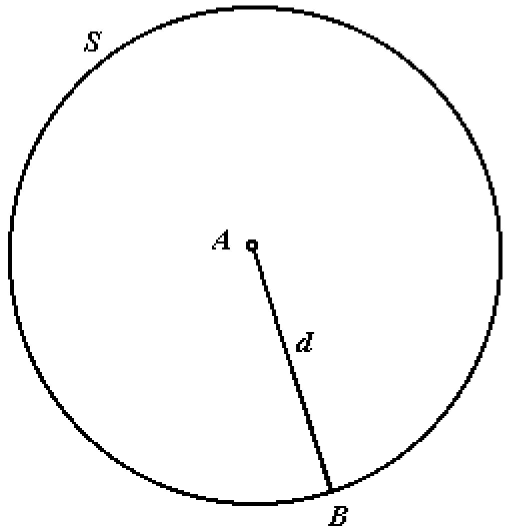

Fig. 9b

Does he have a better choice in these cases, i.e., a route that is shorter than d or $(1+2p)d$ respectively ?

Note: the above two problems originated with the American mathematician R. Bellman (see pp. 161-162 of his book [40]). If the shape of the forest is a circle, it is easy to find the optimal escape route. For other shapes it is not easy at all. V.A. Zalgaller [41] gave a solution for the case in which the shape of the forest is a rectangle and the ratio of its sides is greater than 2.278 (e.g., an infinite strip of finite width). The optimal solution has not yet been found for cases in which the shape is a triangle (e.g., an equilateral triangle), a square, or a rectangle with ratio of sides less than 2.278. Problem 40.b) was analyzed by J. R. Isbell in [42].

Solution of Problem 40

a) There is no better solution: no route shorter than d can provide escape from the forest in every case. Suppose the path AB of the hiker is such that its halfway point coincides with the center O of the forest. Then the distance of an arbitrary point M enroute from O is less than the radius $d/2$ of the forest (Fig. 103). Hence AB lies properly inside the circle, which implies that the hiker cannot escape the forest.

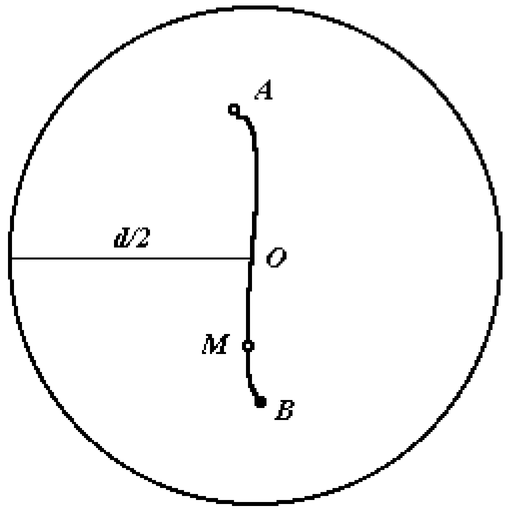

Fig. 103

b) There is a better solution. Suppose the hiker follows a straight-line path of length *d* in an arbitrary direction, then follows a 270° (= *3p/2*) circular arc *S* of radius *d* and centered at *A*, then finally follows a straight-line path *CD* of length *d* tangent to the arc *S* at *C* (Fig. 104/a). It is easy to see that this route intersects every straight line whose distance is *d* from point *A* (i.e., the tangents to circle *S*). Since the edge of the forest is such a line, the path *ABCD* (Fig. 104/a) is surely a valid escape route. The length of this route is

$$AB+BC+CD = d+(3/2)pd+d = (2+(3/2)p)d \text{ » } 6.71d < (1+2p)d \quad (\text{» } 7.28d)$$

Actually, the route *ABCD* can be improved. Suppose the hiker follows a straight-line path of length $AB_1 = (2Ö3/3)d$ in an arbitrary direction. Then he follows a straight-line path B_1C_1 tangent to the circle *S* of radius *d* and centered at *A*. This implies that angle C_1AB_1 is 30°. Then he follows a 210° (= *7p/6*) circular arc *S* of radius *d* and centered at *A*, then finally walks a straight-line path D_1E of length *d* tangent to the arc *S* at *D* (Fig. 104/b). It is clear that the route $AB_1C_1D_1E$ also intersects every tangent of the circle and is hence a valid escape route. The length of this route is

$$AB_1+B_1C_1+C_1D_1+D_1E$$
$$= (2Ö3/3)d+(Ö3/3)d+(7/6)pd+d = (1+Ö3+(7/6)p)d \text{ » } 6.40d$$

which is less than the length of the route on Fig.104/a (that was *(2+(3/2)pd)*).

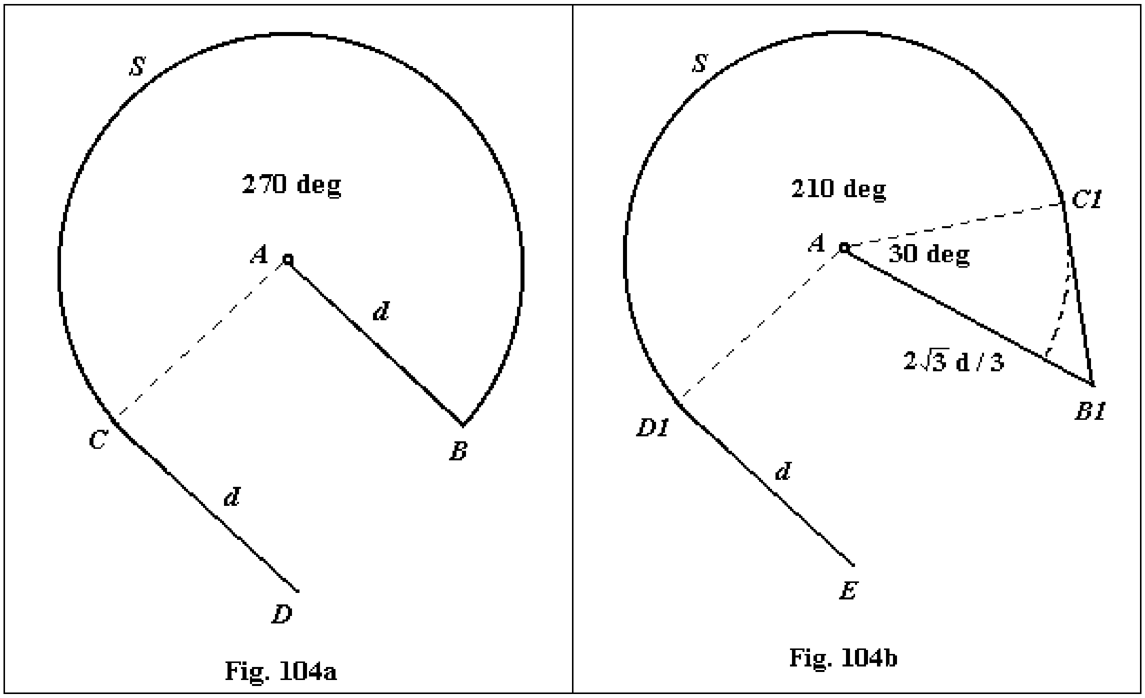

Fig. 104a Fig. 104b

J. R. Isbell [42] has shown that the route $AB_1C_1D_1E$ of Fig.104/b is the shortest one that meets the conditions of problem 40b) and leads out of the forest under any circumstances.

References

40. R. Bellman, *Dynamic Programming*, Princeton University Press, 1957; Minimization problem, *Bulletin of the American Mathematical Society* 62 (1956), p. 270.

41. V. A. Zalgaller, Kak vijtyi iz lesza?, *Matyematyicseszkoje proszvescsenyije* (uj sorozat) 6 (1961) 191-195.

42. J. R. Isbell, An optimal search pattern, *Naval Research Logistics Quarterly* 4 (1957) 357-359.