Worksheet "Dirac pulse - formulae.xmcd"

Files' references

Reference:C:\Users\franc_000\Desktop\M15 Worksheets\global data.xmcd

Files' references

DIRAC PULSE FORMULAE

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DIRAC PULSE

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1 Dirac Pulse and its Derivatives - Definition and Approximation

Dirac pulse definition: $\delta(t) = \begin{pmatrix} \infty & \text{if } t = 0.0 \\ 0.0 & \text{otherwise} \end{pmatrix}$

Some text of electrical engineering, use the symbol: $\mathbf{u_0}(t) := \begin{pmatrix} \infty & \text{if } t = 0.0 \\ 0.0 & \text{otherwise} \end{pmatrix}$

$$\delta(1,t) = \frac{d}{dt}\delta(t) \qquad \delta(2,t) = \frac{d}{dt}\delta(1,t)$$

$$\delta(n,t) = \frac{d^n}{dt^n}\delta(t)$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad \int_{-\infty}^{\infty} \mathbf{u_0}(t)\, dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t)\cdot f(t)\, dt = f(0) \qquad \int_{-\infty}^{\infty} \delta(1,t)\cdot f(t)\, dt = f(0)$$

$$\int_{-\infty}^{\infty} \delta(t)\cdot t\cdot e^{-t^2}\cdot \cos(t)^2\, dt \to 0 \qquad \int_{-\infty}^{\infty} \delta_n(1,t)\cdot t\cdot e^{-t^2}\cdot \cos(t)^2\, dt \to -1$$

$$\int_{-\infty}^{\infty} \delta_n(2,t)\cdot t\cdot e^{-t^2}\cdot \cos(t)\, dt \to 0 \qquad \int_{-\infty}^{\infty} \delta_n(3,t)\cdot t\cdot e^{-t^2}\cdot \cos(t)\, dt \to 9$$

$$\int_{-\infty}^{\infty} \delta_n(4,t)\cdot t\cdot e^{-t^2}\cdot \cos(t)\, dt \to 0 \qquad \int_{-\infty}^{\infty} \delta_n(5,t)\cdot t\cdot e^{-t^2}\cdot \cos(t)\, dt \to -125$$

Let's now approximate the Dirac Pulse in a way that it can be drawn, namely define a time interval as small as desired, for example:

approximation. $\delta_\varepsilon(t,\varepsilon_{gd}) := \begin{pmatrix} \frac{1}{\varepsilon_{gd}} & \text{if } \frac{-\varepsilon_{gd}}{2} \le t \le \frac{\varepsilon_{gd}}{2} \\ 0 & \text{otherwise} \end{pmatrix}$

Dirac Pulse property: $\int_{-\infty}^{\infty} \delta_\varepsilon(t,\varepsilon_{gd})\, dt = \int_{-\frac{\varepsilon_{gd}}{2}}^{\frac{\varepsilon_{gd}}{2}} \delta_\varepsilon(t,\varepsilon_{gd})\, dt = 1$

$$\lim_{\varepsilon \to 0} \int_{-\frac{\varepsilon_{gd}}{2}}^{\frac{\varepsilon_{gd}}{2}} \delta_\varepsilon(t,\varepsilon_{gd})\, dt = 1 \qquad \int_{-\infty}^{\infty} \lim_{\varepsilon \to 0} \delta_\varepsilon(t,\varepsilon_{gd})\, dt = 0$$

or, once defined the rectangle function (**Π** pi Greek uppercase):

$$\mathbf{\Pi}(t) := \begin{pmatrix} 1 & \text{if } |t| < \frac{1}{2} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2} \\ 0 & \text{if } |t| > \frac{1}{2} \end{pmatrix} \qquad [1]$$

(Physics) $\text{rect}(t) := \begin{pmatrix} 1 & \text{if } |t| < \frac{1}{2} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2} \\ 0 & \text{if } |t| > \frac{1}{2} \end{pmatrix}$

Below is defined the same function, but with a different name and with tree parameters, in such away that it can be used in other worksheets.

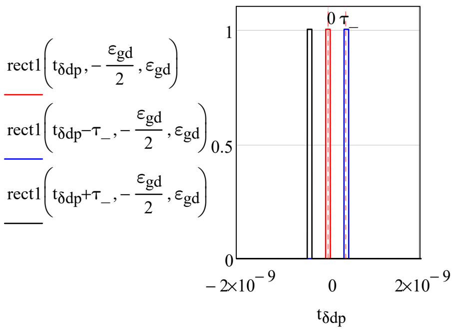
$$\boxed{\text{rect1}(t\_,\text{risingedge},\text{width}) := [\Phi(t\_ - \text{risingedge}) - \Phi[t\_ - (\text{width} + \text{risingedge})]]}$$

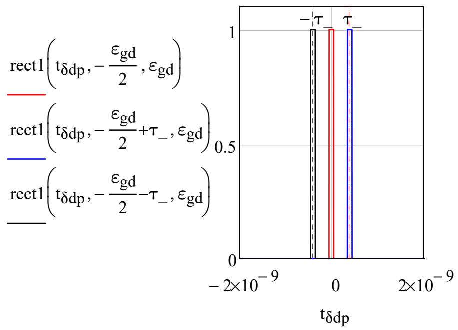
$$\text{rect1}(t\_ - \tau\_,\text{risingedge},\text{width}) = \text{rect1}(t\_,\text{risingedge} + \tau\_,\text{width})$$

$$\text{rect1}(t\_ + \tau\_,\text{risingedge},\text{width}) = \text{rect1}(t\_,\text{risingedge} - \tau\_,\text{width})$$

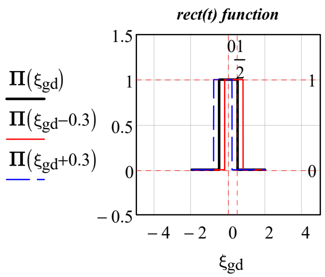
$$\tau\_ := \varepsilon_{gd}\cdot 4 \qquad \tau\_ = 0.4\cdot \text{ns}$$

$$t_{\delta dp} := -10^2 \cdot \varepsilon_{gd}, -10^2 \cdot \varepsilon_{gd} + \frac{2 \cdot \varepsilon_{gd}}{10^2} .. 3 \cdot 10^2 \cdot \varepsilon_{gd}$$



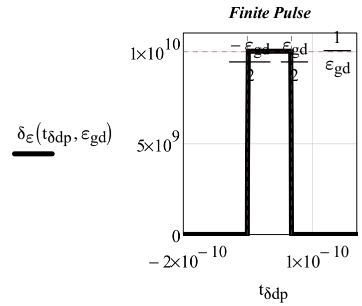


$$\xi_{gd} := -2, -2 + 0.001 .. 2$$

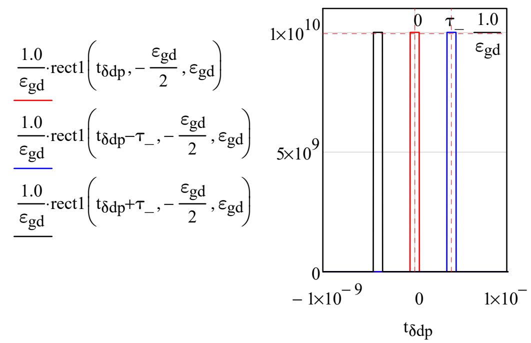


approximate the Dirac Pulse in this way:

$$\delta_\varepsilon(t, \varepsilon_{gd}) := \frac{1}{\varepsilon_{gd}} \cdot \Pi\left(\frac{t}{\varepsilon_{gd}}\right) \qquad \delta_\varepsilon(t, \varepsilon_{gd}) = \frac{1}{\varepsilon_{gd}} \cdot \text{rect1}\left(t_{\delta dp}, -\frac{\varepsilon_{gd}}{2}, \varepsilon_{gd}\right)$$



In the following graph the time shift is subtracted to delay the pulse or summed for a pulse advance to t.

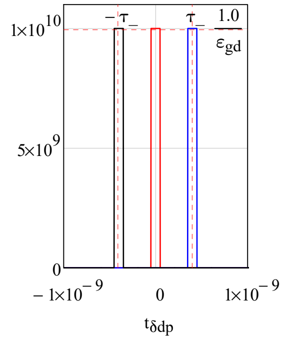


while in the following graph the time shift is subtracted for a pulse advance or summed for a pulse delay, to the second parameter of the function (risingedge).

$$\frac{1.0}{\varepsilon_{gd}}\cdot \mathrm{rect1}\left(t_{\delta dp}, -\frac{\varepsilon_{gd}}{2}, \varepsilon_{gd}\right)$$

$$\frac{1.0}{\varepsilon_{gd}}\cdot \mathrm{rect1}\left(t_{\delta dp}, -\frac{\varepsilon_{gd}}{2}+\tau\_, \varepsilon_{gd}\right)$$

$$\frac{1.0}{\varepsilon_{gd}}\cdot \mathrm{rect1}\left(t_{\delta dp}, -\frac{\varepsilon_{gd}}{2}-\tau\_, \varepsilon_{gd}\right)$$



2 Some derivative approximation

Definition: $\Delta 1_{\varepsilon}(t,\varepsilon_{gd}) = \frac{d}{dt}\delta_{\varepsilon}(t,\varepsilon_{gd})$ $\quad \Delta 2_{\varepsilon}(t,\varepsilon_{gd}) = \frac{d}{dt}\Delta 1_{\varepsilon}(t,\varepsilon_{gd})$ $\quad \Delta 3_{\varepsilon}(t,\varepsilon_{gd}) = \frac{d}{dt}\Delta 2_{\varepsilon}(t,\varepsilon_{gd})$

First derivative: $\Delta 1_{\varepsilon}(t,\varepsilon_{gd}) := \delta_{\varepsilon}\left(\frac{\varepsilon_{gd}}{2}+t,\varepsilon_{gd}\right)-\delta_{\varepsilon}\left(t-\frac{1}{2}\cdot\varepsilon_{gd},\varepsilon_{gd}\right)$

Second derivative: $\Delta 2_{\varepsilon}(t,\varepsilon_{gd}) := \Delta 1_{\varepsilon}\left(\frac{\varepsilon_{gd}}{2}+t,\varepsilon_{gd}\right)-\Delta 1_{\varepsilon}\left(t-\frac{1}{2}\cdot\varepsilon_{gd},\varepsilon_{gd}\right)$

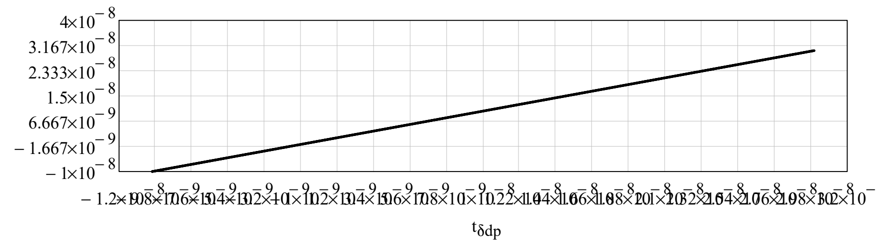
Third derivative: $\Delta 3_{\varepsilon}(t,\varepsilon_{gd}) := \Delta 2_{\varepsilon}\left(\frac{\varepsilon_{gd}}{2}+t,\varepsilon_{gd}\right)-\Delta 2_{\varepsilon}\left(t-\frac{1}{2}\cdot\varepsilon_{gd},\varepsilon_{gd}\right)$

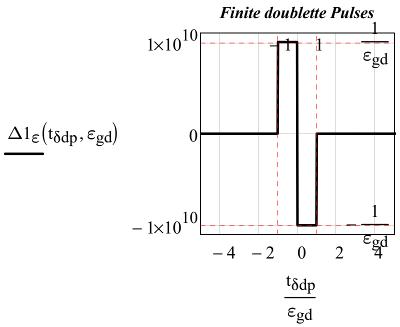
Fourth derivative: $\Delta 4_{\varepsilon}(t,\varepsilon_{gd}) := \Delta 3_{\varepsilon}\left(\frac{\varepsilon_{gd}}{2}+t,\varepsilon_{gd}\right)-\Delta 3_{\varepsilon}\left(t-\frac{1}{2}\cdot\varepsilon_{gd},\varepsilon_{gd}\right)$

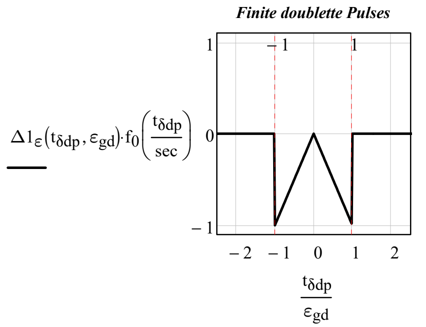
Fifth derivative: $\Delta 5_{\varepsilon}(t,\varepsilon_{gd}) := \Delta 4_{\varepsilon}\left(\frac{\varepsilon_{gd}}{2}+t,\varepsilon_{gd}\right)-\Delta 4_{\varepsilon}\left(t-\frac{1}{2}\cdot\varepsilon_{gd},\varepsilon_{gd}\right)$

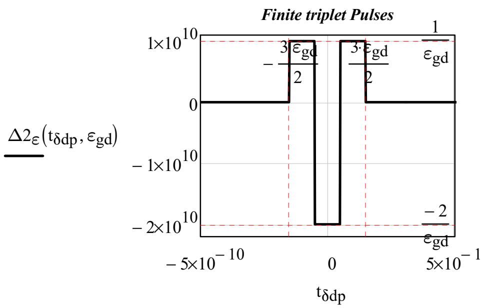
$f_0(t_{\delta dp}) := t_{\delta dp}\cdot e^{-t_{\delta dp}^{2}}\cdot\cos(t_{\delta dp})^2 \qquad \varepsilon_{gd} = 1\times 10^{-10}\,\mathrm{s}$

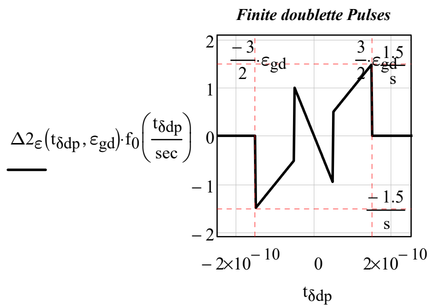
$$f_0\left(\frac{t_{\delta dp}}{s}\right)$$

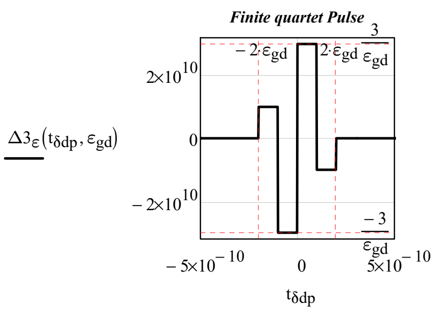


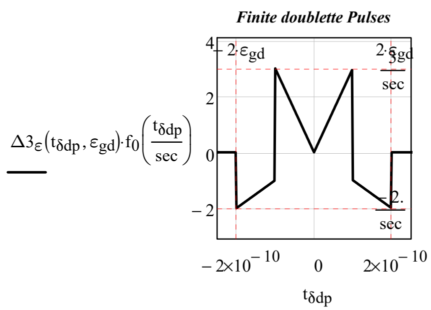

Finite doublette Pulses
$\Delta 1_\varepsilon(t_{\delta dp}, \varepsilon_{gd})$
$\frac{t_{\delta dp}}{\varepsilon_{gd}}$

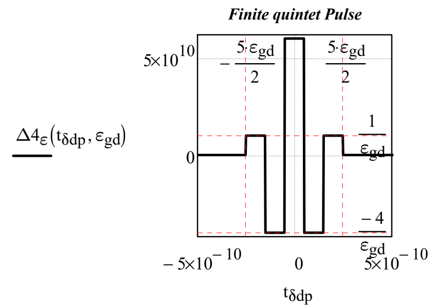

Finite doublette Pulses
$\Delta 1_\varepsilon(t_{\delta dp}, \varepsilon_{gd}) \cdot f0\left(\frac{t_{\delta dp}}{sec}\right)$
$\frac{t_{\delta dp}}{\varepsilon_{gd}}$

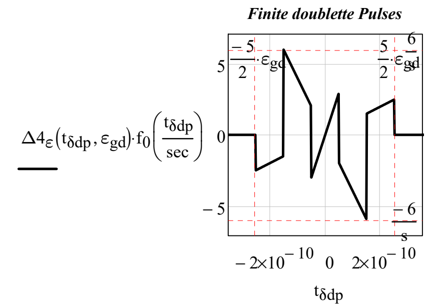

Finite triplet Pulses
$\Delta 2_\varepsilon(t_{\delta dp}, \varepsilon_{gd})$
$t_{\delta dp}$

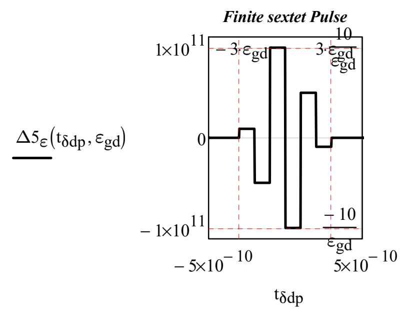

Finite doublette Pulses
$\Delta 2_\varepsilon(t_{\delta dp}, \varepsilon_{gd}) \cdot f0\left(\frac{t_{\delta dp}}{sec}\right)$
$t_{\delta dp}$

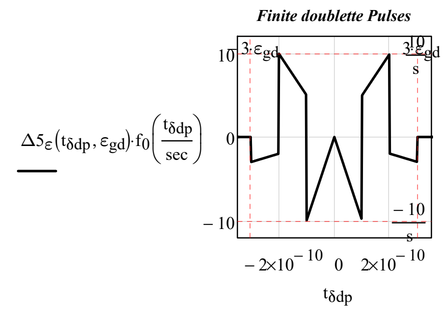

Finite quartet Pulse
$\Delta 3_\varepsilon(t_{\delta dp}, \varepsilon_{gd})$
$t_{\delta dp}$

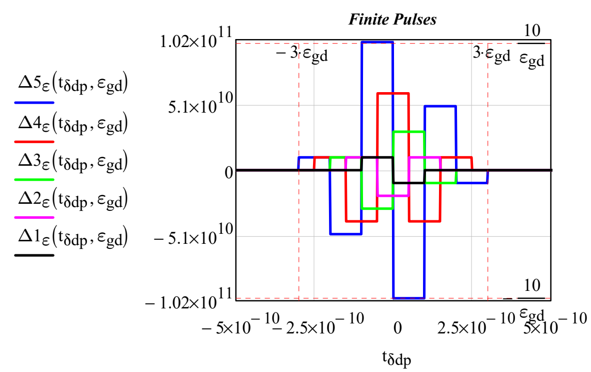

Finite doublette Pulses
$\Delta 3_\varepsilon(t_{\delta dp}, \varepsilon_{gd}) \cdot f0\left(\frac{t_{\delta dp}}{sec}\right)$
$t_{\delta dp}$


Finite quintet Pulse
$\Delta 4_\varepsilon(t_{\delta dp}, \varepsilon_{gd})$
$t_{\delta dp}$


Finite doublette Pulses
$\Delta 4_\varepsilon(t_{\delta dp}, \varepsilon_{gd}) \cdot f0\left(\frac{t_{\delta dp}}{sec}\right)$
$t_{\delta dp}$


Finite sextet Pulse
$\Delta 5_\varepsilon(t_{\delta dp}, \varepsilon_{gd})$
$t_{\delta dp}$


Finite doublette Pulses
$\Delta 5_\varepsilon(t_{\delta dp}, \varepsilon_{gd}) \cdot f0\left(\frac{t_{\delta dp}}{sec}\right)$
$t_{\delta dp}$


Finite Pulses
$\Delta 5_\varepsilon(t_{\delta dp}, \varepsilon_{gd})$
$\Delta 4_\varepsilon(t_{\delta dp}, \varepsilon_{gd})$
$\Delta 3_\varepsilon(t_{\delta dp}, \varepsilon_{gd})$
$\Delta 2_\varepsilon(t_{\delta dp}, \varepsilon_{gd})$
$\Delta 1_\varepsilon(t_{\delta dp}, \varepsilon_{gd})$
$t_{\delta dp}$

3 Dirac pulse obtained from the normalized sinc(t)

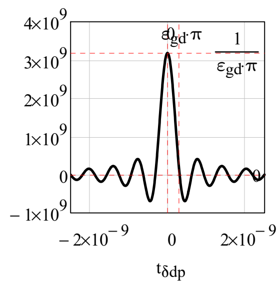
sinc is a Mathcad function, it returns the value of sin(z)/z, with correct behavior in the limit as z approaches 0.

$$f_1(t) := \frac{1}{\pi \cdot \varepsilon_{gd}} \cdot \mathrm{sinc}\left(\frac{t}{\varepsilon_{gd}}\right) \qquad \text{Fourier Optics: } \mathrm{sinc}(\xi) = \frac{\sin(\pi \cdot \xi)}{\pi \cdot \xi}$$

$$\boxed{\lim_{\varepsilon_{gd} \to 0} \lim_{t \to 0} \left(\frac{1}{\pi \cdot \varepsilon_{gd}} \cdot \mathrm{sinc}\left(\frac{t}{\varepsilon_{gd}}\right)\right) = \delta(t) = u_0(t)}$$

$$t_{\delta dp} := -8 \cdot \varepsilon_{gd} \cdot \pi, -8 \cdot \varepsilon_{gd} \cdot \pi + \frac{16 \cdot \varepsilon_{gd} \cdot \pi}{1000} \, .. \, 8 \cdot \varepsilon_{gd} \cdot \pi$$

$f_1(t_{\delta dp})$



$t := t \qquad \varepsilon_{gd} := \varepsilon_{gd} \qquad \xi_{gd} = \frac{t}{\varepsilon_{gd}} \qquad \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \mathrm{sinc}(\xi_{gd})\, d\xi_{gd} = 1 \qquad \int_{-\infty}^{\infty} \frac{\sin(\pi \cdot \xi_{gd})}{\pi \cdot \xi_{gd}}\, d\xi_{gd} = 1$

Sine Integral function: $\mathrm{Si}(t) = \int_0^t \mathrm{sinc}(\tau\_)\, d\tau\_$

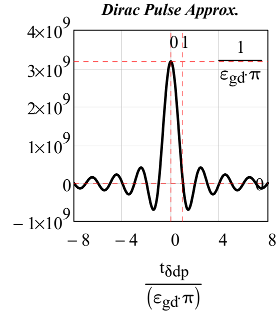
Fourier transform of $\frac{1}{\pi \cdot \varepsilon_{gd}} \cdot \mathrm{sinc}\left(\frac{t}{\varepsilon_{gd}}\right)$ fourier, t $\to \Phi\left(\frac{\omega \cdot \varepsilon_{gd} + 1}{\varepsilon_{gd}}\right) - \Phi\left[\frac{1}{\varepsilon_{gd}} \cdot (\omega \cdot \varepsilon_{gd} - 1)\right]$

$$\mathscr{F}\left\{\frac{1}{\pi \cdot \varepsilon_{gd}} \cdot \mathrm{sinc}\left(\frac{t}{\varepsilon_{gd}}\right)\right\} = \Pi\left(\frac{\varepsilon_{gd}}{2} \cdot \omega\right)$$

$$F_{sinc}(\omega, \varepsilon_{gd}) := \Pi\left(\frac{\varepsilon_{gd}}{2} \cdot \omega\right)$$

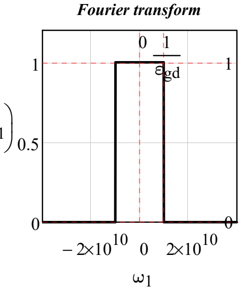
Dirac Pulse Approx.

$\frac{1}{\pi \cdot \varepsilon_{gd}} \cdot \mathrm{sinc}\left(\frac{t_{\delta dp}}{\varepsilon_{gd}}\right)$



Fourier transform

$\Pi\left(\frac{\varepsilon_{gd}}{2} \cdot \omega_1\right)$



$\varepsilon_{gd} := \varepsilon_{gd}$

Fourier transform:

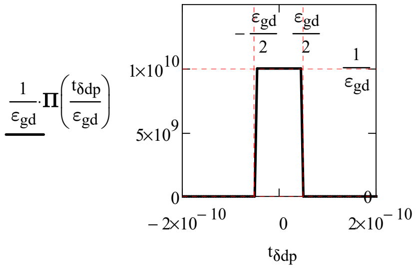
$$\frac{1}{\pi \cdot \varepsilon_{gd}} \cdot \left[\Phi(t + \varepsilon_{gd}) - \Phi((t - \varepsilon_{gd}))\right] \text{ fourier, t} \to -\frac{e^{-\omega \cdot \varepsilon_{gd} \cdot j} \cdot \left(e^{2j \cdot \omega \cdot \varepsilon_{gd}} - 1\right) \cdot (\pi \cdot \omega \cdot \Delta(\omega) + j)}{\pi \cdot \omega \cdot \varepsilon_{gd}}$$

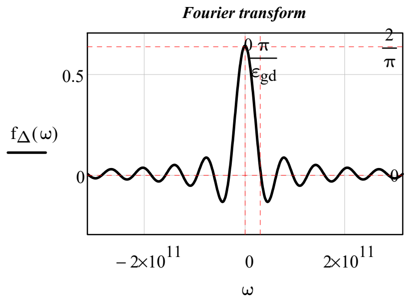
$$\frac{\pi}{\varepsilon_{gd}} = 31.416 \cdot \frac{\text{Grads}}{\text{sec}} \qquad f_\Delta(\omega) := -\frac{e^{-\omega \cdot \varepsilon_{gd} \cdot j} \cdot \left(e^{2j \cdot \omega \cdot \varepsilon_{gd}} - 1\right) \cdot (\pi \cdot \omega \cdot \delta(\omega) + j)}{\pi \cdot \omega \cdot \varepsilon_{gd}}$$

$$\lim_{\omega \to 0} -\frac{e^{-\omega \cdot \varepsilon_{gd} \cdot j} \cdot \left(e^{2j \cdot \omega \cdot \varepsilon_{gd}} - 1\right) \cdot (\pi \cdot \omega \cdot \delta(\omega) + j)}{\pi \cdot \omega \cdot \varepsilon_{gd}} \to \frac{2}{\pi}$$

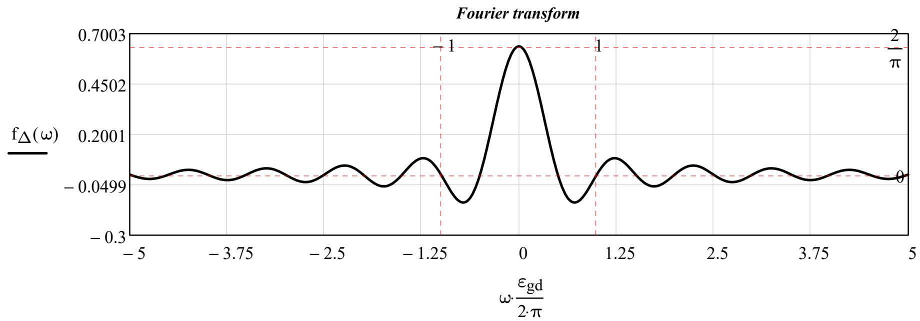
$\omega \neq 0$

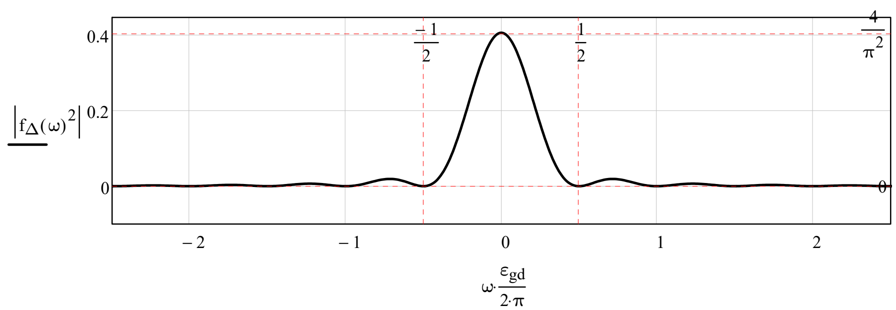
$$\boxed{f_\Delta(\omega) := -\frac{e^{-(\omega \cdot \varepsilon_{gd}) \cdot j}}{\varepsilon_{gd}} \cdot \left[\frac{j \cdot \left(e^{2i \cdot \omega \cdot \varepsilon_{gd}} - 1\right)}{\pi \cdot \omega}\right]} \qquad \left|f_\Delta(\omega)\right| = \frac{2 \cdot \sin(\omega \cdot \varepsilon_{gd})}{\pi \cdot (\omega \cdot \varepsilon_{gd})}$$



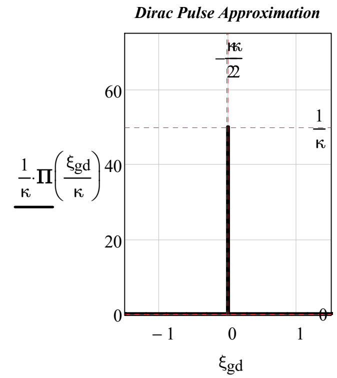


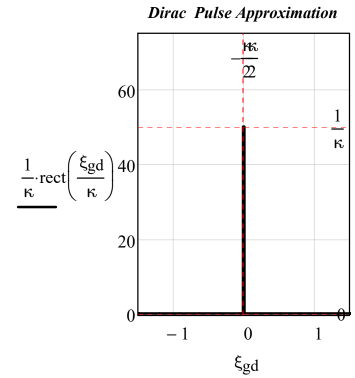
$$\omega := -5\cdot\frac{2\cdot\pi}{\varepsilon_{gd}}, -5\cdot\frac{2\cdot\pi}{\varepsilon_{gd}}+\frac{10}{1000}\cdot\frac{2\cdot\pi}{\varepsilon_{gd}} .. 5\cdot\frac{2\cdot\pi}{\varepsilon_{gd}}$$





$\kappa := 0.02$





$$\int_{-\infty}^{\infty}\frac{1}{\varepsilon_{gd}}\cdot\Pi\left(\frac{t}{\varepsilon_{gd}}\right)dt = 1\ \forall\varepsilon_{gd} < 1$$

$$\int_{-\infty}^{\infty}\frac{1}{\varepsilon_{gd}}\cdot\mathrm{rect}\left(\frac{t}{\varepsilon_{gd}}\right)dt = 1$$

4 Dirac pulse obtained from the normalized Gaussian distribution.

$$f_2(t,\varepsilon_{gd}) := \frac{1}{\varepsilon_{gd}\cdot\sqrt{\pi}}\cdot\exp\left[-\left(\frac{t}{\varepsilon_{gd}}\right)^2\right] \qquad \varepsilon_{gd} = 0.1\cdot ns$$

$$\lim_{\varepsilon_{gd}\to 0}\ \lim_{t\to 0}\left[\frac{1}{\varepsilon_{gd}\cdot\sqrt{\pi}}\cdot\exp\left[-\left(\frac{t}{\varepsilon_{gd}}\right)^2\right]\right] = \delta(t)$$

$t := t$

$$\frac{t}{\varepsilon_{gd}} = \xi_{gd}\quad \frac{1}{\varepsilon_{gd}\cdot\sqrt{\pi}}\cdot\int_{-\infty}^{\infty}\exp\left[-\left(\frac{t}{\varepsilon_{gd}}\right)^2\right]dt = \frac{1}{\sqrt{\pi}}\cdot\int_{-\infty}^{\infty}\exp\left(-\xi_{gd}^{\,2}\right)d\xi_{gd}$$

$\varepsilon_{gd} := \varepsilon_{gd}$

$$\frac{1}{\sqrt{\pi}}\cdot\int_{-\infty}^{\infty}\exp\left(-\xi_{gd}^{\,2}\right)d\xi_{gd}\ \text{simplify}\ \to 1$$

$$\frac{1}{\varepsilon_{gd}\cdot\sqrt{\pi}}\cdot\int_{-\infty}^{\infty}\exp\left[-\left(\frac{t}{\varepsilon_{gd}}\right)^2\right]dt = 1$$

First derivative:

$$f_{21}(t,\varepsilon_{gd}) := -\frac{2\cdot t\cdot e^{-\frac{t^2}{\varepsilon_{gd}^{\,2}}}}{\sqrt{\pi}\cdot\varepsilon_{gd}^{\,3}} \qquad \lim_{\varepsilon_{gd}\to 0} f_{21}(t,\varepsilon_{gd}) = \delta(1,t)$$

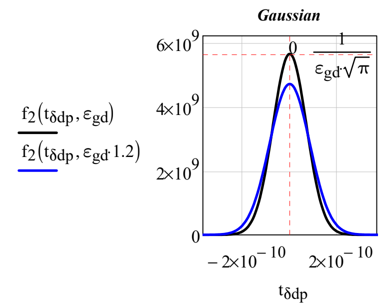
Second derivative:

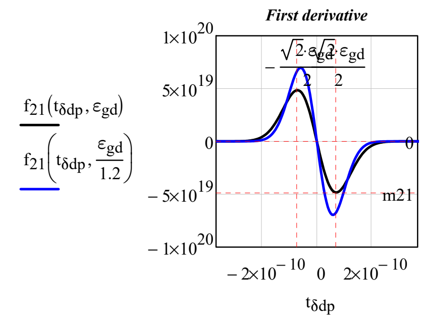
$$f_{22}(t,\varepsilon_{gd}) := -\frac{2\cdot e^{-\frac{t^2}{\varepsilon_{gd}^{\,2}}}\cdot\left(\varepsilon_{gd}^{\,2}-2\cdot t^2\right)}{\sqrt{\pi}\cdot\varepsilon_{gd}^{\,5}} \qquad \lim_{\varepsilon_{gd}\to 0} f_{22}(t,\varepsilon_{gd}) = \delta(2,t)$$

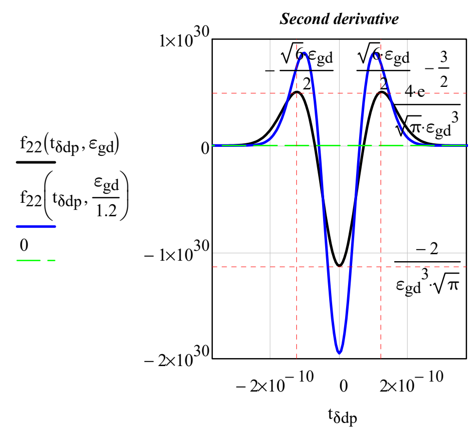
Third derivative

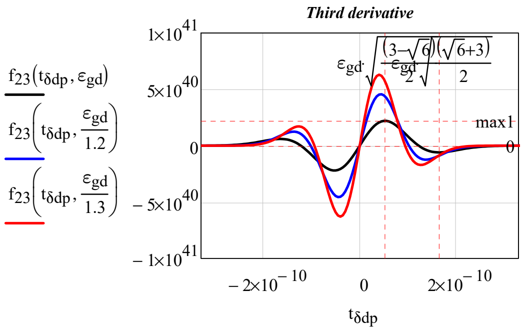
$$f_{23}(t,\varepsilon_{gd}) := \frac{4\cdot e^{-\frac{t^2}{\varepsilon_{gd}^{\,2}}}\cdot t\cdot\left(3\cdot\varepsilon_{gd}^{\,2}-2\cdot t^2\right)}{\sqrt{\pi}\cdot\varepsilon_{gd}^{\,7}} \qquad max1 := \frac{4\cdot\sqrt{6}\cdot e^{\sqrt{\frac{3}{2}}-\frac{3}{2}}\cdot\sqrt{\frac{3}{2}-\sqrt{\frac{3}{2}}}}{\sqrt{\pi}\cdot\varepsilon_{gd}^{\,4}}$$

$$max1 = 2.202\times 10^{4}\cdot ns^{-4} \qquad \lim_{\varepsilon_{gd}\to 0} f_{23}(t,\varepsilon_{gd}) = \delta(3,t) \qquad m21 := -\frac{\sqrt{2}\cdot e^{-\frac{1}{2}}}{\sqrt{\pi}\cdot\varepsilon_{gd}^{\,2}}$$







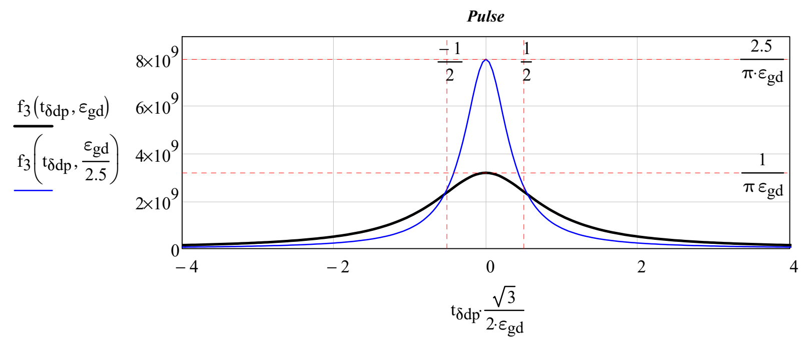


5 Dirac pulse obtained from the resonance function.

$$f_3(t, \varepsilon_{gd}) := \frac{1}{\pi}\cdot\frac{\varepsilon_{gd}}{t^2+\varepsilon_{gd}^2}$$

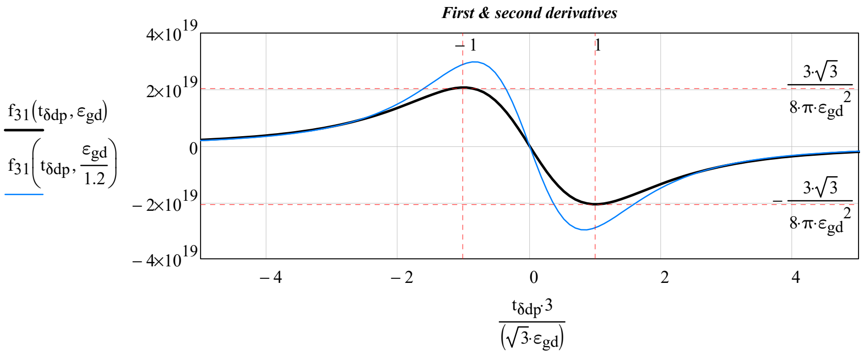
$$\frac{1}{\pi}\cdot\int_{-\infty}^{\infty}\frac{\varepsilon_{gd}}{t^2+\varepsilon_{gd}^2}\,dt \;\Bigg|\; \begin{array}{l}\text{simplify}\\ \text{assume}, \varepsilon_{gd} > 0\end{array} \rightarrow 1$$

$$\boxed{\frac{1}{\pi}\cdot\lim_{\varepsilon_{gd}\to 0}\;\lim_{t\to 0}\frac{\varepsilon_{gd}}{t^2+\varepsilon_{gd}^2} = \delta(t)}$$

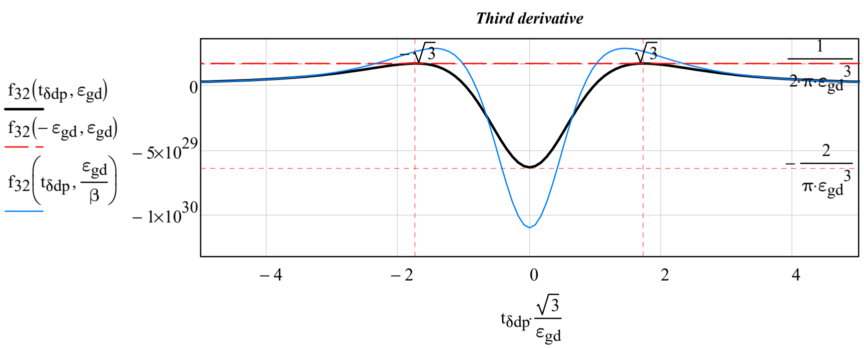


First derivative: $f_{31}(t, \varepsilon_{gd}) := \frac{-1}{\pi}\cdot\frac{2\cdot\varepsilon_{gd}\cdot t}{\left(\varepsilon_{gd}^2+t^2\right)^2}$ II $f_{32}(t, \varepsilon_{gd}) := \frac{-1}{\pi}\cdot\frac{2\cdot\varepsilon_{gd}\cdot\left(\varepsilon_{gd}^2-3\cdot t^2\right)}{\left(\varepsilon_{gd}^2+t^2\right)^3}$

$\lim_{\varepsilon_{gd}\to 0} f_{31}(t, \varepsilon_{gd}) = \delta(1, t)$ $\quad\lim_{\varepsilon_{gd}\to 0} f_{32}(t, \varepsilon_{gd}) = \delta(2, t)$ $\quad\lim_{\varepsilon_{gd}\to 0} f_{33}(t, \varepsilon_{gd}) = \delta(3, t)$



$\beta := 1.2$



6 Properties & Collection of formulas

(Knowing that: $\delta(t) = \lim_{\varepsilon_{gd}\to 0} \delta_\varepsilon(t,\varepsilon_{gd}) = \infty$, or $\delta(t) = \lim_{\varepsilon_{gd}\to 0}\left(\frac{1}{\varepsilon_{gd}}\cdot\Pi\left(\frac{t}{\varepsilon_{gd}}\right)\right) = \infty$ [1]

$$\int_{-\infty}^{\infty}\delta(t)\,dt = 1 \qquad \int_{-\infty}^{\infty}\frac{1}{\varepsilon_{gd}}\cdot\Pi\left(\frac{t}{\varepsilon_{gd}}\right)dt = 1$$

$$\delta(t_0 - t) = \delta(t - t_0)$$

$$\int \delta(t)\,dt = \Phi(t) \qquad = \mathbf{u}_{-1}(t) \quad \text{Heaviside step function}$$

$$\int\int \delta(t)\,dt\,dt = t\cdot\Phi(t) \qquad = \mathbf{u}_{-2}(t)$$

$$\int\int\int \delta(t)\,dt\,dt\,dt = t^2\cdot\Phi(t) \qquad = \mathbf{u}_{-3}(t)$$

⋮ ⋮

$$t^{k-1}\cdot\Phi(t) \qquad = \mathbf{u}_{-k}(t)$$

$$\int_a^b f(t)\cdot\delta(t_0 - t)\,dt = \int_a^b f(t)\cdot\delta(t - t_0)\,dt$$

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\cdot\delta(t-\tau)\,d\tau \qquad \forall t$$

$$\delta(-t) = \delta(t)$$

$$\delta(a \cdot t) = \frac{1}{|a|} \cdot \delta(t)$$

$$f(t) \cdot \delta(t) = f(0)$$

$$\int_{t_1}^{t_2} \delta(t-\tau)\,dt = \begin{cases} 1 & \text{if } t_1 \le \tau \le t_2 \\ 0 & \text{if } t_2 < \tau \end{cases}$$

$$\int_{\tau}^{t} \delta(t-\tau)\,dt = \Phi(t-\tau)$$

$\lim_{t \to \infty} \Phi(t-\tau) \neq 0 \quad \Rightarrow$ it isn't transformable according to Fourier.

Any integrable function s(t) for which $\int_{-\infty}^{\infty} |s(t)|\,dt < \infty$, has a Fourier transform $F(\omega) = \mathscr{F}\{s(t)\}$.

$$\Phi(t-\tau) = \frac{1}{2} + \frac{1}{\pi} \cdot \int_0^{\infty} \frac{\sin[\omega \cdot (t-\tau)]}{\omega}\,d\omega$$

$$\int_{-\infty}^{\infty} t \cdot \delta(t)\,dt = 0$$

$$t \cdot \delta(t) = 0 \quad \Rightarrow A = B \quad \Rightarrow A = B + t \cdot \delta(t) \quad \Rightarrow \quad \frac{A}{t} = \frac{B}{t} + \delta(t)$$

$$t^n \cdot \frac{d^n}{dt^n}\delta(t) = (-1)^n \cdot n! \cdot \delta(t) \qquad n > 0$$

$$t^2 \cdot \frac{d^2}{dt^2}\delta(t) = 2 \cdot \delta(t)$$

$$t^3 \cdot \frac{d^2}{dt^2}\delta(t) = 0 \quad \Rightarrow A = B \Rightarrow A = B + t^3 \cdot \frac{d^2}{dt^2}\delta(t) \quad \Rightarrow \frac{A}{t^3} = \frac{B}{t^3} + \frac{d^2}{dt^2}\delta(t)$$

$$\int \frac{A}{t^3}\,dt = \int \frac{B}{t^3}\,dt + \int \frac{d^2}{dt^2}\delta(t)\,dt \quad \Rightarrow -\frac{A}{2 \cdot t^2} = -\frac{B}{2 \cdot t^2} + \delta(1,t) + \text{cost} \quad \Rightarrow A = B - 2 \cdot t^2 \cdot (\delta(1,t) + \text{cost})$$

$$\delta(a \cdot t + b) = \frac{1}{|a|} \cdot \delta\left(t + \frac{b}{a}\right)$$

$$\delta\left(t^2 - a^2\right) = \delta[(t-a) \cdot (t+a)] = \frac{1}{2} \cdot \frac{1}{|a|} \cdot (\delta(t-a) + \delta(t+a))$$

---

x_k is the k^{th} root of f(x)=0 $\quad \Rightarrow \quad f(\xi_{gd}) = \prod_{k=0}^{n-1} (\xi_{gd} - x_k)$ [1]

$$\delta(f(\xi_{gd})) = \delta\left[\prod_{k=0}^{n-1} (\xi_{gd} - x_k)\right] = \sum_{k=0}^{n-1} \frac{\delta(\xi_{gd} - x_k)}{\lim_{\xi_{gd} \to x_k} \left|\frac{d}{d\xi_{gd}} f(\xi_{gd})\right|} = \sum_{k=0}^{n-1} \frac{\delta(\xi_{gd} - x_k)}{\lim_{\xi_{gd} \to x_k} \left|\frac{d}{d\xi_{gd}} f(\xi_{gd})\right|}$$

$$\frac{1}{2 \cdot \pi} \cdot \int_{-\infty}^{\infty} e^{j \cdot \omega \cdot t}\,dt = \delta(t)$$

$$\int_{-\infty}^{\infty} \delta(t) \cdot e^{j \cdot \omega \cdot t}\,dt = 1$$

$$\int_{-\infty}^{\infty} \left|\frac{d}{dt}\delta(t)\right|\,dt = \infty$$

$$\int_{-\infty}^{\infty} t^2 \cdot \frac{d^2}{dt^2}\delta(t)\,dt = 2$$

$$\int_{-\infty}^{\infty} \frac{d^n}{d\tau^n}\delta(\tau)\cdot f(t-\tau)\,d\tau = \frac{d^n}{dt^n}f(t)$$

$$\int_{-\infty}^{\infty} f(t)\cdot\frac{d^n}{dt^n}\delta(t)\,dt = (-1)^n\cdot\lim_{t\to 0}\frac{d^n}{dt^n}f(t)$$

$$\int \frac{d^k}{dt^k}\delta(t-\tau)\cdot\frac{d^n}{dt^n}\delta(\tau-\sigma)\,d\tau\_ = \frac{d^{k+n}}{dt^{k+n}}\delta(t-\sigma)$$

$$t^{k+1}\cdot\frac{d^k}{dt^k}\delta(t) = 0$$

For some text of electrical engineering:

doublet $\mathbf{u_1}(t) = \frac{d}{dt}\mathbf{u_0}(t)$

$$f(t)\cdot\mathbf{u_1}(t-t_0) = 0 \qquad t \ne t_0$$

$$\int_{-\infty}^{\infty} f(t)\cdot\mathbf{u_1}(t-t_0)\,dt = -\lim_{t\to t_0}\frac{d}{dt}f(t)$$

---

triplet $\mathbf{u_2}(t) = \frac{d}{dt}\mathbf{u_1}(t) = \frac{d^2}{dt^2}\mathbf{u_0}(t)$ $\qquad f(t)\cdot\mathbf{u_2}(t-t_0) = 0 \qquad t \ne t_0$

$$\int_{-\infty}^{\infty} f(t)\cdot\mathbf{u_2}(t-t_0)\,dt = \lim_{t\to t_0}\frac{d^2}{dt^2}f(t)$$

$$\mathbf{u_k}(t) = \frac{d^k}{dt^k}\mathbf{u_0}(t) \qquad f(t)\cdot\mathbf{u_k}(t-t_0) = 0 \qquad t \ne t_0 \qquad k > 0$$

$$\int_{-\infty}^{\infty} f(t)\cdot\mathbf{u_k}(t-t_0)\,dt = (-1)^k\cdot\lim_{t\to t_0}\frac{d^k}{dt^k}f(t)$$

$$\frac{d^k}{dt^k}\mathbf{u_0}(t) = (-1)^k\cdot\frac{d^k}{dt^k}\mathbf{u_0}(-t)$$

$$\frac{d}{dt}\mathbf{u_0}(t) = \frac{j}{2\cdot\pi}\cdot\int_{-\infty}^{\infty}\omega\cdot e^{j\cdot\omega\cdot t}\,d\omega$$

$$\int_{-\infty}^{\infty} f(t)\cdot\frac{d^n}{dt^n}\mathbf{u_0}(t)\,dt = (-1)^n\cdot\lim_{t\to 0}\frac{d^n}{dt^n}f(t)$$

$$\int \frac{d^k}{dt^k}\mathbf{u_0}(t-\tau\_)\cdot\frac{d^n}{d\tau\_^n}\mathbf{u_0}(\tau\_-\sigma)\,d\tau\_ = \frac{d^{k+n}}{dt^{k+n}}\mathbf{u_0}(t-\sigma)$$

$$t^{k+1}\cdot\frac{d^k}{dt^k}\mathbf{u_0}(t) = 0$$

$$\mathscr{L}\left\{\mathrm{rect}(t)\right\} = \frac{e^{-s\cdot(\mathrm{risingedge})}}{s}\cdot\left(1-e^{-\mathrm{width}\cdot s}\right)$$

risingedge $= -\frac{1}{2}$ $\qquad$ width $= 1$

$$\mathscr{L}\left\{\Pi(t)\right\} = \frac{e^{\frac{s}{2}}}{s}\cdot\left(1-e^{-s}\right)$$

7 Fourier transform of $\Delta_\delta(t_\delta,\varepsilon_t)$

$T_s := T_s \qquad \tau_{01} := \tau_{01} \qquad \delta 1 := 2 \qquad \varepsilon_{gd} := \varepsilon_{gd}$

$$\frac{1}{\varepsilon_{gd}}\cdot \mathrm{rect1}\left(t,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right) \;\Big|\; \begin{matrix}\text{fourier}, t\\ \text{simplify}, \max\end{matrix} \rightarrow -\frac{2j\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot(\pi\cdot\omega\cdot\Delta(\omega)+j)}{\omega\cdot\varepsilon_{gd}}$$

$$\frac{1}{\varepsilon_{gd}}\cdot\int_{-\infty}^{\infty}\mathrm{rect1}\left(t,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\cdot e^{-j\cdot\omega\cdot t}\,dt = -\frac{j\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot(\pi\cdot\omega\cdot\delta(\omega)+j)}{\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)}$$

$$\omega\cdot\delta(\omega) = 0$$

sinc is a Mathcad function, it returns the value of sin(z)/z, with correct behavior in the limit as z approaches 0.

$$-\frac{j\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot(\pi\cdot\omega\cdot\delta(\omega)+j)}{\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)} = \frac{\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)}{\frac{\omega\cdot\varepsilon_{gd}}{2}} = \mathrm{sinc}\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)$$

Hence ,

$$\frac{1}{\varepsilon_{gd}}\cdot\int_{-\infty}^{\infty}\mathrm{rect1}\left(t,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\cdot e^{-j\cdot\omega\cdot t}\,dt = \mathrm{sinc}\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)$$

$$\boxed{\mathscr{F}\left\{\frac{1}{\varepsilon_{gd}}\cdot\mathrm{rect1}\left(t,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\} = \mathrm{sinc}\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)} \qquad \text{first null: } \frac{\omega\cdot\varepsilon_{gd}}{2} = \pi$$

$$\mathscr{F}\left\{\frac{1}{\varepsilon_{gd}}\cdot\mathrm{rect1}\left(t-\frac{\varepsilon_{gd}}{2},\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\} = e^{-j\cdot\omega\cdot\frac{\varepsilon_{gd}}{2}}\cdot\mathrm{sinc}\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)$$

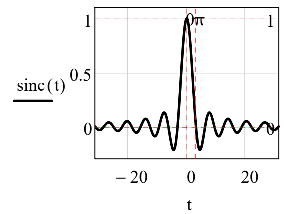
$$\mathscr{F}\left\{\delta_\varepsilon\left(t-\frac{\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\} = e^{-j\cdot\omega\cdot\frac{\varepsilon_{gd}}{2}}\cdot\mathrm{sinc}\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)$$

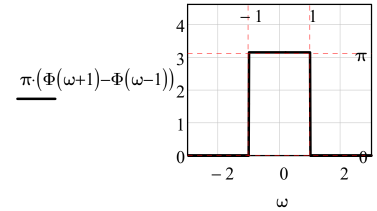
$\frac{2\cdot\pi}{\varepsilon_{gd}} = 62.832\cdot\frac{\text{Grads}}{\text{sec}} \qquad \varepsilon_{gd} := \varepsilon_{gd} \;\; \omega := \omega$

$$\lim_{\omega\to 0}\;\lim_{\varepsilon_{gd}\to 0} -\frac{j\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot(\pi\cdot\omega\cdot\delta(\omega)+j)}{\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)} \;\text{simplify} \rightarrow 1$$

$$\mathrm{sinc}\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\;\text{invfourier}, \omega \rightarrow -\frac{\Phi\left(t-\frac{\varepsilon_{gd}}{2}\right)-\Phi\left(t+\frac{1}{2}\cdot\varepsilon_{gd}\right)}{\varepsilon_{gd}}$$

$t := t \qquad \varepsilon_{gd} := \varepsilon_{gd} \qquad \omega := \omega \quad \mathrm{sinc}(t)\ \text{fourier}, t \rightarrow -\pi\cdot(\Phi(\omega-1)-\Phi(\omega+1))$

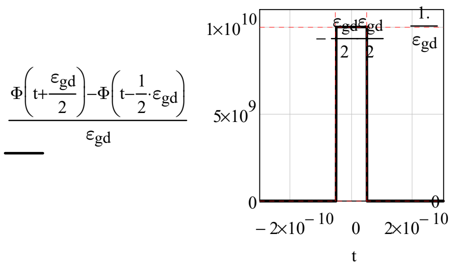


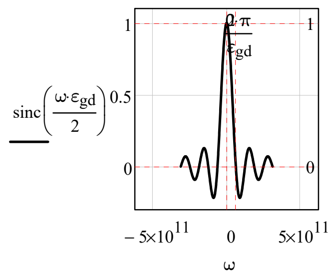


$$\frac{\Phi\left(t+\frac{\varepsilon_{gd}}{2}\right)-\Phi\left(t-\frac{1}{2}\cdot\varepsilon_{gd}\right)}{\varepsilon_{gd}}\;\text{fourier} \rightarrow -\frac{e^{-\frac{\omega\cdot\varepsilon_{gd}\cdot j}{2}}\cdot\left(e^{\omega\cdot\varepsilon_{gd}\cdot j}-1\right)\cdot(\pi\cdot\omega\cdot\Delta(\omega)+j)}{\omega\cdot\varepsilon_{gd}}$$

$$\mathscr{F}\left\{\frac{\Phi\left(t+\frac{\varepsilon_{gd}}{2}\right)-\Phi\left(t-\frac{1}{2}\cdot\varepsilon_{gd}\right)}{\varepsilon_{gd}}\right\} = \frac{2\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot(1-\pi\cdot\omega\cdot\delta(\omega)\cdot i)}{\omega\cdot\varepsilon_{gd}}$$

$$\omega\cdot\delta(\omega) = 0 \qquad \mathscr{F}\left\{\frac{\Phi\left(t+\frac{\varepsilon_{gd}}{2}\right)-\Phi\left(t-\frac{1}{2}\cdot\varepsilon_{gd}\right)}{\varepsilon_{gd}}\right\} = \frac{\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)}{\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)}$$





It is known that: $\delta(t)\ \text{fourier} \rightarrow 1$

8 Laplace transform of $\Delta_\varepsilon(t_\delta,\varepsilon_t)$

Laplace transform: $t := t$ $\varepsilon_{gd} := \varepsilon_{gd}$

$$\frac{1}{\varepsilon_{gd}}\cdot \mathrm{rect1}\left(t,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\ \text{laplace}, t \rightarrow -\frac{\text{laplace}\left(\Phi\left(t-\frac{\varepsilon_{gd}}{2}\right),t,s\right)-\text{laplace}\left(\Phi\left(t+\frac{\varepsilon_{gd}}{2}\right),t,s\right)}{\varepsilon_{gd}}$$

Laplace transform calculations:

$t := t$ $\text{risingedge} := \text{risingedge}$ $\text{width} := \text{width}$

$$(\mathrm{rect1}(t,\text{risingedge},\text{width}))\ \begin{vmatrix}\text{assume}, \text{risingedge} > 0\\ \text{assume}, \text{width} > 0\\ \text{laplace}\\ \text{simplify}, \text{max}\end{vmatrix} \rightarrow -\frac{e^{-\text{risingedge}\cdot s}\cdot\left(e^{-s\cdot\text{width}}-1\right)}{s}$$

$$\text{laplace}(\Phi(t-\text{risingedge}),t,s) = \frac{e^{-s\cdot\text{risingedge}}}{s}$$

$$\text{laplace}(\Phi(t-\text{risingedge}-\text{width}),t,s) = \frac{e^{-s\cdot(\text{risingedge}+\text{width})}}{s}$$

$$\mathscr{L}\left\{\mathrm{rect1}(t,\text{risingedge},\text{width})\right\} = \frac{e^{-s\cdot\text{risingedge}}}{s}-\frac{e^{-s\cdot(\text{risingedge}+\text{width})}}{s}$$

$$\boxed{\mathscr{L}\left\{\mathrm{rect1}(t,\text{risingedge},\text{width})\right\} = \frac{e^{-s\cdot\text{risingedge}}}{s}\cdot\left(1-e^{-\text{width}\cdot s}\right)}$$

$$\text{risingedge} = \frac{-\varepsilon_{gd}}{2} \qquad \text{width} = \varepsilon_{gd}$$

$$\mathscr{L}\left\{\frac{1}{\varepsilon_{gd}}\cdot\mathrm{rect1}\left(t,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\} = \frac{e^{\frac{\varepsilon_{gd}}{2}\cdot s}}{\varepsilon_{gd}\cdot s}\cdot\left(1-e^{-\varepsilon_{gd}\cdot s}\right)$$

$$\delta(t)\ \text{laplace}, t \rightarrow 1 \qquad \lim_{\varepsilon_{gd}\rightarrow 0}\left[\frac{e^{\frac{\varepsilon_{gd}}{2}\cdot s}}{\varepsilon_{gd}\cdot s}\cdot\left(1-e^{-\varepsilon_{gd}\cdot s}\right)\right]\ \text{simplify} \rightarrow 1$$

9 Even impulse pair

II(written as two I uppercase) [1]

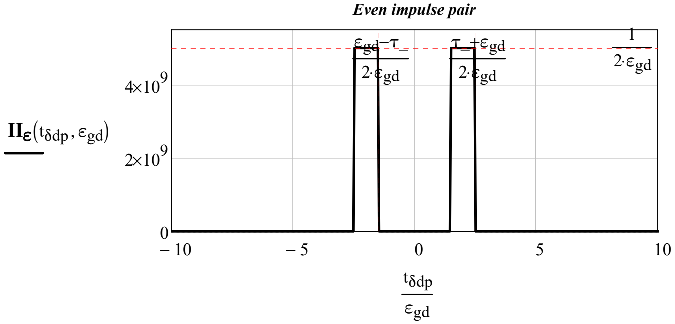
$$\mathbf{II}(t,\tau\_) = \frac{1}{2}\cdot\delta\left(t+\frac{1}{2}\cdot\tau\_\right)+\frac{1}{2}\cdot\delta\left(t-\frac{1}{2}\cdot\tau\_\right) \quad \text{or} \quad \mathbf{II}(t,\tau\_) = \frac{1}{2}\cdot u_0\left(t+\frac{1}{2}\cdot\tau\_\right)+\frac{1}{2}\cdot u_0\left(t-\frac{1}{2}\cdot\tau\_\right)$$

$$\mathbf{II}(t) = \delta\left(2\cdot t^2-\frac{\tau\_^2}{2}\right) = \delta\left[2\cdot\left(t-\frac{\tau}{2}\right)\cdot\left(t+\frac{\tau}{2}\right)\right]$$

Approximation. $\tau\_ := 4\cdot\varepsilon_{gd}$ $\quad \varepsilon_{gd} = 0.1\cdot ns$

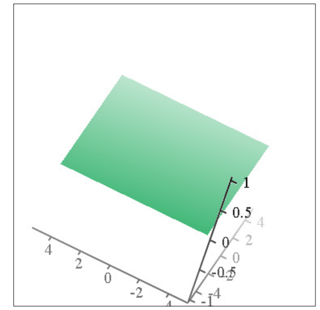
$$\mathbf{II}_{\varepsilon}(t,\varepsilon_{gd}) := \frac{1}{2}\cdot\delta_{\varepsilon}\left(t+\frac{1}{2}\cdot\tau\_,\varepsilon_{gd}\right)+\frac{1}{2}\cdot\delta_{\varepsilon}\left(t-\frac{1}{2}\cdot\tau\_,\varepsilon_{gd}\right)$$

Even impulse pair



Two dimensional $u := -10\cdot\varepsilon_{gd}, -10\cdot\varepsilon_{gd}+\frac{20\cdot\varepsilon_{gd}}{2000} .. 10\cdot\varepsilon_{gd}$ $\quad v := -10\cdot\varepsilon_{gd}, -10\cdot\varepsilon_{gd}+\frac{20\cdot\varepsilon_{gd}}{2000} .. 10\cdot\varepsilon_{gd}$

$$\mathbf{II}_{2\varepsilon}(u,v) := \mathbf{II}_{\varepsilon}(u,\varepsilon_{gd})\cdot\mathbf{II}_{\varepsilon}(v,\varepsilon_{gd})$$

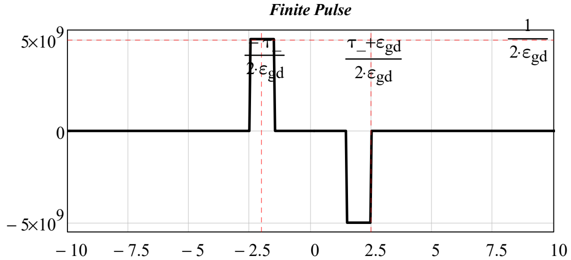


$\mathbf{II}_{2\varepsilon}$

10 Odd impulse pair

$$\mathbf{I_I}(t) = \frac{1}{2}\cdot\delta\left(t+\frac{1}{2}\cdot\tau\_\right)-\frac{1}{2}\cdot\delta\left(t-\frac{1}{2}\cdot\tau\_\right) \qquad \mathbf{I_I}(t) = \frac{1}{2}\cdot u_0\left(t+\frac{1}{2}\cdot\tau\_\right)-\frac{1}{2}\cdot u_0\left(t-\frac{1}{2}\cdot\tau\_\right)$$

approximation. $\mathbf{I_{I.\varepsilon}}\left(t,\varepsilon_{gd}\right) := \frac{1}{2}\cdot\delta_{\varepsilon}\left(t+\frac{1}{2}\cdot\tau\_,\varepsilon_{gd}\right)-\frac{1}{2}\cdot\delta_{\varepsilon}\left(t-\frac{1}{2}\cdot\tau\_,\varepsilon_{gd}\right)$



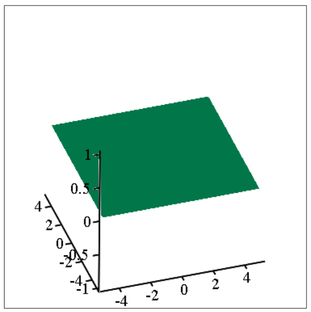
$\mathbf{I_{I.\varepsilon}}\left(t_{\delta dp},\varepsilon_{gd}\right)$

$\frac{t_{\delta dp}}{\varepsilon_{gd}}$

$$\frac{-\tau\_}{2\cdot\varepsilon_{gd}} = -2$$

$$\frac{\tau\_+\varepsilon_{gd}}{2\cdot\varepsilon_{gd}} = 2.5$$

$\varepsilon_{gd} = 0.1\cdot ns$ $\qquad \mathbf{I_{I.2\varepsilon}}(u,v) := \mathbf{I_{I.\varepsilon}}\left(u,\varepsilon_{gd}\right)\cdot\mathbf{I_{I.\varepsilon}}\left(v,\varepsilon_{gd}\right)$



$\mathbf{I_{I.2\varepsilon}}$

11 Sampling or replicating symbol

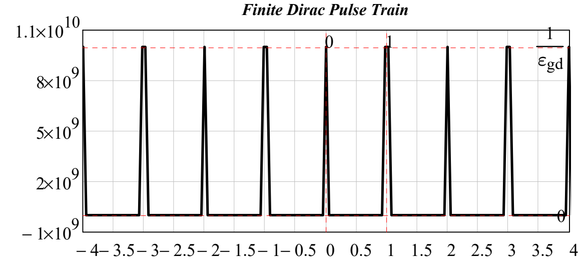
Read "Shah" $\quad \mathbf{III}(t) = \sum_{k=-\infty}^{\infty}\delta(t-k\cdot T) \qquad \mathbf{III}(t) = \sum_{k=-\infty}^{\infty}u_0(t-k\cdot T)$ [1]

12 Sampling function Approximation

$N0_{gd} = 256 \qquad \mathbf{III_{\varepsilon}}\left(t,T_{0\_},\varepsilon_{gd}\right) := \sum_{k=-N0_{gd}}^{N0_{gd}}\delta_{\varepsilon}\left(t-k\cdot T_{0\_},\varepsilon_{gd}\right)$

$T_{0\_} := 4\cdot\tau\_ \qquad t_{\Delta} := -N0_{gd}\cdot T_{0\_}, -N0_{gd}\cdot T_{0\_}+\frac{N0_{gd}\cdot T_{0\_}+N0_{gd}\cdot T_{0\_}}{10000} .. N0_{gd}\cdot T_{0\_}$

$\tau\_ = 4\times10^{-10}s$



$\mathbf{III_{\varepsilon}}\left(t_{\Delta},T_{0\_},\varepsilon_{gd}\right)$

$\frac{t_{\Delta}}{T_{0\_}}$

Laplace transform-

Ideal case: $\quad \mathscr{L}\left\{\mathbf{III}(t)\right\} = \sum_{k=0}^{N0_{gd}}\mathscr{L}(\delta(t-\tau\_\cdot k))$

$$\mathscr{L}\left\{\sum_{k=0}^{\infty}\delta(t-k\cdot\tau\_)\right\} = \frac{1}{1-e^{-\tau\_\cdot s}}$$

$$\mathscr{L}\left\{\mathbf{III}(t)\right\} = \frac{1}{1-e^{-T_{0\_}\cdot s}}$$

Approximated pulse case:

$$\mathscr{L}\left\{\mathbf{III}_{\varepsilon}(t,T_{0\_},\varepsilon_{gd})\right\} = \frac{1}{\varepsilon_{gd}}\cdot\sum_{k=0}^{N0_{gd}}\mathscr{L}\left\{\mathrm{rect1}\left(t-T_{0\_}\cdot k,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\}$$

$$\boxed{\mathscr{L}\left\{\mathrm{rect1}(t,\mathrm{risingedge},\mathrm{width})\right\} = \frac{e^{-s(\mathrm{risingedge})}}{s}\cdot\left(1-e^{-\mathrm{width}\cdot s}\right)}$$

$$\text{delay theorem:}\quad \mathscr{L}\left\{(f(t-\tau\_)\cdot\Phi(t-\tau\_))\right\} = e^{-\tau\_\cdot s}\cdot F(s)$$

$$\boxed{\mathscr{L}\left\{\mathrm{rect1}(t-\tau\_,\mathrm{risingedge},\mathrm{width})\right\} = e^{-\tau\_\cdot s}\cdot\mathscr{L}(\mathrm{rect1}(t,\mathrm{risingedge},\mathrm{width}))}$$

$$\mathscr{L}\left\{\mathrm{rect1}\left(t,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\} = \frac{e^{\frac{\varepsilon_{gd}}{2}\cdot s}}{s}\cdot\left(1-e^{-\varepsilon_{gd}\cdot s}\right)$$

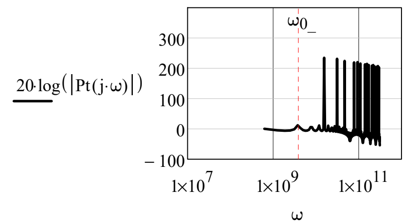
$$\mathscr{L}\left\{\sum_{k=0}^{\infty}F(t-k\cdot T)\right\} = \frac{F(s)}{1-e^{-T\cdot s}}$$

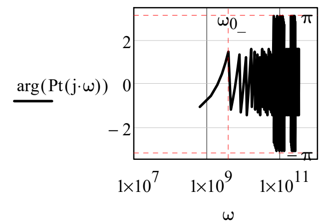
$$\frac{1}{\varepsilon_{gd}}\cdot\sum_{k=0}^{\infty}\mathscr{L}\left\{\mathrm{rect1}\left(t-T_{0\_}\cdot k,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\} = \frac{e^{\frac{\varepsilon_{gd}}{2}\cdot s}\cdot\left(1-e^{-\varepsilon_{gd}\cdot s}\right)}{\varepsilon_{gd}\cdot s}\cdot\frac{1}{1-e^{-T_{0\_}\cdot s}}$$

$$\boxed{\mathscr{L}\left\{\mathbf{III}_{\varepsilon}(t,T_{0\_},\varepsilon_{gd})\right\} = \frac{e^{\frac{\varepsilon_{gd}}{2}\cdot s}\cdot\left(1-e^{-\varepsilon_{gd}\cdot s}\right)}{\varepsilon_{gd}\cdot s\cdot\left(1-e^{-T_{0\_}\cdot s}\right)}}\qquad Pt(s) := \frac{e^{\frac{\varepsilon_{gd}}{2}\cdot s}\cdot\left(1-e^{-\varepsilon_{gd}\cdot s}\right)}{\varepsilon_{gd}\cdot s\cdot\left(1-e^{-T_{0\_}\cdot s}\right)}$$

$$\mathscr{L}\left\{\mathbf{III}_{\varepsilon}(t,T_{0\_},\varepsilon_{gd})\right\} = \frac{e^{\frac{\varepsilon_{gd}}{2}\cdot s}\cdot\left(1-e^{-\varepsilon_{gd}\cdot s}\right)\cdot\left(1+\tanh\left(\frac{T_{0\_}\cdot s}{2}\right)\right)}{\varepsilon_{gd}\cdot s\cdot 2\cdot\tanh\left(\frac{T_{0\_}\cdot s}{2}\right)}$$

$$\omega_{0\_} := \frac{2\cdot\pi}{T_{0\_}}$$





causing ε_t tend to zero, the sequence of pulses results composed by Dirac pulses with infinite amplitude and unitary area. So it results:

$T_{0\_} := T_{0\_}$ $\qquad\qquad \varepsilon_{gd} := \varepsilon_{gd} \quad s := s$

$$\lim_{\varepsilon_{gd}\to 0}\frac{e^{\frac{\varepsilon_{gd}}{2}\cdot s}\cdot\left(1-e^{-\varepsilon_{gd}\cdot s}\right)}{\varepsilon_{gd}\cdot s\cdot\left(1-e^{-T_{0\_}\cdot s}\right)}\ \text{simplify}\ \rightarrow -\frac{1}{e^{-T_{0\_}\cdot s}-1}$$

$$\mathscr{L}\left\{\mathbf{III}(t)\right\} = \frac{1}{1-e^{-T_{0\_}\cdot s}} = \frac{1+\tanh\left(\frac{T_{0\_}\cdot s}{2}\right)}{2\cdot\tanh\left(\frac{T_{0\_}\cdot s}{2}\right)}$$

Fourier transform:

For the definition and a deepening see the file "Fourier Transform.xmcd"

Approximated pulse case: $$\mathscr{F}\left\{\mathbf{III}_{\varepsilon}(t,T_{0\_},\varepsilon_{gd})\right\} = \mathscr{F}\left\{\sum_{k=0}^{N0_{gd}}\delta_{\varepsilon}(t-T_{0\_}\cdot k,\varepsilon_{gd})\right\}$$

$$\mathscr{F}\left\{\mathbf{III}_{\varepsilon}(t,T_{0\_},\varepsilon_{gd})\right\} = \frac{1}{\varepsilon_{gd}}\cdot\mathscr{F}\left\{\sum_{k=0}^{N0_{gd}}\mathrm{rect1}\left(t-T_{0\_}\cdot k,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\}$$

$$\frac{1}{\varepsilon_{gd}}\cdot\mathcal{F}\left\{\sum_{k=0}^{N0_{gd}} \text{rect1}\left(t-T_{0\_}\cdot k,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\} = \frac{1}{\varepsilon_{gd}}\cdot\sum_{k=0}^{N0_{gd}}\mathcal{F}\left\{\text{rect1}\left(t-T_{0\_}\cdot k,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\}$$

$$\mathcal{F}\left\{\sum_{k=0}^{\infty} f(t-k\cdot T)\right\} = \frac{\mathcal{F}\left\{f(t)\right\}}{1-e^{-T\cdot j\cdot\omega}}$$

$$\boxed{\mathcal{F}\left\{\text{rect1}(t,\text{risingedge},\text{width})\right\} = \frac{e^{-j\cdot\omega\cdot\text{risingedge}}}{j\cdot\omega}\cdot\left(1-e^{-\text{width}\cdot j\cdot\omega}\right)}$$

$$\boxed{\mathcal{F}\left\{\text{rect1}(t-\tau,\text{risingedge},\text{width})\right\} = \frac{e^{-j\cdot\omega\cdot(\text{risingedge}+\tau)}}{j\cdot\omega}\cdot\left(1-e^{-\text{width}\cdot j\cdot\omega}\right)}$$

$t := t \quad T_{0\_} := T_{0\_} \quad k := k \quad \varepsilon_{gd} := \varepsilon_{gd}$

$$\text{rect1}\left(t-T_{0\_}\cdot k,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\ \begin{vmatrix}\text{fourier}, t\\ \text{simplify}\end{vmatrix} \rightarrow -\frac{2j\cdot e^{T_{0\_}\cdot\omega\cdot k\cdot j}\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot(\pi\cdot\omega\cdot\Delta(\omega)+j)}{\omega}$$

$$\mathcal{F}\left\{\text{rect1}\left(t-T_{0\_}\cdot k,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\} = -\frac{2j\cdot e^{T_{0\_}\cdot\omega\cdot k\cdot j}\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot(\pi\cdot\omega\cdot\delta(\omega)+j)}{\omega}$$

$$\omega\cdot\delta(\omega) = 0$$

$$\mathcal{F}\left\{\text{rect1}\left(t-T_{0\_}\cdot k,\frac{-\varepsilon_{gd}}{2},\varepsilon_{gd}\right)\right\} = \frac{2\cdot e^{T_{0\_}\cdot\omega\cdot k\cdot j}\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)}{\omega}$$

$N0_{gd} = 256 \qquad F_{\Delta\varepsilon}(\omega) := \frac{2}{\omega\cdot\varepsilon_{gd}}\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot\sum_{k=0}^{N0_{gd}}\left(e^{-T_{0\_}\cdot\omega\cdot k\cdot j}\right)$

$$q = \frac{e^{\left(-T_{0\_}\cdot\omega\cdot N0_{gd}\cdot j\right)}}{e^{\left[-T_{0\_}\cdot\omega\cdot\left(N0_{gd}-1\right)\cdot j\right]}} = e^{-T_{0\_}\cdot\omega\cdot j}$$

assuming |q|<1 $\qquad \sum_{k=0}^{N0_{gd}} e^{\left(-T_{0\_}\cdot\omega\cdot k\cdot j\right)} = \frac{1+e^{-N0_{gd}\cdot T_{0\_}\cdot\omega\cdot j}}{1-e^{-T_{0\_}\cdot\omega\cdot j}}$

$$\mathcal{F}\left\{\mathbf{III}_{\varepsilon}\left(t,T_{0\_},\varepsilon_{gd}\right)\right\} = \frac{2}{\omega\cdot\varepsilon_{gd}}\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot\frac{1+e^{-N0_{gd}\cdot T_{0\_}\cdot\omega\cdot j}}{1-e^{-T_{0\_}\cdot\omega\cdot j}}$$

for $\omega \neq 0 \qquad F_{\Delta\varepsilon}(\omega) := \frac{2}{\omega\cdot\varepsilon_{gd}}\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot\frac{1+e^{-N0_{gd}\cdot T_{0\_}\cdot\omega\cdot j}}{1-e^{-T_{0\_}\cdot\omega\cdot j}}$

$T_{0\_} := T_{0\_} \quad \delta1 := 2 \quad \varepsilon_{gd} := \varepsilon_{gd} \quad \omega := \omega \quad V_{i\_} := V_{i\_} \quad N0_{gd} := N0_{gd}$

$$\lim_{\varepsilon_{gd}\to 0}\left(\frac{2}{\omega\cdot\varepsilon_{gd}}\cdot\sin\left(\frac{\omega\cdot\varepsilon_{gd}}{2}\right)\cdot\frac{1+e^{-N0_{gd}\cdot T_{0\_}\cdot\omega\cdot j}}{1-e^{-T_{0\_}\cdot\omega\cdot j}}\right)\ \text{simplify} \rightarrow -\frac{e^{-N0_{gd}\cdot T_{0\_}\cdot\omega\cdot j}+1}{e^{-T_{0\_}\cdot\omega\cdot j}-1}$$

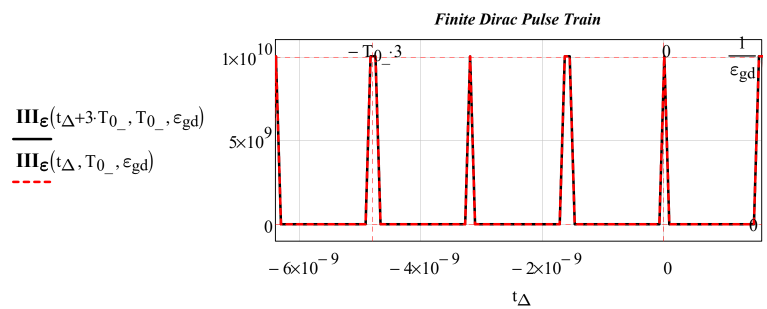
$$F_{\Delta}(\omega) := \frac{e^{-N0_{gd}\cdot T_{0\_}\cdot\omega\cdot j}+1}{1-e^{-T_{0\_}\cdot\omega\cdot j}}$$

$$\mathcal{F}\left\{\mathbf{III}(t)\right\} = \frac{1+e^{-N0_{gd}\cdot T_{0\_}\cdot\omega\cdot j}}{1-e^{-T_{0\_}\cdot\omega\cdot j}}$$

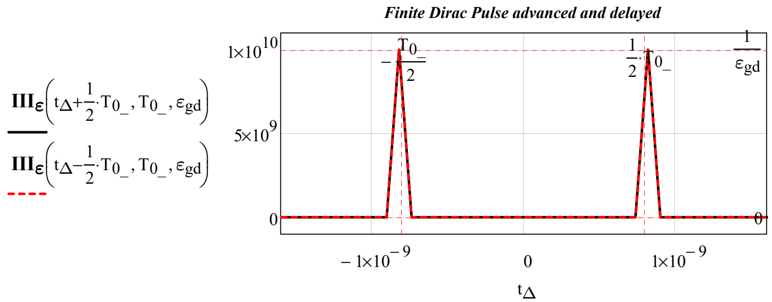
Collection of formulas:

$$\mathbf{III}(-t) = \mathbf{III}(t)$$

$$\mathbf{III}\left(t+k\cdot T_{0\_}\right) = \mathbf{III}(t) \qquad k \text{ integer}$$



$$\mathbf{III}\left(t-\frac{1}{2}\cdot T_{0\_}\right) = \mathbf{III}\left(t+\frac{1}{2}\cdot T_{0\_}\right)$$



$$\int_{\left(k-\frac{1}{2}\right)\cdot T_{0\_}}^{\left(k+\frac{1}{2}\right)\cdot T_{0\_}} \mathbf{III}(t)\,dt = 1$$

$$\mathbf{III}(t) = \sum_{k=-\infty}^{\infty} \delta(t-k\cdot T) = 0 \quad \text{if } t \neq k\cdot T$$

$$\mathbf{III}(a\cdot t) = \frac{1}{|a|}\cdot\sum_{k=-\infty}^{\infty}\delta\left(t-k\cdot\frac{T}{a}\right)$$

$$f(t)\cdot \mathbf{III}(t) = \sum_{k=-\infty}^{\infty}(f(k\cdot T)\cdot\delta(t-k\cdot T))$$

$$\frac{f(t)}{\tau}\cdot\mathbf{III}\left(\frac{t}{\tau}\right)=\sum_{k} (f(k\cdot\tau)\cdot\delta(t-k\cdot\tau))$$ τ is the sampling interval.

Convolution:

$$f * g = \int_0^x f(\tau)\cdot g(x-\tau)\,d\tau$$

$$conv(g(t),f(t)) = \int_0^t f(\xi_{gd})\cdot g(t-\xi_{gd})\,d\xi_{gd} = \int_0^t g(\xi_{gd})\cdot f(t-\xi_{gd})\,d\xi_{gd}$$

$$f * g = conv(s(t),g(t)) = \int_0^t s(\xi_{gd})\cdot g(t-\xi_{gd})\,d\xi_{gd} = \mathcal{F}^{-1}(S(\omega)\cdot G(\omega)) = \mathcal{F}^{-1}(\mathcal{F}(s(t))\cdot\mathcal{F}(g(t)))$$

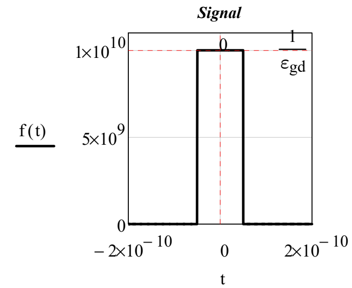
Replicating spectrum at interval τ^{-1}

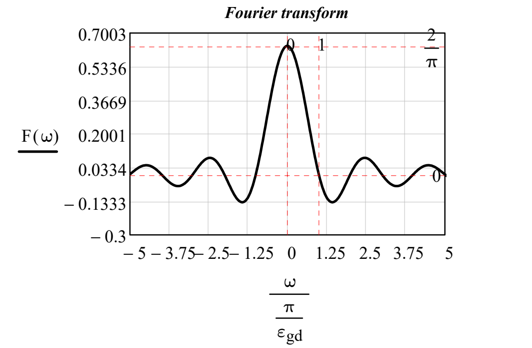
$$\mathcal{F}\left\{f(t)\cdot\mathbf{III}\left(\frac{t}{\tau}\right)\right\} = conv(\tau\cdot\mathbf{III}(\tau\cdot\omega),F(\omega)) = \int_0^\infty \tau\cdot\mathbf{III}(\tau\cdot x)\cdot F(\omega-x)\,dx$$

$$\tau\cdot\sum_{k=-\infty}^{\infty}\int_0^\infty \delta(\tau\cdot x-k\cdot\tau\cdot\omega)\cdot F(\omega-x)\,dx = \tau\cdot\sum_{k=-\infty}^{\infty}\int_0^\omega\delta(\tau\cdot\Omega-k\cdot\tau\cdot\Omega_0)\cdot F(\Omega_0-\Omega)\,d\Omega$$

Example: $f(t) := \delta_\varepsilon(t,\varepsilon_{gd})$ $\quad F(\omega) := -\frac{e^{-\omega\cdot\varepsilon_{gd}\cdot j}}{\varepsilon_{gd}}\cdot\left[\frac{j\cdot\left(e^{2i\cdot\omega\cdot\varepsilon_{gd}}-1\right)}{\pi\cdot\omega}\right]$ $\quad |F(\omega)| = \frac{2\cdot\sin(\omega\cdot\varepsilon_{gd})}{\pi\cdot(\omega\cdot\varepsilon_{gd})}$

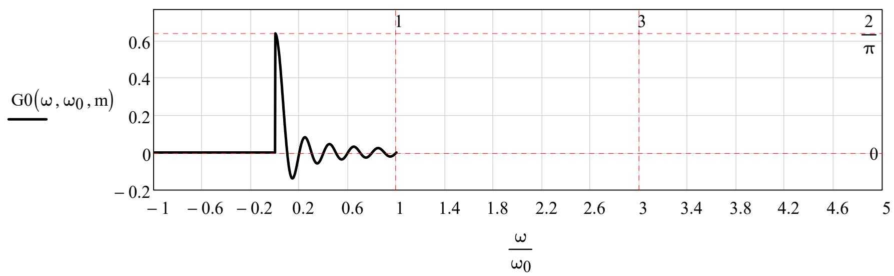
$$\mathcal{F}\left\{\delta_\varepsilon(t,\varepsilon_{gd})\cdot\mathbf{III}\left(\frac{t}{\tau}\right)\right\} = conv1\left[\tau\cdot\mathbf{III}(\tau\cdot\omega), \frac{e^{-(\omega\cdot\varepsilon_{gd})\cdot j}}{\varepsilon_{gd}}\cdot\left[\frac{j\cdot\left(e^{2i\cdot\omega\cdot\varepsilon_{gd}}-1\right)}{\pi\cdot\omega}\right]\right]$$

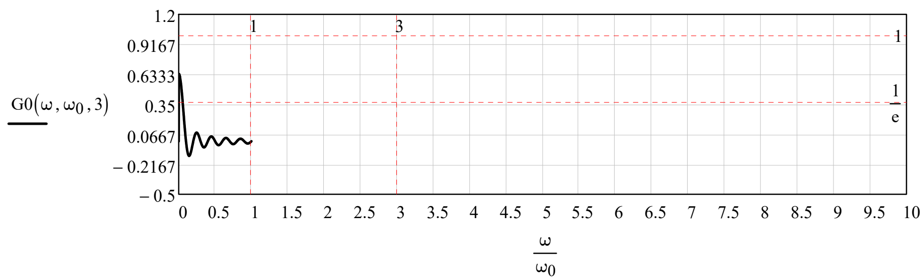


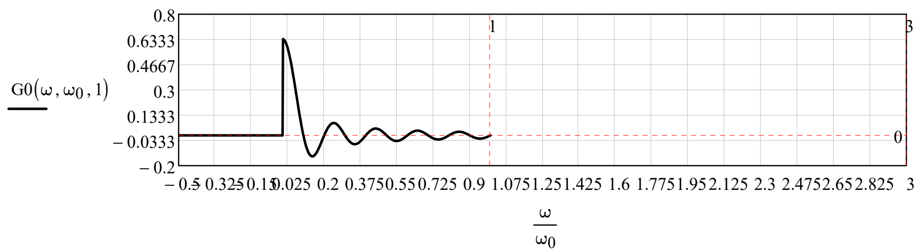


$$\omega_0 := 10\cdot\frac{\pi}{\varepsilon_{gd}}$$

$m := 1$ $\quad G0(\omega,\tau,m) := \sum_{k=0}^{100}\left(F(\omega-k\cdot m\cdot\omega_0)\cdot\Phi(\omega-k\cdot m\cdot\omega_0)\right)$







Sampling theorem: $s(t) = \sum_{n=-\infty}^{\infty} \left[s\left(n \cdot \frac{\pi}{\omega_1}\right) \cdot \left(\frac{\sin(\omega_1 \cdot t - n \cdot \pi)}{\omega_1 \cdot t - n \cdot \pi} \right) \right]$ s(t) is bandlimited

Instants at which the signal must be sampled: $t_n = n \cdot \frac{\pi}{\omega_1} = n \cdot \frac{1}{2 \cdot f_1}$

Sampling period: $T_s = (n+1) \cdot \frac{1}{2 \cdot f_1} - n \cdot \frac{1}{2 \cdot f_1} = \frac{1}{2 \cdot f_1}$

Sampling frequency: $f_s \ge 2 \cdot f_1$

Sampling function $\mathrm{sinc}(\omega_1 \cdot t - n \cdot \pi) = \frac{\sin(\omega_1 \cdot t - n \cdot \pi)}{\omega_1 \cdot t - n \cdot \pi}$

sinc is a Mathcad function, it returns the value of sin(z)/z, with correct behavior in the limit as z approaches 0.

$$s(t) = \sum_{n=-\infty}^{\infty} \left(s\left(n \cdot \frac{\pi}{\omega_1}\right) \cdot \mathrm{sinc}(\omega_1 \cdot t - n \cdot \pi) \right)$$

s(t) is bandlimited

(Orthogonal function expansion)

Frequency domain sampling: f(t) is time limited, namely: $f(t) \ne 0$, for $. -T \le t \le T$,

$$s_p(t) = \frac{1}{2 \cdot T} \cdot \sum_{n=-\infty}^{\infty} \left(S(n \cdot \omega_1) \cdot e^{j \cdot \omega_1 \cdot n \cdot t} \right)$$

$$S(\omega) = \sum_{n=-\infty}^{\infty} \left(S\left(n \cdot \frac{2 \cdot \pi}{T}\right) \cdot \mathrm{sinc}(\omega \cdot T - n \cdot \pi) \right)$$

Frequencies at which the spectrum should be sampled: $\omega_n = n \cdot \omega_1$

$$\mathbf{III}(t) = \pi \cdot \delta(\sin(\pi \cdot t)) \qquad \omega = \pi \qquad f = \frac{1}{2}$$

$$\delta(\sin(t)) = \frac{1}{\pi} \cdot \mathbf{III}\left(\frac{t}{\pi}\right)$$

$$\mathbf{III}_{2\varepsilon}(u, v) := \mathbf{III}_{\varepsilon}(u, T_{0\_}, \varepsilon_{gd}) \cdot \mathbf{III}_{\varepsilon}(v, T_{0\_}, \varepsilon_{gd})$$

13 The Dirac Pulse in two dimensions

Cartesian coordinates:

$$\delta_2(x,y) \equiv \left(\begin{array}{|l} 0 \ \text{ if } x^2+y^2 \neq 0 \\ \infty \ \text{ if } x^2+y^2 = 0 \end{array} \right)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_2(x,y)\,dx\,dy = 1$$

$$\delta_2(x,y) = \delta(x)\cdot\delta(y)$$

$$\delta_2(a\cdot x, b\cdot y) = \frac{1}{|a\cdot b|}\cdot\delta_2(x,y)$$

$$\delta_2(x,y) = \frac{\delta(x)+\delta(y)}{\sqrt{x^2+y^2}}$$

Polar coordinates: $x = r\cdot\cos(\theta)$ $y = r\cdot\sin(\theta)$ $r^2 = x^2+y^2$ $\delta_2(x,y) = \frac{\delta(r)}{\pi\cdot r}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_2(x,y)\,dx\,dy = \frac{1}{\pi}\cdot\int_{0}^{\pi}\int_{0}^{\infty}\delta(r)\,dr\,d\theta = 1$$

$$\mathbf{III_2}(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta_2(x-m, y-n)$$

$\nu := 10^1$ $\varepsilon_x := 0.1$ $\varepsilon_y := 0.1$

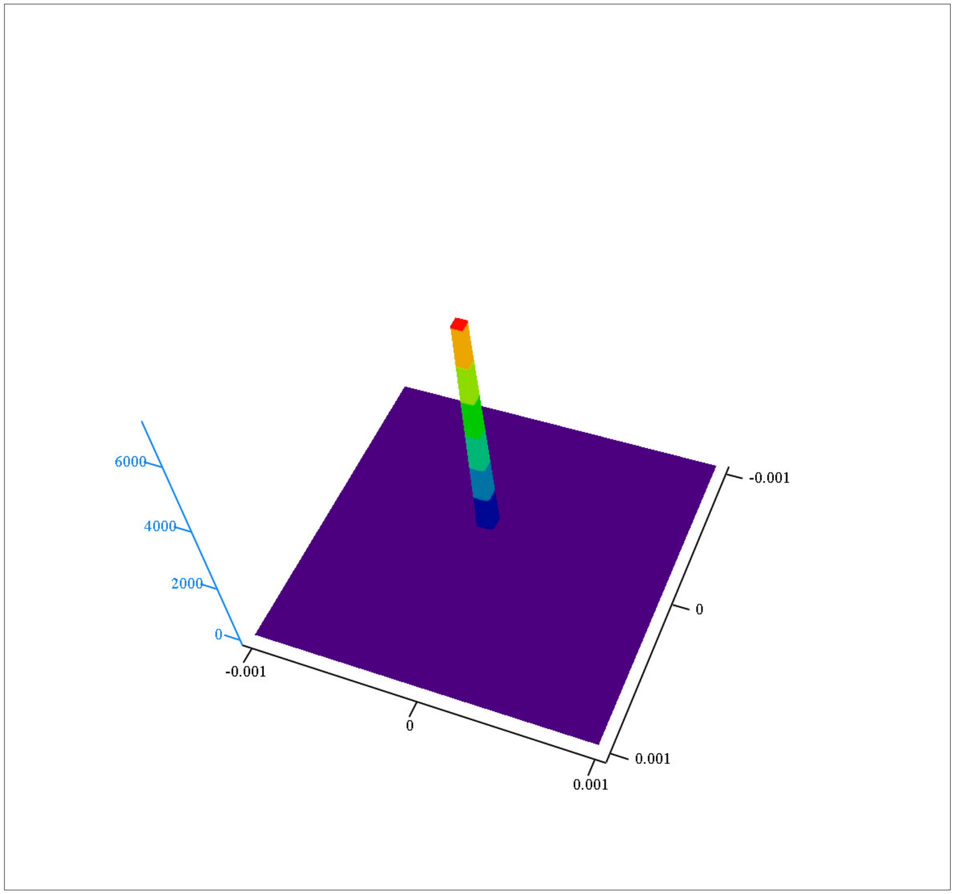
$\varepsilon := 10^{-4}$ $\nu\cdot\varepsilon = 1\times 10^{-3}$

$$\delta_\varepsilon(t,\varepsilon_{gd}) := \begin{array}{|l} \frac{1}{\varepsilon_{gd}} \ \text{ if } \frac{-\varepsilon_{gd}}{2} \le t \le \frac{\varepsilon_{gd}}{2} \\ 0 \ \text{ otherwise} \end{array}$$

$$\delta_{2\varepsilon}(x,y) := \begin{array}{|l} \varepsilon_x \leftarrow \varepsilon \\ \varepsilon_y \leftarrow \varepsilon \\ \frac{1}{\sqrt{\varepsilon_x^2+\varepsilon_y^2}} \ \text{ if } \frac{-\varepsilon_x}{2} \le x \le \frac{\varepsilon_x}{2} \wedge \frac{-\varepsilon_y}{2} \le y \le \frac{\varepsilon_y}{2} \\ 0 \ \text{ otherwise} \end{array}$$

Bound := 100

$$\delta_2 := \text{CreateMesh}(\delta_{2\varepsilon}, -\nu\cdot\varepsilon, \nu\cdot\varepsilon, -\nu\cdot\varepsilon, \nu\cdot\varepsilon, \text{Bound}, \text{Bound})$$

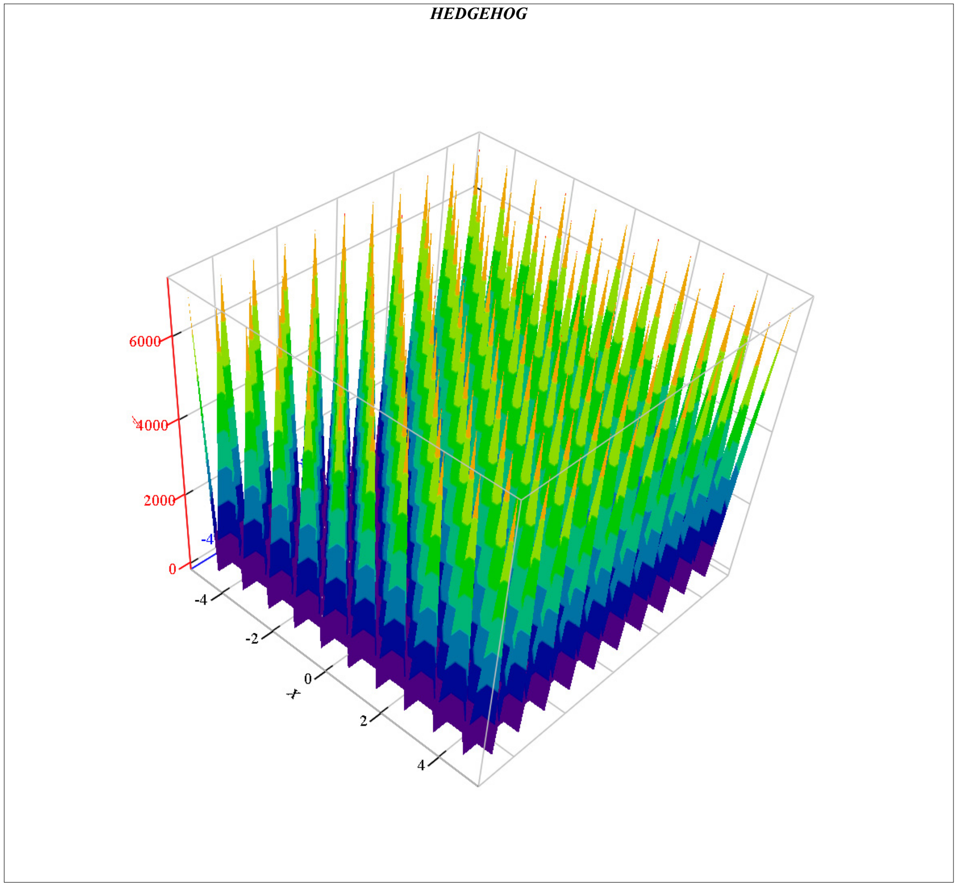


δ_2

$N0_{gd} := 10$

$sp := 1 \qquad \mathbf{III}_{2\varepsilon}(x,y) := \sum_{m=-N0_{gd}}^{N0_{gd}} \sum_{n=-N0_{gd}}^{N0_{gd}} \delta_{2\varepsilon}(x - m \cdot sp, y - n \cdot sp) \qquad N0_{gd} = 10$

$$\delta_{2HD} := \mathrm{CreateMesh}(\delta_{2\varepsilon}, -1.5, 1.5, -1.5, 1.5, \mathrm{Bound}, \mathrm{Bound})$$



$\mathbf{III}_{2\varepsilon}$

14 The Dirac Pulse in three dimensions

$$\delta_3(x,y,z) := \begin{cases} 0 & \text{if } x^2 + y^2 + z^2 \neq 0 \\ \infty & \text{if } x^2 + y^2 + z^2 = 0 \end{cases} \qquad [2][4][5]$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_3(x,y,z)\,dx\,dy\,dz = 1$$

$$\delta_3(x,y,z) = \delta(x)\cdot\delta(y)\cdot\delta(z) = \delta_2(x\cdot y)\cdot\delta(z)$$

$$\mathbf{III}_{2\varepsilon}(x,y) = \mathbf{III}_{\varepsilon}(x)\cdot\mathbf{III}_{\varepsilon}(y)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot\mathbf{III}_2(x,y)\,dx\,dy = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(m,n)$$

Cylindrical coordinates: $r^2 = x^2 + y^2$

$$\delta_3(x,y,z) = \frac{\delta(r)\cdot\delta(z)}{\pi\cdot|r|}$$

Spherical coordinates: $\rho^2 = x^2 + y^2 + z^2$ [2][4][5]

$$\delta_3(x,y,z) = \frac{\delta(\rho)}{2\cdot\pi\cdot\rho^2}$$

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