Bellman's Problem

The problem of R. Bellman, an American mathematician, is presented in a book by D. O. Shklarsky, N. N. Chentzov and I. M. Yaglom, *Válogatott feladatok és tétlek az elemi matematika köréböl* 2/2: *Geometriai egyenlötlenségek és szélsöértékek* (*Selected problems and statements from elementary mathematics 2/2: Geometrical inequality and limit values*), problem 40.

A hiker is lost in a forest. He knows the shape and the size of the forest, but doesn't know where he is. Design the shortest route that will lead him out of the forest for all starting positions.

This problem will be considered for several cases, depending on the shape of the forest. Solutions are given for the first two cases in the cited problem collection.

1. The shape of the forest is a circle of diameter *d*. The hiker should follow a straight-line path in an arbitrary direction. Walking a distance not more than *d* leads out of the forest. Now suppose there is a shorter route. Position its midpoint (the halfway point of its length) at the center *O* of the circle. The two endpoints must be properly inside the circle since they are at a distance less than *d/2* from *O*. Hence starting from one endpoint of the route, the hiker can not escape the forest, i.e., there is no route shorter than *d* (Fig. 1.)

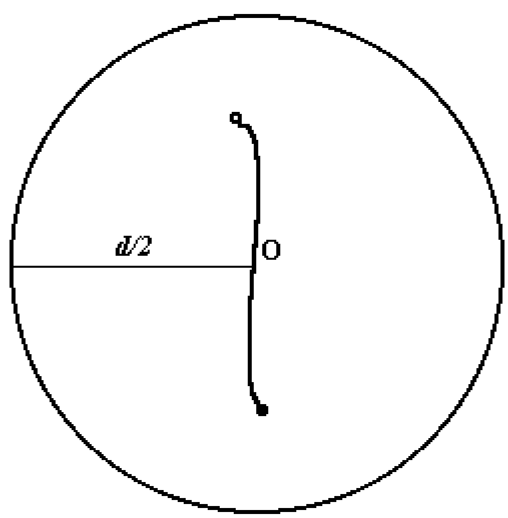

Fig. 1

2. The shape of the forest is a half-plane and the hiker knows that he is not farther than *d* from the edge. The shortest route is shown in Fig. 2. The length of route *OABCD* is about 6.40 *d*, consisting of *OA=2d/Ö3*, *AB= d/Ö3*, *BC=7dp/6* and *CD=d* . A proof of this fact can be found in an article by J. R. Isbell, An optimal search pattern (*Nava1Research Logistics Quarterly* 4 (1957) 357-359).

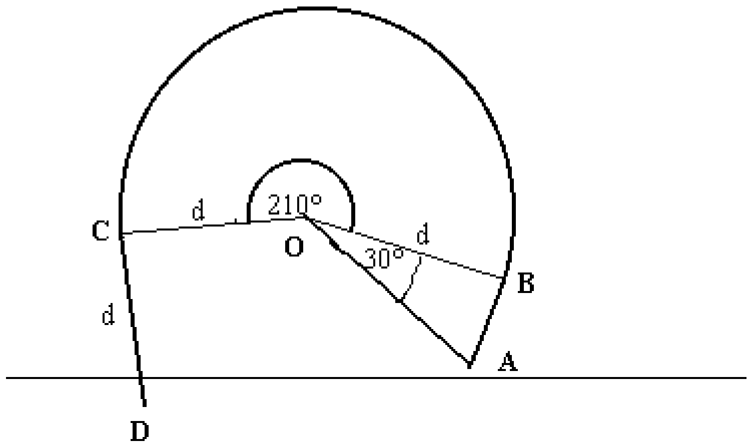

Fig. 2.

3. The shape of the forest is an infinite strip of width *d*, or a rectangle of width *d* and length at least 2.278 *d*. V.A. Zalgaller gave a solution for this case, but I was unable to find it.

4. The shape of the forest is a square of side *d*. The hiker should follow a straight-line path in an arbitrary direction. Walking a distance not more than *dÖ2* leads out of the forest. Suppose that a route *U* always leads out of the square. We will now prove that the length of *U* is at least *dÖ2*.

If *U* can be positioned within the square so that it doesn't touch any of the sides, then the route is inadmissible and cannot lead out of the forest. Denote the vertices of the square by *A, B, C,* and *D*. Place *U* in the quarter-plane bordered by the lines *AB* and *AD* in such a way that

- *U* intersects both *AB* and *AD*, and
- *U* is within the quarter-plane containing *C*.

In other words, *AB* and *AD* are supporting lines of *U*. Then *U* intersects either *BC* or *DC* or both. Otherwise displacing *U* by a sufficiently small distance parallel to the diagonal *AC*, *U* would lie entirely inside the square. For reasons of symmetry we may suppose that *U* intersects *BC*. So *U* has (at least) one point *X* on *AB*, one point *Y* on *BC* and one point *T* on *AD* (Fig. 3.)

Rotate *U* counterclockwise and displace it appropriately so that *AB* and *AD* remain supporting lines. From the continuity of *U*, we infer that this "rotating-displacing" motion is continuous and the following argument holds. Initially *U* intersects *BC*. At each stage of the rotation, *U* intersects at least one of *BC* and *DC*. After a rotation of 90°, *T* lies on *AB* and *Y* lies on *DC*, hence *U* intersects *DC* (Fig. 4.) Consequently, during the rotation, there is an intermediate position of *U* which intersects both *BC* and *DC*. In this position, *U* has a point on each straight line defined by the sides of the square, since *AB* and *AD* are supporting lines.

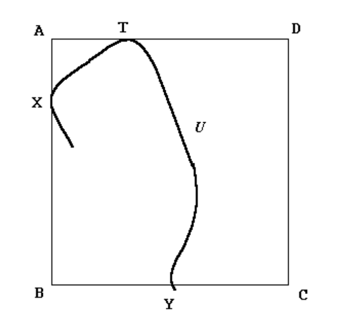

Fig. 3.

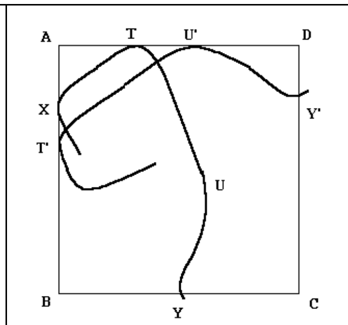

Fig. 4.

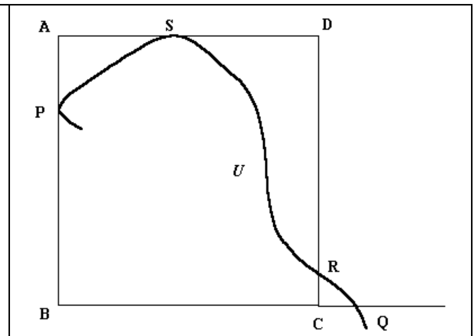

Fig. 5.

Denote the four points of intersection by *P*, *Q*, *R* and *S*, lying on the sides *AB*, *BC*, *DC* and *AD* respectively (a square side can be extended, Fig. 5.) Consider the order in which *U* connects these points. We assume P to be the first. There are basically two cases to distinguish:

1. The second point is on the side opposite to *AB*, so the second point is *R*; we may further assume that the third point is *S* and the last point is *Q* (Fig. 6.) Clearly $PR \geq AD$ and $SQ \geq AB$, so the length of $U \geq PR+RS+SQ \geq AD+AB \geq BD$.

2. The second point is on a side adjacent to *AB*, say it is *Q*. Now two cases remain:

2a. The third point is *R* and the fourth point is *S*. Considering the right-triangles *RDS* and *PBQ*, we deduce $RD \leq RS$ and $BQ \leq PQ$.

2b. The third point is *S* and the fourth point is *R*. Then we deduce $SD \leq RS$ and $BQ \leq PQ$.

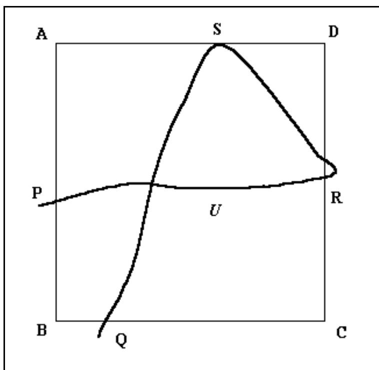

Fig. 6.
case 1

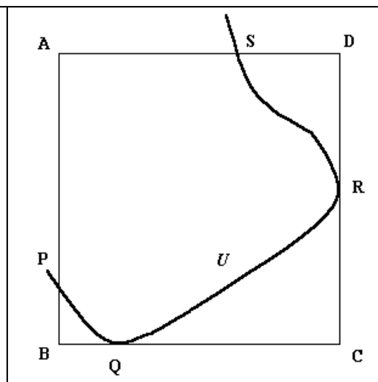

Fig. 6.
case 2a

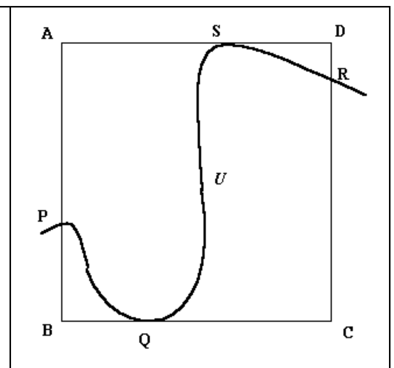

Fig. 6.
case 2b

Thus, in case 2, the length of the route decreases (or at least doesn't increase) if we travel from *B* to *Q* rather than from *P*, and finish the route at *D* rather than at *S* or *R*. But then the length of the route is at least *BD*, so the length of $U \geq BD$.

For each case we have found that the length of *U* is not less than the length of the diagonal, and that's what we wished to prove.

This proof can be generalized from squares to rectangles with some restrictions.

If we assume that *U* has such a position where it intersects simultaneously the opposite sides farthest apart, the above proof can be directly applied, i.e., *U* is not shorter than the diagonal. If this is not true, then at any stage of the rotation, *U* lies between the opposite sides farthest apart, so *U* has to intersect the opposite sides closer together. Denote their distance by *h*, then the solution is the same as for the strip of width *h*. This has the following consequence.

For a rectangle, depending on the ratio of the sides, the solution is either the diagonal or it coincides with the solution for the strip (in an intermediate case, the two may yield an identical result). I unfortunately don't know Zalgaller's solution, so I don't know what the limiting ratio is, but it is definitely less than the value given by him (2.278) and greater than *Ö3* ≈ 1.732. Thus, if the ratio of the sides is less than $\sqrt{3}$, the hiker's best strategy is to follow an arbitrary straight-line path and he will escape after walking at most the length of the diagonal. Such a rectangle can be inscribed in each of the regular even-sided polygons by selecting four appropriate vertices. In the regular hexagon the ratio of the sides is exactly $\sqrt{3}$.

5. The shape of the forest is a regular even-sided polygon. The best escape from such a polygonal forest is also an escape from the inscribed rectangle, so the length of the route is at least the length of the diagonal, i.e., the diameter of the circle circumscribing the polygon. This can be realized, since starting in an arbitrary direction, the hiker escapes after walking a distance not greater than the diameter.

Summary: In the case of a circle, rectangle (ratio of sides $<= \sqrt{3}$) or regular even-sided polygon, the best route is a straight-line path in an arbitrary direction. The distance traversed is at most the length of the diagonal of the circumscribed circle.

The cases for an equilateral triangle and, more generally, for regular odd-sided polygons, remain unsolved. It is interesting that if the forest is an equilateral triangle, the hiker can escape following a route shorter than a side. Several such routes have been found, but the shortest one is still unknown. The problem can also be cast in three dimensions. Which is the shortest space-curve that leads out of a body? For the sphere, the solution is similar to that for the planar circle, but for the cube, the solution can not be transferred.

Gabor Toth
(written when he was a senior high school student in Budapest)