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ADVANCED MATHEMETICS

DIFFERENCE EQUATIONS

BY F. M.

The aim of this work is to give a simple and concise (but not exhaustive) presentation of the argument to all who are interested in it. I've created this worksheet to implement digital filters (IIR) similar to the classics ones for DSP applications. I assume that the reader already knows both the concepts and the use of the Laplace transform in the study of linear and invariant lumped electrical circuits and everything about the z-transform.

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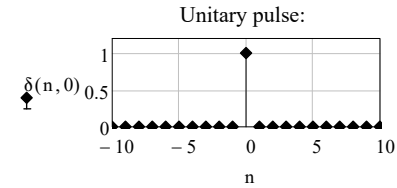
INTRODUCTION

The most common signals used when dealing with signal processing are those here illustrated:

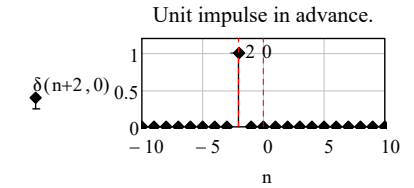
$$\text{Unitary pulse: } \delta(\nu, k) = \begin{cases} 1 & \text{if } \nu = k \\ 0 & \text{otherwise} \end{cases}, \quad (1.1)$$

which is an abstract mathematical entity.

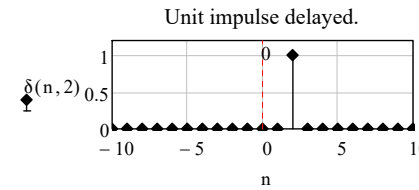
In electronics I use very short voltage pulses of limited amplitude in agreement with the DSP data.



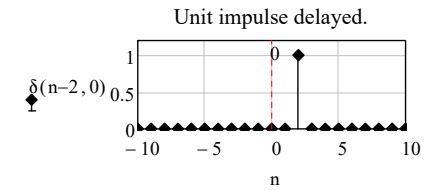
(fig. 1.0.1)



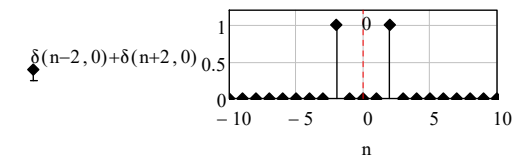
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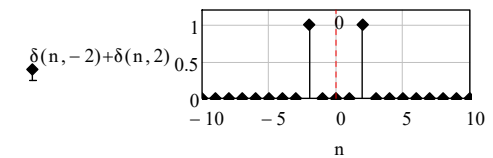
(fig. 1.0.3)



(fig. 1.0.4)



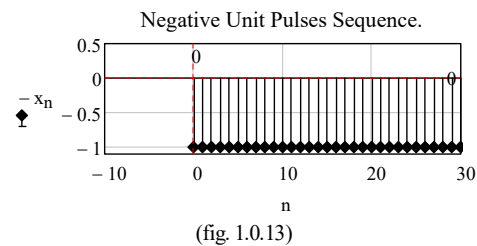
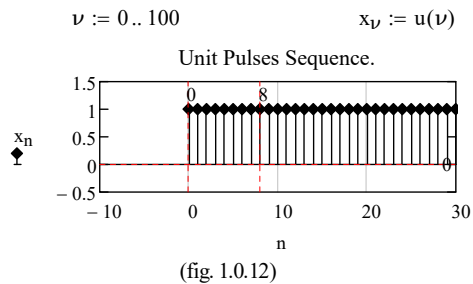
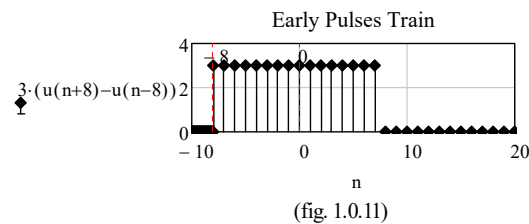
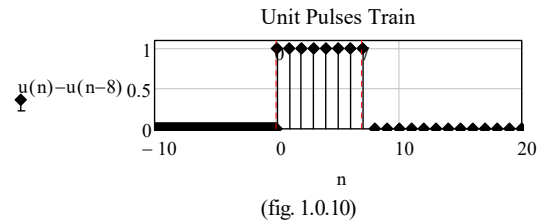
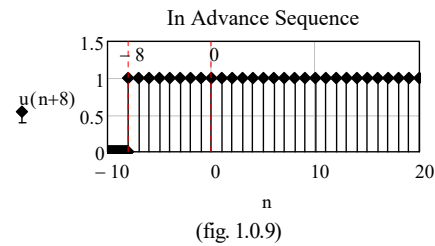
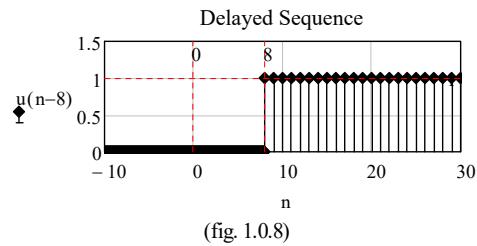
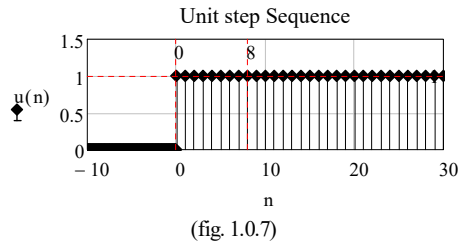
(fig. 1.0.5)



(fig. 1.0.6)

$$\text{Unitary step: } u(n) := \begin{cases} \sum_{k=0}^n \delta(n, k) & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad (1.2)$$

$$\text{Generic sequence: } x_0(n) = \sum_{k=-\infty}^{\infty} (x(k) \cdot \delta(k, n-k)) \quad (1.3)$$



§1.1) The Z transform of a generic causal sequence h_n ,

(non causal: unitary pulse response $h_n \neq 0$ for $n < 0$) is so defined:

$$H(z) = \int_{-\infty}^{\infty} \phi(t) \cdot z^{-t} dt = \sum_{n=0}^{\infty} (h_n \cdot z^{-n}), \quad (1.1.1)$$

where: $\phi(t) = \sum_{n=0}^{\infty} (h(n) \cdot \Delta(t-n))$ and $h(n)$ are the sample values of $h(t)$ at $t=0,1,2,\dots,n$, and $\Delta(t)$ is the well known

Dirac pulse.

Example 1): $h(n) = \frac{4}{5} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots \right)$,

(geometric series)

z transform: $H(z) = \sum_{n=0}^{\infty} (h_n \cdot z^{-n}) = \frac{4}{5} \cdot \left(1 + \frac{z^{-1}}{2} + \frac{z^{-2}}{4} + \frac{z^{-3}}{8} + \dots + \frac{z^{-n}}{2^n} + \dots \right)$, $q = \frac{z^{-1}}{2}$

$$H(z) = \frac{4}{5} \cdot \sum_{n=0}^{\infty} \left(\frac{z^{-n}}{2^n} \right) = \frac{4}{5} \cdot \lim_{N \rightarrow \infty} \frac{1 - \left(\frac{z^{-1}}{2} \right)^N}{1 - \frac{z^{-1}}{2}}$$

$$H(z) = \frac{4}{5} \cdot \frac{2}{2 - z^{-1}}$$

Example 2): $h(n) = \frac{4}{5} \cdot (1 + 0.75 + 0.75^2 + 0.75^3 + \dots + 0.75^n + \dots)$,

$$q = \frac{3}{4} \cdot z^{-1}, \quad H(z) = \frac{4}{5} \cdot \sum_{n=0}^{\infty} \left[\left(\frac{3}{4} \right)^n \cdot z^{-n} \right]$$

$$\frac{4}{5} \cdot \sum_{n=0}^{\infty} \left[\left(\frac{3}{4} \right)^n \cdot z^{-n} \right] = \frac{4}{5} \cdot \lim_{N \rightarrow \infty} \frac{1 - (0.75 \cdot z^{-1})^N}{1 - 0.75 \cdot z^{-1}}$$

$$\lim_{N \rightarrow \infty} \frac{1 - (0.75 \cdot z^{-1})^N}{1 - 0.75 \cdot z^{-1}} = \frac{4}{5} \cdot \frac{1}{\frac{3}{4} \cdot z^{-1} - 1}$$

$$H(z) = \frac{16 \cdot z}{20 \cdot z - 15} = \frac{4}{5} \cdot \frac{4}{4 - 3 \cdot z^{-1}}$$

Example 3): $h(n) = \frac{4}{5} \cdot \left[1 + 0.75 + 0.75^2 + 0.75^3 + \dots + 0.75^n - \left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} \right) \right]$,

$$H(z) = \frac{4}{5} \cdot \left[\sum_{n=0}^{\infty} \left[\left(\frac{3}{4} \right)^n \cdot z^{-n} \right] + \sum_{n=0}^{\infty} \left(\frac{z^{-n}}{2^n} \right) \right]$$

for the linearity of the z transform, namely

$$Z(a \cdot x1_n + b \cdot x2_n) = a \cdot Y1(z) + b \cdot Y2(z),$$

I can write:

$$H(z) = \frac{4}{5} \cdot \left(\frac{4}{4-3 \cdot z^{-1}} - \frac{2}{2-z^{-1}} \right) = \frac{4}{5} \cdot \frac{2 \cdot z^{-1}}{3 \cdot z^{-2} - 10 \cdot z^{-1} + 8}$$

$$H(z) = \frac{4}{5} \cdot \frac{2 \cdot z^{-1}}{3 \cdot z^{-2} - 10 \cdot z^{-1} + 8}$$

In this worksheet and all others I will use the operator **ztrans**.

Examples of Z transform:

$$1) \quad n \text{ ztrans} \rightarrow \frac{z}{(z-1)^2},$$

$$2) \quad \frac{1}{2^n} \text{ ztrans} \rightarrow \frac{2 \cdot z}{2 \cdot z - 1},$$

$$3) \quad k := k \quad k \cdot n \text{ ztrans} \rightarrow \frac{k \cdot z}{(z-1)^2},$$

$$4) \quad k := k, k \cdot n^6 \left| \begin{array}{l} \text{ztrans} \\ \text{collect, k} \end{array} \right. \rightarrow \frac{z^6 + 57 \cdot z^5 + 302 \cdot z^4 + 302 \cdot z^3 + 57 \cdot z^2 + z}{(z-1)^7} \cdot k,$$

$$5) \quad k \cdot (-n + n^2 - n^3 + n^4) \text{ ztrans} \rightarrow \frac{2 \cdot k \cdot z (5 \cdot z^2 + 5 \cdot z + 2)}{(z-1)^5},$$

$$6) \quad a \cdot (n-k) + b \cdot (n+k) \left| \begin{array}{l} \text{ztrans} \\ \text{collect, z, k} \end{array} \right. \rightarrow \frac{z[a+b+k \cdot (a-b)] - k \cdot z^2 \cdot (a-b)}{z^2 - 2 \cdot z + 1},$$

$$7) \quad a \cdot (n-k+m) + b \cdot (n+k+m) \left| \begin{array}{l} \text{ztrans} \\ \text{simplify} \\ \text{collect, k, m, z} \end{array} \right. \rightarrow \left[-\frac{z(a-b)}{z-1} \right] \cdot k + \frac{z(a+b)}{z-1} \cdot m + \frac{z(a+b)}{z^2 - 2 \cdot z + 1}.$$

$$8) \quad \sin(\omega \cdot n) \text{ ztrans} \rightarrow \frac{z \sin(\omega)}{z^2 - 2 \cdot \cos(\omega) \cdot z + 1},$$

$$9) \quad \cos(\omega \cdot n) \text{ ztrans} \rightarrow \frac{z(z - \cos(\omega))}{z^2 - 2 \cdot \cos(\omega) \cdot z + 1},$$

$$10) \quad e^{-(\omega_1 \cdot n)} \cdot \sin(\omega_1 \cdot n) \left| \begin{array}{l} \text{ztrans} \\ \text{simplify} \\ \text{factor} \end{array} \right. \rightarrow \frac{z \cdot e^{\omega_1} \cdot \sin(\omega_1)}{e^{2 \cdot \omega_1} \cdot z^2 - 2 \cdot e^{\omega_1} \cdot \cos(\omega_1) \cdot z + 1},$$

$$11) \quad k \cdot a^n \text{ ztrans} \rightarrow -\frac{k \cdot z}{a-z}$$

$$12) \quad k \cdot e^{-a \cdot n} \text{ ztrans} \rightarrow \frac{k \cdot z}{z - e^{-a}}$$

$$13) \quad k \cdot n \cdot e^{-a \cdot n} \text{ ztrans} \rightarrow \frac{k \cdot z \cdot e^a}{(z e^a - 1)^2},$$

$$14) \quad 1 - e^{a \cdot n} \text{ ztrans} \rightarrow -\frac{z(e^a - 1)}{(z - e^a) \cdot (z - 1)},$$

$$15) \quad k \cdot n \cdot \alpha^n \text{ ztrans} \rightarrow \frac{\alpha \cdot k \cdot z}{(\alpha - z)^2},$$

$$16) \quad \sinh(\alpha \cdot n) \left| \begin{array}{l} \text{ztrans} \\ \text{simplify} \\ \text{collect, } e^\alpha, z \end{array} \right. \rightarrow \frac{z(e^{2 \cdot \alpha} - 1)}{e^\alpha \cdot (2 \cdot z^2 + 2) - z(2 \cdot e^{2 \cdot \alpha} + 2)},$$

$$17) \quad \cosh(\alpha \cdot n) \left| \begin{array}{l} \text{ztrans} \\ \text{simplify} \end{array} \right. \rightarrow \frac{z \cosh(\alpha) - z^2}{z^2 - 2 \cdot \cosh(\alpha) \cdot z + 1},$$

$$18) \quad (-1)^{n+1} \cdot \frac{a^n}{n} \left| \begin{array}{l} \text{ztrans} \\ \text{simplify} \end{array} \right. \rightarrow -\text{ztrans}\left(\frac{1}{n}, n, -\frac{z}{a}\right) \log(1 + a \cdot z^{-1}) \quad |a| < |z|,$$

$$19) \quad \frac{a \cdot (a^n + b^n)}{a-b} \text{ ztrans} \rightarrow \frac{a \cdot z(a+b-2 \cdot z)}{(a-b) \cdot (a-z) \cdot (b-z)},$$

$$20) \quad \frac{a \cdot (a^n - b^n)}{a-b} \text{ ztrans} \rightarrow \frac{a \cdot z}{(a-z) \cdot (b-z)}$$

$$21) \quad \frac{n}{2 \cdot n + 1} \text{ ztrans} \rightarrow \begin{cases} \frac{z + \sqrt{z} \cdot \operatorname{atanh}\left(\frac{-1}{z^2}\right) - z^{\frac{3}{2}} \cdot \operatorname{atanh}\left(\frac{-1}{z^2}\right)}{2 \cdot z - 2} & \text{if } 1 < |z| \\ \text{undefined otherwise} \end{cases}$$

§1.2) The inverse Z transform of H(z)

It is a contour integration of $F(z) = z^{n-1}H(z)$ on a counterclockwise contour Γ traced on the Gauss plane, enclosing all the poles of $F(z)$:

$$h_n = \frac{1}{2 \cdot \pi \cdot j} \oint_{\Gamma} z^{n-1} H(z) d\xi, \quad (1.2.1)$$

Expand to see some classical examples of contour integration

§1.2.1) RESIDUES METHOD

The *Laurent* series of the function $F(z)$ about a point z_0 , holomorphic in $\tau_1 = \tau \cup \sigma_1 \cup -\sigma_2$, as known, is given by:

$$F(z) = \sum_{k=-\infty}^{\infty} [c_k \cdot (z - z_0)^k],$$

$$c_k = \frac{1}{2 \cdot \pi \cdot j} \oint_{\Gamma} \frac{F(\xi)}{(\xi - z_0)^{k+1}} d\xi,$$

the first few terms of the series are:

$$F(z) = \sum_{k=-\infty}^{\infty} [c_k \cdot (z - z_i)^k] = c_0 + c_1 \cdot (z - z_i) + c_2 \cdot (z - z_i)^2 + \dots + \frac{c_{-1}}{(z - z_i)} + \frac{c_{-2}}{(z - z_i)^2} + \dots$$

The contour integral of $F(z)$ then is:

$$\oint_{\Gamma} F(z) dz = \sum_{k=1}^N \oint_{\Gamma_k} F(z_i) dz = 2 \cdot \pi \cdot j \cdot \sum_{k=0}^{P-1} (\text{Res}(F(z_i))).$$

$$\text{Residue for the pole } z_i: c_{-1} = \frac{1}{2 \cdot \pi \cdot j} \oint_{\sigma} \frac{F(\xi)}{(\xi - z_i)^0} d\xi = \frac{1}{2 \cdot \pi \cdot j} \oint_{\sigma} F(\xi) d\xi = \text{Res}(F(z_i)),$$

so that one can write:

$$\frac{1}{2 \cdot \pi \cdot j} \oint_{\sigma} F(z) dz = \text{Res}(F(z_i))$$

For a *rational function* $F(z, n) = z^{n-1}H(z) = \frac{A(z)}{B(z)}$ whose poles are known, the integral can be analytically calculated with the residues method. Stated that " m_k " is the order of a single pole z_k , " μ " the number of such poles and P is the total number of poles:

the contour integral:
$$h_n = \frac{1}{2 \cdot \pi \cdot j} \oint_{\Gamma} F(z, n) dz = \sum_{k=0}^{P-1} (\text{Res}(F(z, n))), \quad (1.2.1.1)$$

If $F(z, n)$ is a rational function with P poles of order m_k , with $\sum_{k=0}^{P-1} m_k = P$,

$$F(z, n) = \frac{A(z)}{B(z)} = \frac{A(z)}{(z - p_0)^{m_0} \cdot (z - p_1)^{m_1} \cdot \dots \cdot (z - p_{q-1})^{m_{p-1}}}, \quad (1.2.1.2)$$

the theorem gives the residue for the pole p_k of order m_k :

$$\text{Res}(F(z, n)) = \lim_{z \rightarrow p_k} \left[\frac{1}{(m_k - 1)!} \cdot \frac{\partial^{m_k-1}}{\partial z^{m_k-1}} [(z - p_k)^{m_k} \cdot (F(z, n))] \right], \quad (1.2.1.3)$$

(If the system has 6 poles, for example, with 1 one of third order and the other (3) of first order, the sum of the residues is composed by the sum of the residues of the poles of the first order plus the residue of the pole of third order, namely it results the sum of four residues.)

For simple poles ($m_k=1, \forall k$) I have:

$$\sum_{k=0}^{P-1} (\text{Res}(F(z, n))) = \sum_{k=0}^{P-1} \left[\lim_{z \rightarrow p_k} [(z - p_k) \cdot (F(z, n))] \right], \quad (1.2.1.4)$$

If $F(z, n)$ is a rational function:
$$F(z, n) = \frac{A(z)}{B(z)} = \frac{A(z)}{(z - p_1) \cdot (z - p_2) \cdot \dots \cdot (z - p_n)} \quad (1.2.1.5)$$

with all simple poles, then:

$$h_n = \sum_{k=0}^{P-1} \lim_{z \rightarrow p_k} \frac{A(z)}{\frac{\partial}{\partial z} B(z)} \quad (1.2.1.6)$$

Expand to see some examples

Example 1) two simple poles

I compute the inverse Z transform of the given function:
$$H_3(z) := \frac{z}{(z - 0.75) \cdot (z + 0.5)}, \quad (1.2.1.7)$$

$$F_3(z, n) = z^{n-1} \cdot H(z) \quad (1.2.1.8)$$

$$h_{3n} = \sum_{j=0}^P \left[\lim_{z \rightarrow p_j} [(z - p_j) \cdot (F_3(z, n))] \right] \quad (1.2.1.9)$$

Creation of the coefficient's vector of the polynomial denominator:

$$z := z \quad n := n \quad F_3(z, n) := \frac{z^n}{(z - 0.75) \cdot (z + 0.5)} \quad P = 2 \quad (1.2.1.10)$$

$$v := \text{denom}(F_3(z, n)) \text{ coeffs}, z \rightarrow \begin{pmatrix} -3.0 \\ -2.0 \\ 8.0 \end{pmatrix} \quad (1.2.1.11)$$

Search of the poles of $F_3(z, n)$: $\text{poles3} := \text{polyroots}(v)$,

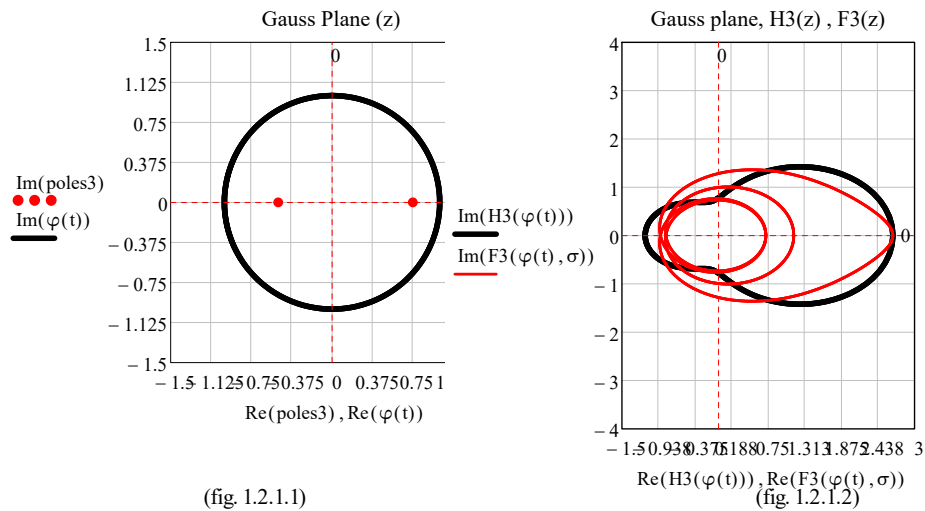
$$\text{poles3}^T = (-0.5 \quad 0.75) \quad p := \text{poles3}$$

$$M1 := 100 \quad h_{3n} = \lim_{z \rightarrow p_0} [(z - p_0) \cdot F_3(z, n)] + \lim_{z \rightarrow p_1} [(z - p_1) \cdot F_3(z, n)] \quad (1.2.1.12)$$

Representation of the contour of integration on the Gaussian plane of the complex variable z .

Radius of the circle that encloses all the poles of $F_3(z, v)$: $r := \text{ceil}(\max(|\text{poles3}|)) \cdot 1.0$, $r = 1$

$$\xi(t) := r \cdot \cos(t) \quad \psi(t) := r \cdot \sin(t) \quad \varphi(t) := \xi(t) + j \cdot \psi(t) \quad \sigma := 6 \quad (1.2.1.13)$$



(fig. 1.2.1.1)

$$h_{3n} = \lim_{z \rightarrow 0.75} \left[(z - 0.75) \cdot z^{n-1} \cdot \frac{z}{(z - 0.75) \cdot (z + 0.5)} \right] + \lim_{z \rightarrow -0.5} \left[(z + 0.5) \cdot z^{n-1} \cdot \frac{z}{(z - 0.75) \cdot (z + 0.5)} \right] \quad (1.2.1.14)$$

Finally I get the sequence sought:

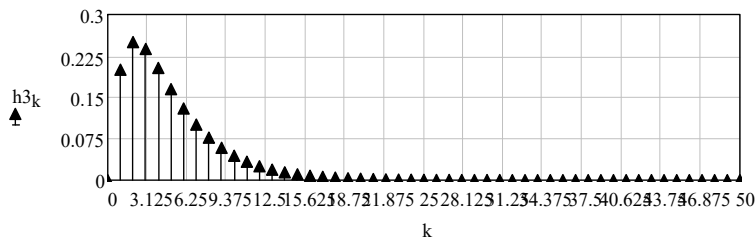
$$h_{3n} = \left(\frac{0.75^n}{0.75 + 0.5} \right) + \left[\frac{(-0.5)^n}{-0.5 - 0.75} \right] \quad (1.2.1.15)$$

$$n := 0..M1 - 1$$

$$h_{3n} := \frac{4}{5} \cdot (0.75^n - 0.5^n) \quad (1.2.1.16)$$

$$h_{3n}^T = [0 \quad 0.2 \quad 0.25 \quad 0.238 \quad 0.203 \quad 0.165 \quad \dots]$$

$$\text{Energy } E := \sum_{k=0}^{\text{rows}(h3)-1} (|h_{3k}|)^2 \quad E = 0.268 \quad (1.2.1.17)$$



(fig. 1.2.1.3)

Example 2) c.c. poles:

I compute the inverse Z transform of the given rational function:

$$\alpha := 0.3561 \quad H4(z) := \frac{1 + 2 \cdot z^{-1} + z^{-2}}{1 - z^{-1} + \alpha \cdot z^{-2}} \quad (1.2.1.18)$$

$$P = 3 \quad F4(z) = z^{n-1} \cdot H4(z) \Rightarrow F4(z, n) := \frac{z^n \cdot (z + 1)^2}{z \cdot (z^2 - z + \alpha)} \quad (1.2.1.19)$$

$$h_{4n} = \sum_{k=0}^2 \left[\lim_{z \rightarrow p_k} [(z - p_k) \cdot F4(z)] \right] \quad (1.2.1.20)$$

$$\alpha := \alpha \quad z := z \quad n := n \quad v4 := \text{denom}(F4(z, n)) \text{ coeffs}, z \rightarrow \begin{pmatrix} 0 \\ 3561.0 \\ -10000.0 \\ 10000.0 \end{pmatrix} \quad (1.2.1.21)$$

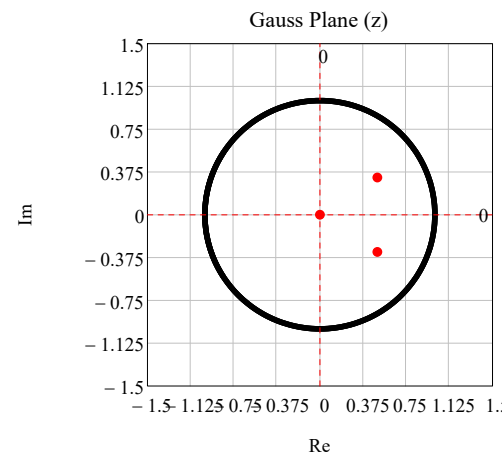
$$\text{Search of the poles of } F4(z, n): \text{poles4} := \text{polyroots}(v4), \quad \text{poles4}^T = (0 \quad 0.5 - 0.326j \quad 0.5 + 0.326j)$$

$$\text{Radius of the circle that encloses all the poles of } F4(z): r4 := \text{ceil}(\max(|\text{poles4}|)) \cdot 1.0, \quad (1.2.1.23)$$

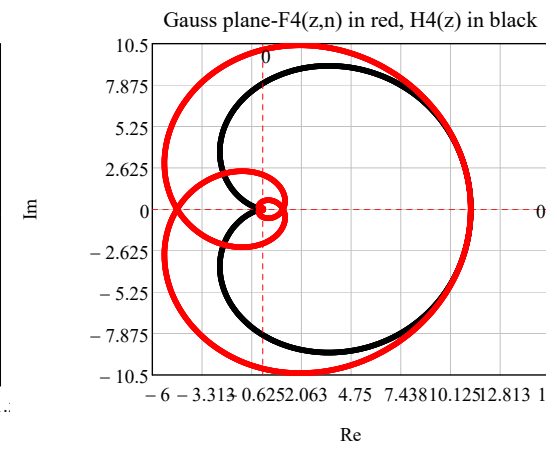
Representation of the contour of integration on the Gaussian plane of the complex variable z.

$$\xi(t) := r4 \cdot \cos(t) \quad \psi(t) := r4 \cdot \sin(t) \quad \varphi(t)(t) := \xi(t) + j \cdot \psi(t) \quad (1.2.1.22)$$

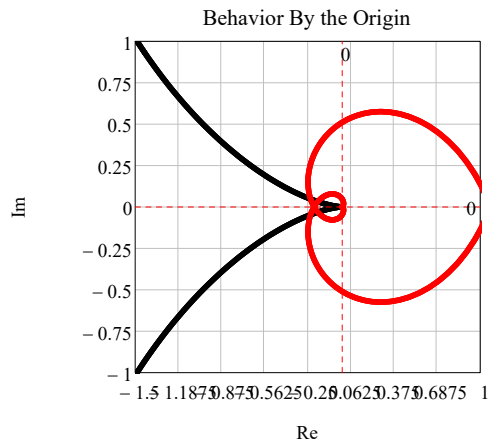
$$t_0 := 0 \quad t_{fin} := 2 \cdot \pi \quad \sigma = 6$$



(fig. 1.2.1.4)



(fig. 1.2.1.5)



(fig. 1.2.1.6)

$$\begin{aligned}
 \text{poles4}_0 &= 0 & \text{poles4}_1 &= 0.5 - 0.326j & \text{poles4}_2 &= 0.5 + 0.326j \\
 \text{p0} &:= \text{poles4}_0 & \text{p1} &:= \frac{\sqrt{1-4\alpha}}{2} + \frac{1}{2} & \text{p2} &:= \frac{1}{2} - \frac{\sqrt{1-4\alpha}}{2} \\
 \text{p0} &= 0 & \text{p1} &= 0.5 + 0.326j & \text{p2} &= 0.5 - 0.326j
 \end{aligned}
 \tag{1.2.1.24}$$

The sum of the residues gives:

$$\begin{aligned}
 h_{4n} &= \lim_{z \rightarrow 0} \left[z \cdot z^{n-1} \cdot \frac{(z+1)^2}{(z-p1) \cdot (z-p2)} \right] \dots \\
 &+ \lim_{z \rightarrow p2} \left[\frac{z^{n-1} \cdot (z+1)^2}{(z-p1)} \right] \dots \\
 &+ \lim_{z \rightarrow p1} \left[\frac{z^{n-1} \cdot (z+1)^2}{(z-p2)} \right]
 \end{aligned}
 \tag{1.2.1.25}$$

$$\lim_{z \rightarrow 0} \left[z \cdot z^{n-1} \cdot \frac{(z+1)^2}{(z-p1) \cdot (z-p2)} \right] = \frac{\delta(n,0)}{p1 \cdot p2}
 \tag{1.2.1.26}$$

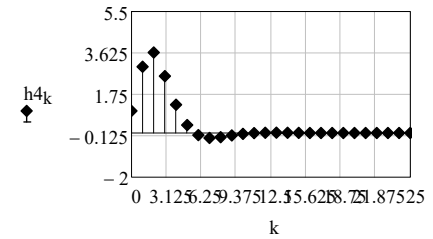
$$\frac{1}{p1 \cdot p2} = \left(\frac{\sqrt{1-4\alpha}}{2} + \frac{1}{2} \right) \cdot \left(\frac{1}{2} - \frac{\sqrt{1-4\alpha}}{2} \right) = \frac{1}{\alpha} \quad \frac{1}{\alpha} = 2.808
 \tag{1.2.1.27}$$

$$\lim_{z \rightarrow p1} \left[\frac{z^{n-1} \cdot (z+1)^2}{(z-p2)} \right] = \frac{p1^{n-1} \cdot (p1+1)^2}{p1-p2} \quad \lim_{z \rightarrow p2} \left[\frac{z^{n-1} \cdot (z+1)^2}{(z-p1)} \right] = \frac{p2^{n-1} \cdot (p2+1)^2}{p2-p1}$$

and finally:

$$h_{4\nu} := \frac{1}{\alpha} \cdot \delta(\nu, 0) + \frac{p1^{\nu-1} \cdot (p1+1)^2}{p1-p2} + \frac{p2^{\nu-1} \cdot (p2+1)^2}{p2-p1}
 \tag{1.2.1.28}$$

h_4^T	0	1	2	3	4	5	
	0	1	3	3.644	2.576	1.278	...



(fig. 1.2.1.7)

Example 3) one second-order pole:

I compute the inverse Z transform of the given rational function :

$$\alpha_2 := 1, \beta_2 := \sqrt{2} \quad H5(z) := \alpha_2 \cdot \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_2)^2}
 \tag{1.2.1.29}$$

$$F5(z, n) = z^{n-1} \cdot H5(z) \Rightarrow F5(z, n) := \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot (\beta_2 \cdot z + 1)^2}
 \tag{1.2.1.30}$$

$$h_{5n} = \text{Res}(F5(z, n), p0) + \text{Res}(F5(z, n), p1)
 \tag{1.2.1.31}$$

$$h_{5n} = \lim_{z \rightarrow 0} \left[z \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot (\beta_2 \cdot z + 1)^2} \right] + \lim_{z \rightarrow p1} \frac{\partial}{\partial z} \left[(z-p1)^2 \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot (\beta_2 \cdot z + 1)^2} \right]
 \tag{1.2.1.32}$$

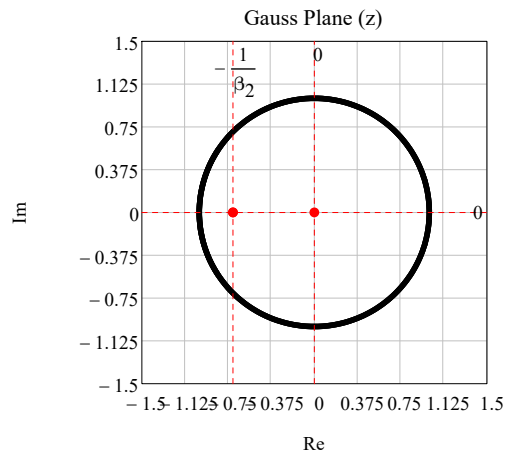
$$\alpha_2 := \alpha_2 \quad \beta_2 := \beta_2$$

Search of the poles of F5(z,n): , poles5 := z (β₂·z + 1)² solve, z →

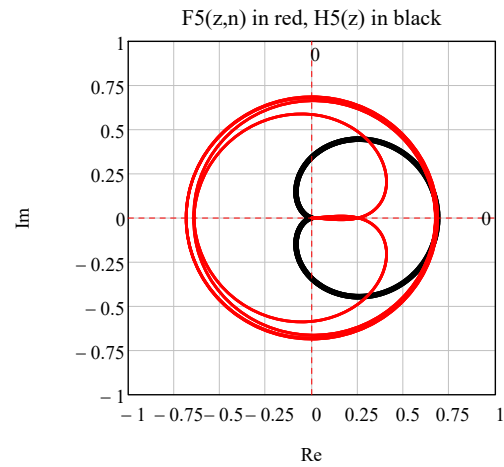
$$\begin{pmatrix} 0 \\ \frac{1}{\beta_2} \\ \frac{1}{\beta_2} \end{pmatrix}
 \tag{1.2.1.33}$$

$$\text{poles5}_0 := 0 \quad \text{poles5}_1 := -\frac{1}{\beta_2} \quad F5(z, n) = \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{(z-p0) \cdot (z-p1)^2}
 \tag{1.2.1.34}$$

The function has a second order pole = $-\frac{1}{\beta_2}$ σ = 6



(fig. 1.2.1.8)



(fig. 1.2.1.9)

$$\text{Res}(F5(z,n),p0) = \lim_{z \rightarrow 0} \left[z \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot (\beta_2 \cdot z + 1)^2} \right] = \lim_{z \rightarrow 0} \left[\frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{(\beta_2 \cdot z + 1)^2} \right] = \alpha_2 \cdot \delta(n,0) \quad (1.2.1.35)$$

$$\text{Res}(F5(z,n),p1) = \lim_{z \rightarrow p1} \frac{\partial}{\partial z} \left[(z-p1)^2 \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot (\beta_2 \cdot z + 1)^2} \right] = \lim_{z \rightarrow p1} \frac{\partial}{\partial z} \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z} \quad (1.2.1.36)$$

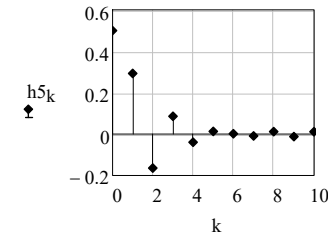
$$n := n$$

$$\frac{\partial}{\partial z} \left[\left(z + \frac{1}{\beta_2} \right)^2 \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{z \cdot (\beta_2 \cdot z + 1)^2} \right] \text{ simplify, max} \rightarrow \frac{\alpha_2 \cdot z^{n-2} \cdot (z+1) \cdot (n+z+n \cdot z-1)}{\beta_2^2} \quad (1.2.1.37)$$

$$\text{Res}(F5(z,n),p1) = \lim_{z \rightarrow p1} \frac{\alpha_2 \cdot z^{n-2} \cdot (z+1) \cdot [n+z \cdot (1+n)-1]}{\beta_2^2} \quad (1.2.1.38)$$

$$\lim_{z \rightarrow -\frac{1}{\beta_2}} \frac{\alpha_2 \cdot z^{n-2} \cdot (z+1) \cdot [n+z \cdot (1+n)-1]}{\beta_2^2} \text{ simplify, max} \rightarrow \frac{\alpha_2 \cdot (\beta_2 - 1) \cdot \left(-\frac{1}{\beta_2} \right)^n \cdot (n + \beta_2 - n \cdot \beta_2 + 1)}{\beta_2^2}$$

$$h5_n := \alpha_2 \cdot \delta(n,0) - \frac{\alpha_2 \cdot (\beta_2 - 1) \cdot \left(-\frac{1}{\beta_2} \right)^n \cdot [n + \beta_2 \cdot (1-n) + 1]}{\beta_2^2} \quad (1.2.1.39)$$



(fig. 1.2.1.10)

$$\alpha_2 \cdot \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_2)^2} \begin{cases} \text{invztrans} \\ \text{simplify} \\ \text{collect, } \left(\frac{1}{\beta_2} \right)^n \end{cases} \rightarrow \frac{\alpha_2 \cdot (n + n \cdot \beta_2^2 - \beta_2^2 - 2 \cdot n \cdot \beta_2 + 1)}{\beta_2^2} \cdot \left(-\frac{1}{\beta_2} \right)^n + \alpha_2 \cdot \delta(n,0) \quad (1.2.1.40)$$

Example 4) one second order pole:

I compute the inverse Z transform of the given rational function :

$$\alpha_2 := 1, \beta_2 := \sqrt{2} \quad H55(z) := \alpha_2 \cdot \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_2)^3}, \quad (1.2.1.41)$$

$$F55(z,n) = z^{n-1} \cdot H55(z) \Rightarrow F55(z,n) := \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{(\beta_2 \cdot z + 1)^3} \quad (1.2.1.42)$$

$$\alpha_2 := \alpha_2 \quad \beta_2 := \beta_2$$

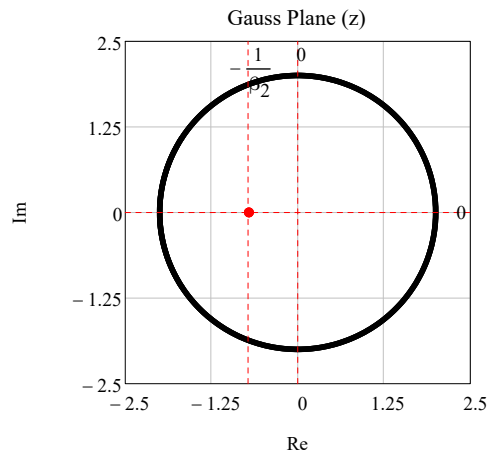
$$\text{Search of the poles of } F55(z,n): \text{ , poles55 := } (\beta_2 \cdot z + 1)^3 \text{ solve, } z \rightarrow \begin{pmatrix} -\frac{1}{\beta_2} \\ -\frac{1}{\beta_2} \\ -\frac{1}{\beta_2} \end{pmatrix}, \quad (1.2.1.43)$$

The function has a third order pole = $-\frac{1}{\beta_2}$

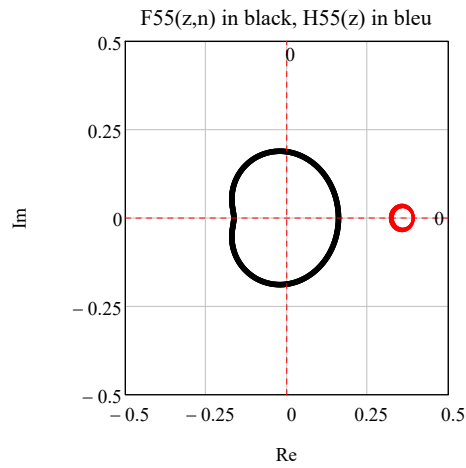
Representation of the contour of integration on the Gaussian plane of the complex variable z.

Radius of the circle enclosing all the poles of F55(z,n): r55 := ceil(max(|poles55|)) - 1.0, r55 = 2

$$\begin{aligned} \xi(t) &:= r55 \cdot \cos(t) & \psi(t) &:= r55 \cdot \sin(t) \\ \varphi(t) &:= \xi(t) + j \cdot \psi(t) & \sigma &:= 0 \end{aligned} \quad (1.2.1.44)$$



(fig. 1.2.1.8)



(fig. 1.2.1.9)

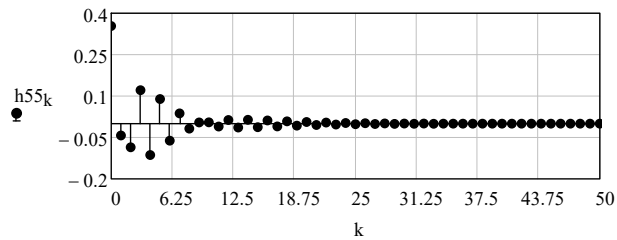
$$h55_n = \frac{1}{2} \cdot \alpha_2 \cdot \left(\frac{1}{\beta_2}\right)^3 \cdot \lim_{z \rightarrow -\frac{1}{\beta_2}} \frac{\partial^2}{\partial z^2} [z^n \cdot (z+1)^2] \quad (1.2.1.46)$$

$$h55_n = \text{Res}(F5(z, n), p0) = \lim_{z \rightarrow -\frac{1}{\beta_2}} \left[\frac{1}{2} \cdot \frac{\partial^2}{\partial z^2} \left[\left(z + \frac{1}{\beta_2}\right)^3 \cdot \frac{\alpha_2 \cdot z^n \cdot (z+1)^2}{(\beta_2 \cdot z + 1)^3} \right] \right] \quad (1.2.1.45)$$

$$\alpha_2 := \alpha_2 \quad \beta_2 := \beta_2 \quad z := z \quad n := n$$

$$h55_n := \frac{1}{2} \cdot \alpha_2 \cdot \left(\frac{1}{\beta_2}\right)^3 \cdot \lim_{z \rightarrow -\frac{1}{\beta_2}} \frac{\partial^2}{\partial z^2} [z^n \cdot (z+1)^2] \quad \left| \begin{array}{l} \text{simplify, max} \\ \text{collect, n} \\ \text{factor} \end{array} \right. \rightarrow \frac{\left(-\frac{1}{\beta_2}\right)^n \cdot \alpha_2 \cdot (n^2 \cdot \beta_2^2 - 2 \cdot n^2 \cdot \beta_2 + n^2 - n \cdot \beta_2^2 - 2 \cdot n \cdot \beta_2 + 3 \cdot n + 2)}{2 \cdot \beta_2^3}$$

$$h55_n := \frac{\alpha_2 \cdot \left(-\frac{1}{\beta_2}\right)^n \cdot [n \cdot (1 - \beta_2) \cdot [(1 - n) \cdot \beta_2 + n + 3] + 2]}{2 \cdot \beta_2^3} \quad (1.2.1.47)$$



(fig. 1.2.1.10)

Verify:

$$\alpha_2 \cdot \frac{(z^{-1} + 1)^2}{(z^{-1} + \beta_2)^3} \quad \left| \begin{array}{l} \text{invztrans} \\ \text{simplify, max} \end{array} \right. \rightarrow \frac{\alpha_2 \cdot \left(-\frac{1}{\beta_2}\right)^n \cdot (n^2 \cdot \beta_2^2 - 2 \cdot n^2 \cdot \beta_2 + n^2 - n \cdot \beta_2^2 - 2 \cdot n \cdot \beta_2 + 3 \cdot n + 2)}{2 \cdot \beta_2^3} \quad (1.2.1.48)$$

$$n := n \quad z := z$$

$$\delta(n, 0) - 1 \quad \left| \begin{array}{l} \text{assume, n = integer} \\ \text{ztrans} \end{array} \right. \rightarrow -\frac{1}{z - 1}$$

$$\frac{1}{1 - z} \quad \left| \begin{array}{l} \text{invztrans} \\ \text{simplify, max} \end{array} \right. \rightarrow \delta(n, 0) - 1$$

$$\frac{1}{1 - z} = \sum_{k=0}^{\infty} z^k \quad \boxed{|z| < 1}$$

$$\frac{1}{1 - z} = \frac{1 - z + z}{1 - z} = 1 + \frac{z}{1 - z} = 1 + \frac{1}{\frac{1}{z} - 1} = 1 - \frac{1}{1 - \frac{1}{z}}$$

$$\zeta = \frac{1}{z} \quad 1 - \frac{1}{1 - \frac{1}{z}} = 1 - \frac{1}{1 - \zeta} = 1 - \sum_{k=0}^{\infty} \zeta^k = \sum_{k=1}^{\infty} z^{-k} \quad |z| > 1$$

$$\frac{1}{1 - z} = \sum_{k=1}^{\infty} z^{-k} \quad \boxed{|z| > 1}$$

Expand to see some examples

§1.2.2) POWER SERIES EXPANSION METHOD.

I now consider a generic Z transform of a causal sequence, corresponding to a rational function, like this (IIR):

$$H(z) = \frac{\sum_{k=0}^N (b_k \cdot z^{-k})}{\sum_{k=0}^M (a_k \cdot z^{-k})}, \quad (1.2.2.1)$$

The division of the two polynomials at the numerator and denominator generate the series:

$$H(z) = \sum_{n=0}^{\infty} (h_n \cdot z^{-n}), \quad (1.2.2.1')$$

whose coefficients are given by the following algorithm:

$$\text{with: } h_0 = \frac{b_0}{a_0}, \quad h_n = \frac{1}{a_0} \cdot \left[b_n - \sum_{i=1}^n (h_{n-i} \cdot a_i) \right] \quad (1.2.2.2)$$

$$\text{where for } N=2: H(z) = \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{a_0 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} \quad (1.2.2.3)$$

$$\text{or } H(z) = \frac{b_0 \cdot z^2 + b_1 \cdot z + b_2}{a_0 \cdot z^2 + a_1 \cdot z + a_2} \quad (1.2.2.3')$$

$$H(z) = \sum_{n=0}^{\infty} (h_n \cdot z^{-n}),$$

$$\text{with: } h_0 = \frac{b_0}{a_0}, \quad h_n = \frac{1}{a_0} \cdot \left[b_n - \sum_{i=1}^n (h_{n-i} \cdot a_i) \right]. \quad (1.2.2.2')$$

Expand to see some examples on POWER SERIES EXPANSION METHOD

Example 1):

$$M1 = 100 \quad n := 1..M1 \quad H6(z) := \frac{z^2 - 1.618 \cdot z + 4.0}{z^2 - 1.5161 \cdot z + 0.878}, \quad (1.2.2.4)$$

$$b_n := 0, \quad a_n := 0,$$

$$\text{rows}(b) = 101 \quad \beta := \text{numer}(H6(z)) \text{ coeffs} \rightarrow \begin{pmatrix} 2.0e7 \\ -8.09e6 \\ 5.0e6 \end{pmatrix}, \quad \mu := \text{rows}(\beta), \quad (1.2.2.5)$$

$$i := 0.. \mu - 1 \quad b_i := \beta_{\mu-i-1}$$

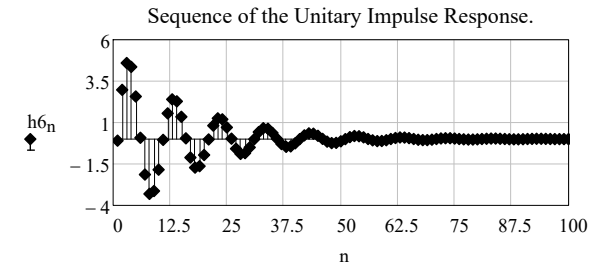
$$\text{rows}(\beta) = 3$$

$$\alpha := \text{denom}(H6(z)) \text{ coeffs} \rightarrow \begin{pmatrix} 4.39e6 \\ -7.5805e6 \\ 5.0e6 \end{pmatrix}, \quad nr := \text{rows}(\alpha)$$

$$j := 0..nr - 1$$

$$a_j := \alpha_{nr-j-1}$$

$$h6_0 := \frac{b_0}{a_0}, \quad h6_n := \frac{1}{a_0} \cdot \left[b_n - \sum_{i=1}^n (h6_{n-i} \cdot a_i) \right]. \quad (1.2.2.7)$$



(fig. 1.2.2.1)

Example 2):

$$\text{Numerator } \underline{N_n} := 2 \quad H7(z) := \frac{1 + 2 \cdot z^{-1} + z^{-2}}{1 - z^{-1} + 0.3561 \cdot z^{-2}} \quad (1.2.2.8)$$

Denominator $M := 2$

$N1 := N + M$ $n := 1..M1$ $j := 0..N1$ $M1 = 100$ $k := 0..M1$

Numerator Denominator

$b_n := 0.0$ $a_n := 0.0$

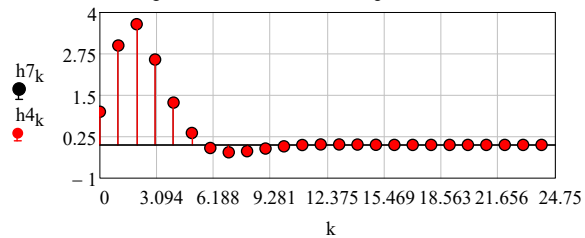
$b_0 := 1$ $a_0 := 1$

$b_1 := 2$ $a_1 := -1$

$b_2 := 1$ $a_2 := 0.3561$

$$N1 = 4 \quad h7_0 := \frac{b_0}{a_0} \quad h7_n := \frac{1}{a_0} \left[b_n - \sum_{i=1}^n (h7_{n-i} \cdot a_i) \right] \quad (1.2.2.9)$$

Comparison Between The Sequences h4 and h7.



(fig. 1.2.2.2)

Expand to see some examples on POWER SERIES EXPANSION METHOD

§1.2.3) PARTIAL FRACTION EXPANSION METHOD.

This analytic method consists in expanding $H(z)$ in a sum of simple partial fractions. The inverse transform is then found using the elementary table of Z transform.

If I suppose $N=M$ and the poles of $H(z)$ are of the first order, then:

$$H(z) = \frac{\sum_{k=0}^N (b_k \cdot z^{-k})}{\sum_{k=0}^M (a_k \cdot z^{-k})} \quad ((1.2.2.1))$$

is expanded as:

$$H(z) = B_0 + \sum_{k=1}^M \left(C_k \cdot \frac{z}{z - p_k} \right), \quad \text{where } B_0 = \frac{b_N}{a_N}. \quad (1.2.3.1)$$

if $N < M$ I have:

$$H(z) = \sum_{k=1}^M \left(C_k \cdot \frac{z}{z - p_k} \right) \quad (1.2.3.2)$$

where:

$$C_k = \lim_{z \rightarrow p_k} \left[\frac{H(z)}{z} \cdot (z - p_k) \right]. \quad (1.2.3.3)$$

If $H(z)$ has one or more multiple-order poles (coincident) (for an m^{th} -order pole at $z = p_i$), then,

$$H(z) = B_0 + \sum_{k=1}^M \left(C_k \cdot \frac{z}{z - p_k} \right) + \sum_{k=1}^m \frac{D_k}{(z - p_i)^k}, \quad (1.2.2.4)$$

where:

$$D_k = \frac{1}{(m_k - k)!} \cdot \lim_{z \rightarrow p_i} \frac{\partial^{m_k - k}}{\partial z^{m_k - k}} \left[(z - p_i)^{m_k} \cdot \frac{H(z)}{z} \right]. \quad (1.2.2.5)$$

Expand to see some examples on PARTIAL FRACTION EXPANSION METHOD

Example 1) (two real and distinct poles): $H8(z) = \frac{z^{-1}}{1 - \frac{z^{-1}}{4} - \frac{3}{8} \cdot z^{-2}}, \quad (1.2.2.6)$

$H8(z)$ can be written also: $H8(z) = \frac{z}{z^2 - \frac{z}{4} - \frac{3}{8}} = \frac{z}{\left(z - \frac{3}{4}\right) \cdot \left(z + \frac{1}{2}\right)}$, it has a zero at the origin and two simple poles in

$$p_0 = \frac{3}{4}, \text{ and } p_1 = -\frac{1}{2}.$$

Since the numerator's order is less than the order of the denominator, I can write the partial fraction in this form:

$$H8(z) = \frac{z}{\left(z - \frac{3}{4}\right) \cdot \left(z + \frac{1}{2}\right)} = \frac{C_0 \cdot z}{z - \frac{3}{4}} + \frac{C_1 \cdot z}{z + \frac{1}{2}}, \quad (1.2.2.7)$$

so:

$$C_0 = \lim_{z \rightarrow p_0} \left[\frac{H8(z)}{z} \cdot (z - p_0) \right],$$

$$C_0 = \lim_{z \rightarrow p_0} \left[\frac{H8(z)}{z} \cdot (z - p_0) \right] = \lim_{z \rightarrow p_0} \left[\frac{1}{\left(z - \frac{3}{4}\right) \cdot \left(z + \frac{1}{2}\right)} \cdot \left(z - \frac{3}{4}\right) \right] = \lim_{z \rightarrow \frac{3}{4}} \left[\frac{1}{\left(z + \frac{1}{2}\right)} \right] = \frac{4}{5},$$

$$C_1 = \lim_{z \rightarrow p_1} \left[\frac{H8(z)}{z} \cdot (z - p_1) \right],$$

$$C_1 = \lim_{z \rightarrow p_1} \left[\frac{H8(z)}{z} \cdot (z - p_1) \right] = \lim_{z \rightarrow p_1} \left[\frac{1}{\left(z - \frac{3}{4}\right) \cdot \left(z + \frac{1}{2}\right)} \cdot \left(z + \frac{1}{2}\right) \right] = \lim_{z \rightarrow -\frac{1}{2}} \left[\frac{1}{\left(z - \frac{3}{4}\right)} \right] = -\frac{4}{5},$$

$$H8(z) = \frac{4}{5} \cdot \left(\frac{z}{z - \frac{3}{4}} - \frac{z}{z + \frac{1}{2}} \right), \quad (1.2.2.8)$$

I know that: $a := a$ $z := z$, the Z inverse transform of each term in (1.2.2.8) are:

$$\frac{z}{z - a} \text{ invztrans} \rightarrow a^n, \quad \frac{z}{z + a} \text{ invztrans} \rightarrow (-a)^n$$

that is the Z inverse transform of (1.2.2.8) ultimately is:

$$h8(n) := \frac{4}{5} \cdot \left[\left(\frac{3}{4}\right)^n - \left(-\frac{1}{2}\right)^n \right]. \quad (1.2.2.9)$$

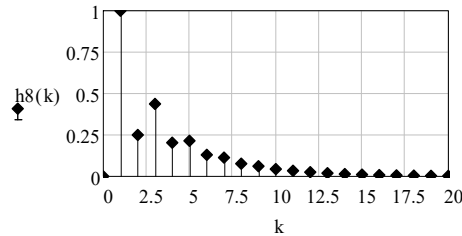


fig.(1.2.3.1)

$$\frac{z^{-1}}{1 - \frac{z^{-1}}{4} - \frac{3}{8} \cdot z^{-2}} \text{ invztrans} \rightarrow \frac{4 \cdot \left(\frac{3}{4}\right)^n}{5} - \frac{4 \cdot \left(\frac{1}{2}\right)^n}{5} \quad (1.2.2.10)$$

Example 2) (first order complex conjugate poles): $H9(z) = \frac{1 + a \cdot z^{-1} + z^{-2}}{1 - z^{-1} + b \cdot z^{-2}}$, $a := 2$, $b := \frac{1}{e}$

or:

$$H9(z) := \frac{z^2 + a \cdot z + 1}{z^2 - z + b} \quad (1.2.2.11)$$

vector of the denominator's coefficients

$$v := \text{denom}(H9(z)) \text{ coeffs, } z \rightarrow \begin{pmatrix} e^{-1} \\ -1 \\ 1 \end{pmatrix} \quad (1.2.2.12)$$

Poles' calculation:

$$a := a \quad b := b \quad \text{poles9} := \text{polyroots}(v) \quad \text{poles9}^T = (0.5 - 0.343j \quad 0.5 + 0.343j)$$

$$p_0 := \text{poles9}_0 \quad p_0 = 0.5 - 0.343j \quad p_1 := \text{poles9}_1 \quad p_1 = 0.5 + 0.343j$$

Radius of the circle enclosing all the poles of H9(z,v): $r_9 := \text{ceil}(\max(|\text{poles9}|)) \cdot 1.0$, $r_9 = 1$

$$\xi(t) := r_9 \cdot \cos(t) \quad \psi(t) := r_9 \cdot \sin(t) \quad \zeta(t) := \xi(t) + j \cdot \psi(t) \quad (1.2.2.13)$$

$$t_{0v} := 0 \quad t_{\text{max}} := 2 \cdot \pi$$

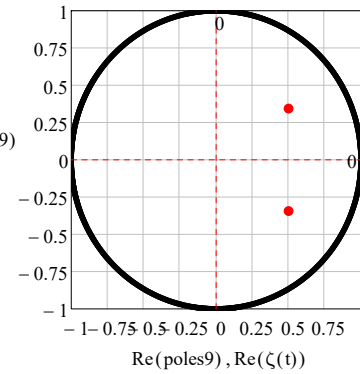


fig.(1.2.3.2)

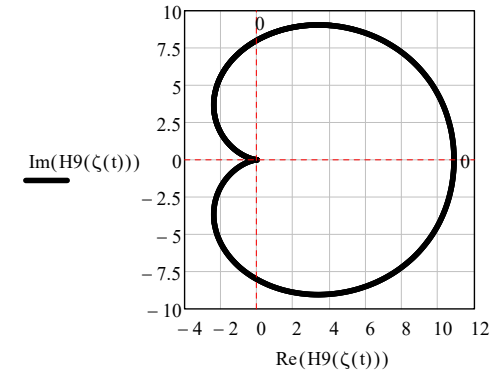


fig.(1.2.3.3)

so, I can write:

$$H9(z) = \frac{z^2 + 2 \cdot z + 1}{(z - p_0) \cdot (z - p_1)} \quad (1.2.2.14)$$

$$p_0 = 0.5 - 0.343j$$

$$p_1 = 0.5 + 0.343j$$

Since numerator and denominator are of the same order, the partial fraction expansion has the form:

$$\frac{H9(z)}{z} = \frac{B_0}{z} + \frac{C_1}{z - p_0} + \frac{C_2}{z - p_0}, \quad B_0 = \frac{b_N}{a_N} = \frac{1}{b} = e, \quad (1.2.2.15)$$

$$B_0 := e, \quad C_1 = \lim_{z \rightarrow p_0} \left[\frac{H9(z)}{z} \cdot (z - p_0) \right],$$

$$C_1 = \lim_{z \rightarrow p_0} \left[\frac{z^2 + 2 \cdot z + 1}{z \cdot (z - p_0) \cdot (z - p_1)} \cdot (z - p_0) \right] \quad (1.2.2.16)$$

$$(p_0 - \bar{p}_0) = -0.687j$$

$$p_0 := p_0 \quad C_1 := \lim_{z \rightarrow p_0} \left[\frac{z^2 + 2 \cdot z + 1}{z \cdot (z - \bar{p}_0)} \right] \rightarrow \frac{(p_0 + 1)^2}{p_0 \cdot (p_0 - \bar{p}_0)} \quad (1.2.2.17)$$

$$p_1 := p_1 \quad p_0 := p_0 \quad C_1 = -0.859 + 5.62j$$

$$C_2 := \lim_{z \rightarrow p_0} \left[\frac{z^2 + 2.0 \cdot z + 1}{z \cdot (z - p_0)} \right] \rightarrow -\frac{(\overline{p_0})^2 + 2.0 \cdot \overline{p_0} + 1.0}{\overline{p_0} \cdot (\overline{p_0} - 1.0 \cdot \overline{p_0})} \quad (1.2.2.18)$$

$$C_2 = -0.859 - 5.62j$$

$$H9(z) = e + \frac{C_1 \cdot z}{z - p_0} + \frac{C_2 \cdot z}{z - p_0} \quad (1.2.2.19)$$

$$e \text{ invztrans} \rightarrow e \cdot \delta(n, 0)$$

$$\frac{C_1 \cdot z}{z - p_0} \text{ invztrans} \rightarrow \frac{p_0^{n-1} \cdot (p_0 + 1)^2}{p_0 - p_0} \quad (1.2.2.20)$$

$$\frac{C_2 \cdot z}{z - p_0} \text{ invztrans} \rightarrow -\frac{(\overline{p_0})^{n-1} \cdot (\overline{p_0} + 1)^2}{p_0 - p_0} \quad (1.2.2.21)$$

$$k := 0..99 \quad h9_k := e \cdot \delta(k, 0) + \frac{p_0^{k-1} \cdot (p_0 + 1)^2}{p_0 - p_0} - \frac{(\overline{p_0})^{k-1} \cdot (\overline{p_0} + 1)^2}{p_0 - p_0} \quad (1.2.2.22)$$

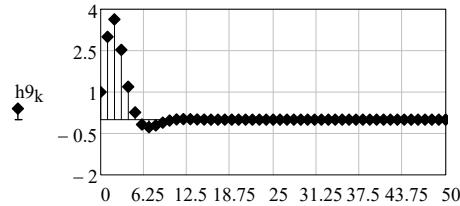
$$h9^T = \begin{bmatrix} 1 & 3 & 3.632 & 2.528 & 1.192 & 0.262 & -0.176 & \dots \end{bmatrix}$$


fig.(1.2.3.4)

Example 3) ($H10(z)$ has a second order pole): $H10(z) = \frac{z^2}{z^3 - \frac{5 \cdot z^2}{2} + 2 \cdot z - \frac{1}{2}} = \frac{z^2}{\left(z - \frac{1}{2}\right) \cdot (z - 1)^2}$

Poles:

$$p_{0v} := \frac{1}{2} \quad \text{First order,}$$

$$p_{1v} := 1 \quad \text{Second order.}$$

$$\frac{H10(z)}{z} = \frac{A}{z - \frac{1}{2}} + \frac{B}{z - 1} + \frac{C}{(z - 1)^2} \quad (1.2.2.23)$$

remembering that $C_k = \lim_{z \rightarrow p_k} \left[\frac{H10(z)}{z} \cdot (z - p_k) \right]$,

$$A = \lim_{z \rightarrow p_0} \left[\frac{H10(z)}{z} \cdot (z - p_0) \right] = \lim_{z \rightarrow \frac{1}{2}} \frac{\left(z - \frac{1}{2}\right) \cdot z^2}{z \cdot \left(z - \frac{1}{2}\right) \cdot (z - 1)^2} = \frac{\left(\frac{1}{2}\right)^2}{\frac{1}{2} \cdot \left(\frac{-1}{2}\right)^2} = 2, \quad (1.2.2.24)$$

$$B = \lim_{z \rightarrow p_1} \frac{\partial}{\partial z} \left[\frac{H10(z)}{z} \cdot (z - p_1)^2 \right] = \lim_{z \rightarrow 1} \frac{\partial}{\partial z} \left[\frac{z^2}{z \cdot \left[\left(z - \frac{1}{2}\right) \cdot (z - 1)^2\right]} \cdot (z - 1)^2 \right] = -2, \quad (1.2.2.25)$$

$$C = \lim_{z \rightarrow p_1} \left[\frac{H10(z)}{z} \cdot (z - p_1)^2 \right] = \lim_{z \rightarrow 1} \left[\frac{z^2}{z \cdot \left[\left(z - \frac{1}{2}\right) \cdot (z - 1)^2\right]} \cdot (z - 1)^2 \right] = 2. \quad (1.2.2.26)$$

$$H10(z) = \frac{2 \cdot z}{z - \frac{1}{2}} - \frac{2 \cdot z}{z - 1} + \frac{2 \cdot z}{(z - 1)^2}, \quad (1.2.2.27)$$

$$\frac{2 \cdot z}{z - \frac{1}{2}} \text{ invztrans} \rightarrow 2 \cdot \left(\frac{1}{2}\right)^n, \quad \frac{2 \cdot z}{z - 1} \text{ invztrans} \rightarrow 2, \quad \frac{2 \cdot z}{(z - 1)^2} \text{ invztrans} \rightarrow 2 \cdot n,$$

$$h10_n := 2 \cdot \left[\left(\frac{1}{2}\right)^n + n - 1 \right], \quad n \geq 0 \quad (1.2.2.28)$$

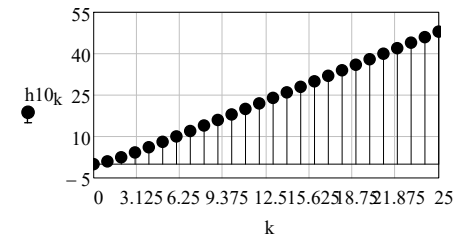


fig.(1.2.3.5)

$n := n$

$$\text{test: } 2 \cdot \left[\left(\frac{1}{2}\right)^n + n - 1 \right] \text{ ztrans} \rightarrow \frac{2 \cdot z^2}{(z - 1)^2 \cdot (2 \cdot z - 1)} \quad (1.2.2.29)$$

Expand to see some examples on PARTIAL FRACTION EXPANSION METHOD

§1.2.4) USING THE "INVZTRANS" OPERATOR.

I will use this operator in all worksheets.

Examples:

$$n := n \quad \frac{z \cdot (z + 1)}{(z - 1)^2} \left| \begin{array}{l} \text{invztrans} \\ \text{simplify} \end{array} \right. \rightarrow 2 \cdot n + 1 \quad (1.2.4.1)$$

$$n := n \quad \left[\frac{-2 \cdot z}{(z - 1)^3} \right] \left| \begin{array}{l} \text{invztrans} \\ \text{simplify} \rightarrow -n \cdot (n - 1) \\ \text{factor} \end{array} \right. \quad (1.2.4.2)$$

$$a := a \quad \left(\frac{a \cdot z}{a \cdot z - 1} \right) \left| \begin{array}{l} \text{invztrans} \\ \text{simplify} \end{array} \right. \rightarrow \left(\frac{1}{a} \right)^n \quad (1.2.4.3)$$

$$k := k \quad \left[\left(\frac{z}{z - 1} \right) \cdot k + \frac{z}{(z - 1)^2} \right] \left| \begin{array}{l} \text{invztrans} \\ \text{simplify} \end{array} \right. \rightarrow n - k \quad (1.2.4.4)$$

$$k := k \quad \left[\frac{z \cdot (k - 1) - k \cdot z^2}{z^2 - 2 \cdot z + 1} \right] \left| \begin{array}{l} \text{invztrans} \\ \text{simplify} \end{array} \right. \rightarrow k + n \quad (1.2.4.5)$$

$$m := m \quad k := k \quad a := a \quad b := b \quad \frac{z \cdot [a + b + k \cdot (a - b)] - k \cdot z^2 \cdot (a - b)}{z^2 - 2 \cdot z + 1} \left| \begin{array}{l} \text{invztrans} \\ \text{simplify} \rightarrow (n - k) \cdot a + (k + n) \cdot b \\ \text{collect, a, b} \end{array} \right. \quad (1.2.4.6)$$

$$m := m \quad k := k \quad a := a \quad b := b \quad \left[\frac{z \cdot (a - b)}{z - 1} \right] \cdot k + \frac{z \cdot (a + b)}{z - 1} \cdot m + \frac{z \cdot (a + b)}{z^2 - 2 \cdot z + 1} \left| \begin{array}{l} \text{invztrans} \\ \text{simplify} \rightarrow (m - k + n) \cdot a + (k + m + n) \cdot b \\ \text{collect, a, b} \end{array} \right. \quad (1.2.4.7)$$

§1.2.5) Transformation $s \rightarrow z, \Rightarrow F(s) \rightarrow G(z)$

1) First transformation ($s \rightarrow z$).

Place $z^{-1} = e^{-s \cdot T_s} = \sum_{k=0}^{\infty} \left[\frac{(-s \cdot T_s)^k}{k!} \right]$, ($s = \sigma + j \cdot \omega$) and truncate the power series to the first order, namely;

$z^{-1} = 1 - s \cdot T_s$, from which

$$s = \frac{1 - z^{-1}}{T_s}, \quad (1.2.5.1)$$

Although, it is a very inaccurate approximation, nevertheless using this approximation and notwithstanding the previously mentioned defect, the results produced are very acceptable, as will be seen later).

The previous transformation is a conformal mapping because it is analytical:

Now place $z = x + j \cdot y$ and consider the function $f(x, y, T_s) := \frac{1 - (x + j \cdot y)^{-1}}{T_s}$. To see if it is analytical or not, I

apply the Cauchy-Riemann conditions:

$$\frac{\partial}{\partial y} f(x, y, T_s) \rightarrow \frac{j}{T_s \cdot (x + j \cdot y)^2}, \quad j \cdot \left(\frac{\partial}{\partial x} f(x, y, T_s) \right) \rightarrow \frac{j}{T_s \cdot (x + j \cdot y)^2},$$

$$\text{hence } \frac{\partial}{\partial y} f(x, y, T_s) = j \cdot \frac{\partial}{\partial x} f(x, y, T_s).$$

moreover it satisfies the condition:

$$\lim_{z \rightarrow z_0} \left| \frac{\partial}{\partial z} f(z) \right| \neq 0$$

$$z := z, \quad z_0 := z_0, \quad T_s := T_s, \quad \lim_{z \rightarrow z_0} \left| \frac{\partial}{\partial z} \frac{1 - z^{-1}}{T_s} \right| \rightarrow \begin{cases} \frac{\text{signum}(z_0, 0)^2}{z_0^2 \cdot |T_s|} & \text{if } z_0 \neq 0 \\ \infty & \text{if } z_0 = 0 \end{cases}$$

this result let the previous transformation be a conformal mapping (as requested by the theorem thereof).

Complex plane representation of s and z.

$$\underline{N} := 100 \quad j := 0..N - 1 \quad \rho := 1.0$$

$$\theta_j := \frac{2 \cdot \pi}{N} \cdot j \quad \text{rows}(\theta) = 100$$

$$z_j := \rho \cdot e^{j \cdot \theta_j}$$

$$T_s := 1 \quad \underline{s}_j := \frac{1 - (z_j)^{-1}}{T_s}$$

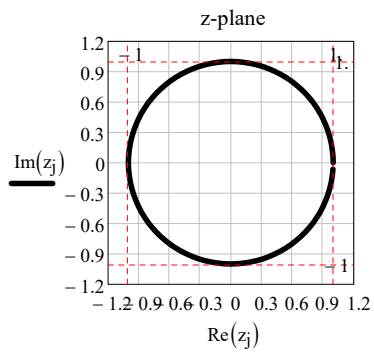


fig.(1.2.5.1)

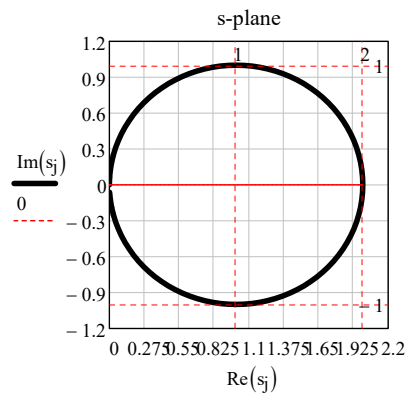


fig.(1.2.5.2)

Example: knowing the function $f(t)$, I search the corresponding sequence via Laplace transformation.

The function needs some parameters: $\mu := 1000$, $\beta := \frac{1}{\mu}$, $\alpha := 2 \cdot \pi \cdot \mu$,

the function is: $f_{00}(t) := -t \cdot \alpha^2 \cdot e^{-t \cdot \alpha} \cdot \Phi(t)$, (1.2.5.2)

Sampling frequency: $\mu_0 := 100 \cdot \mu$, $\tau_0 := \frac{1}{\mu_0}$, $\tau_0 = 1 \times 10^{-5}$

$t := 0, 0 + \frac{2 \cdot \beta}{N} .. 2 \cdot \beta$ $\frac{\beta}{\tau_0 \cdot N} = 1$

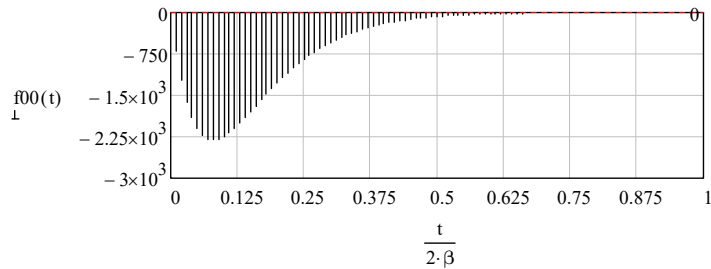


fig.(1.2.5.3)

Laplace transform of the given function:

$t := t$ $T_{\text{ov}} := \tau_0$ $s := s$ $\rho := \rho$ $\alpha := \alpha$

$-t \cdot \alpha^2 \cdot e^{-t \cdot \alpha} \cdot \Phi(t) \text{ laplace, } t \rightarrow -\frac{\alpha^2}{(\alpha + s)^2}$ (1.2.5.3)

$F_6(s) = \mathcal{L}(-t \cdot \alpha^2 \cdot e^{-t \cdot \alpha} \cdot \Phi(t)) = \frac{-\alpha^2}{(s + \alpha)^2}$ (1.2.5.4)

$$F_6(s) := \frac{-\alpha^2}{(s + \alpha)^2} \tag{1.2.5.5}$$

Z transform applying the given approximation:

$\tau_0 := \tau_0$ $s := s$ $z := z$ $\alpha := \alpha$

$G_6(z) := \frac{-\alpha^2}{(s + \alpha)^2} \left| \begin{array}{l} \text{substitute, } s = \frac{1 - z^{-1}}{\tau_0} \\ \text{collect, } z \end{array} \right. \rightarrow -\frac{\alpha^2 \cdot \tau_0^2 \cdot z^2}{(\alpha^2 \cdot \tau_0^2 + 2 \cdot \alpha \cdot \tau_0 + 1) \cdot z^2 + (-2 \cdot \alpha \cdot \tau_0 - 2) \cdot z + 1}$

or $G_6(z) = \frac{-(\tau_0^2 \cdot \alpha^2)}{(\tau_0^2 \cdot \alpha^2 + 2 \cdot \tau_0 \cdot \alpha + 1) - 2 \cdot (\tau_0 \cdot \alpha + 1) \cdot z^{-1} + z^{-2}}$ (1.2.5.6)

Define the following parameters:

$A1 := -1 \cdot \tau_0^2 \cdot \alpha^2$ $B1 := 2 \cdot (\tau_0 \cdot \alpha + 1)$ $C1 := \tau_0^2 \cdot \alpha^2 + 2 \cdot \tau_0 \cdot \alpha + 1$ (1.2.5.7)

$A1 = -3.948 \times 10^{-3}$ $B1 = 2.126$ $C1 = 1.13$

$G_6(z) := \frac{A1}{z^{-2} - B1 \cdot z^{-1} + C1}$ (1.2.5.8)

Plotting magnitude in dB and phase of the two functions $F(s)$ and $G(z)$:

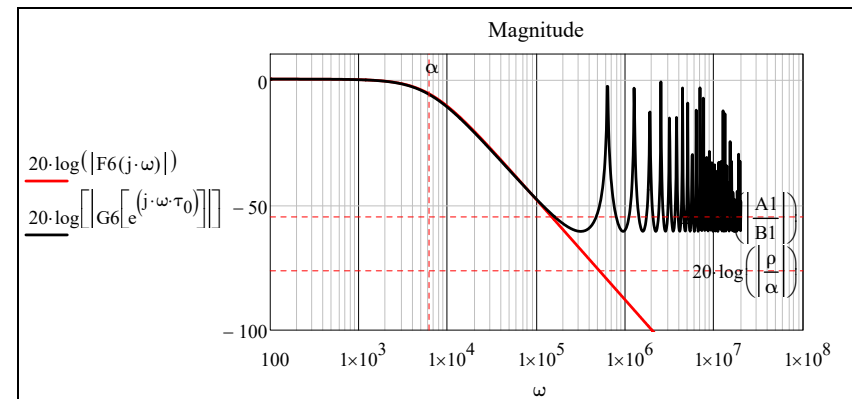


fig.(1.2.5.4)

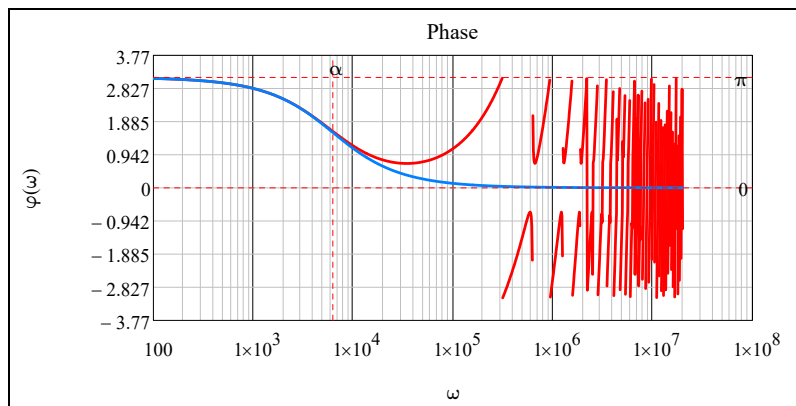


fig.(1.2.5.5)

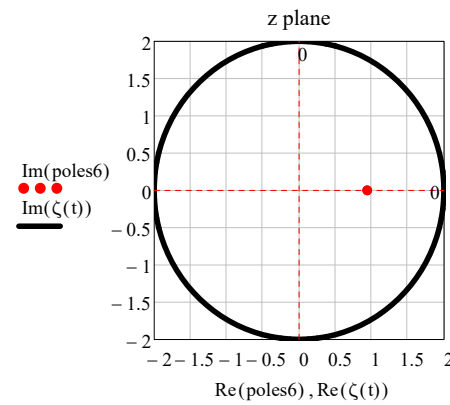


fig.(1.2.5.6)

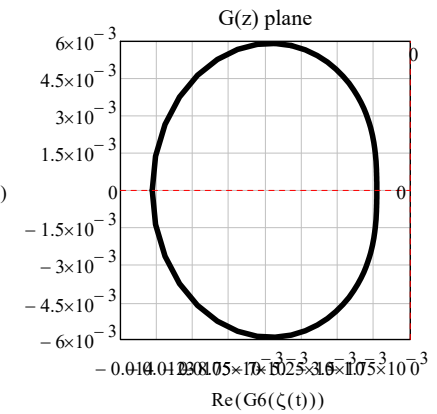


fig.(1.2.5.7)

Poles calculation of G(z):

$$v := \text{denom}(G6(z)) \text{ coeffs}, z \rightarrow \begin{pmatrix} 1 \\ -2 \cdot \alpha \cdot \tau_0 - 2 \\ \alpha^2 \cdot \tau_0^2 + 2 \cdot \alpha \cdot \tau_0 + 1 \end{pmatrix} \quad (1.2.5.9)$$

$$\text{poles6} := \text{polyroots}(v) \quad \text{poles6}^T = (0.941 \quad 0.941)$$

$$p_0 := \text{poles6}_0 \quad p_0 = 0.941 \quad p_1 := \text{poles6}_1 \quad p_1 = 0.941$$

$$t := t \quad \begin{aligned} r &:= \text{ceil}(\max(|\text{poles6}|)) \cdot 1.0 \\ \xi(t) &:= r \cdot \cos(t) & r &= 2 & |p| \\ \psi(t) &:= r \cdot \sin(t) \\ \zeta(t) &:= \xi(t) + j \cdot \psi(t) \\ t_0 &:= 0 \quad t_{\text{fin}} := 2 \cdot \pi & t &:= t_0, t_0 + \frac{t_{\text{fin}} - t_0}{100} \dots t_{\text{fin}} \end{aligned}$$

Calculation of the sequence corresponding to G6(z):

$$\text{out}_n := \frac{A1}{z^{-2} - B1 \cdot z^{-1} + C1} \begin{array}{l} z := z \quad A1 := A1 \quad B1 := B1 \\ \text{invztrans} \\ \text{simplify, max} \end{array} \rightarrow \quad (1.2.5.10)$$

parameters definition:

$$\gamma_1 := \frac{B1 - \sqrt{B1^2 - 4 \cdot C1}}{C1} \quad \gamma_2 := \frac{B1 + \sqrt{B1^2 - 4 \cdot C1}}{C1} \quad (1.2.5.11)$$

$$\sigma_1 := 2 \cdot C1 - B1^2 + B1 \cdot \sqrt{B1^2 - 4 \cdot C1} \quad \sigma_2 := B1^2 - 2 \cdot C1 + B1 \cdot \sqrt{B1^2 - 4 \cdot C1} \quad (1.2.5.12)$$

searched sequence

$$\text{out}_n := \left(\frac{1}{2}\right)^n \cdot \frac{A1}{(C1^2 \cdot \sqrt{B1^2 - 4 \cdot C1})} \cdot (\gamma_1^{n-1} \cdot \sigma_1 + \gamma_2^{n-1} \cdot \sigma_2) \quad (1.2.5.13)$$

$$n := 0..N-1$$

The approximation, as shown by the plot below, reduces the amplitude of the given function, although it looks very like to the original one. Hence, the amplitude of the original function is scaled down.

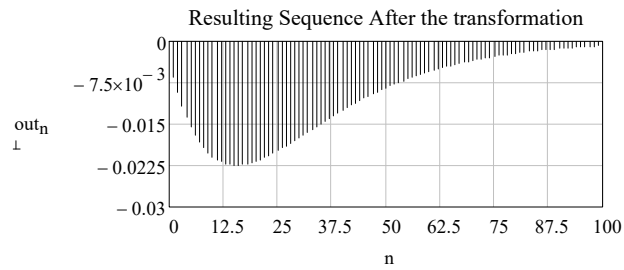


fig.(1.2.5.8)

$$t := 0, 0 + \frac{2 \cdot \beta}{N} .. 2 \cdot \beta$$

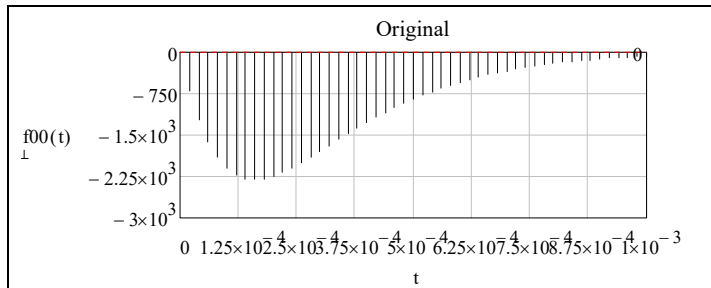


fig.(1.2.5.9)

2) Bilinear transformation:

$$s = \left(\frac{1}{\tau_0} \cdot \ln(z) \right) \approx \left[\frac{2}{\tau_0} \cdot \left[\frac{z-1}{z+1} + \frac{(z-1)^3}{3 \cdot (z+1)^3} + \dots \right] \right] \quad (1.2.5.14)$$

(See: **R. Isermann**, *Digitale Regelsysteme*, 3.2-74 pag.:32) in a first approximation one can write:

$$s \approx \left[\frac{2}{\tau_0} \cdot \left(\frac{z-1}{z+1} \right) \right]$$

Bilinear transformation:

$$s = \frac{2}{\tau_0} \cdot \frac{1-z^{-1}}{1+z^{-1}} \quad (1.2.5.15)$$

The previous transformation is a conformal mapping because it is analytical, in fact:

place $(z = x + j \cdot y)$, define $f(x, y, \tau_0) := \frac{2}{\tau_0} \cdot \frac{1 - (x + j \cdot y)^{-1}}{1 + (x + j \cdot y)^{-1}}$,

To see if it is analytical or not, I apply the Cauchy-Riemann conditions:

$$x := x \quad y := y \quad \tau_0 := \tau_0$$

$$\frac{\partial}{\partial y} f(x, y, \tau_0) \text{ simplifymax} \rightarrow \frac{4j}{\tau_0 \cdot (x + 1 + y \cdot j)^2},$$

$$j \cdot \frac{\partial}{\partial x} f(x, y, \tau_0) \text{ simplifymax} \rightarrow \frac{4j}{\tau_0 \cdot (x + 1 + y \cdot j)^2},$$

$$\text{hence } \frac{\partial}{\partial y} f(x, y, \tau_0) = j \cdot \frac{\partial}{\partial x} f(x, y, \tau_0).$$

$$\lim_{z \rightarrow z_0} \left| \frac{\partial}{\partial z} \left(\frac{2}{\tau_0} \cdot \frac{1-z^{-1}}{1+z^{-1}} \right) \right| \left| \begin{array}{l} \text{simplify} \\ \text{assume, } z_0 \neq 0 \end{array} \right. \rightarrow \frac{4}{(|z_0 + 1|)^2 \cdot |\tau_0|}$$

this result let the previous transformation be a conformal mapping (as requested by the theorem thereof.)

Bilinear transformation s and z planes:

$$\overset{\text{www}}{N} := 100 \quad j := 0 .. N - 1 \quad \rho := 0.1 \quad \theta_j := \frac{2 \cdot \pi}{N} \cdot j \quad \text{rows}(\theta) = 100$$

$$z_j := \rho \cdot e^{j \cdot \theta_j}$$

$$s_j := 1 \cdot \frac{1 - (z_j)^{-1}}{1 + (z_j)^{-1}} \quad (1.2.5.16)$$

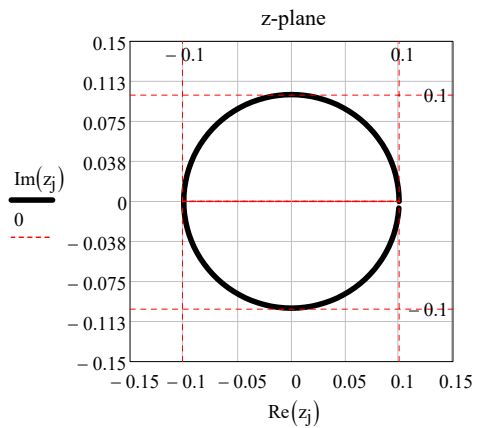


fig.(1.2.5.10)

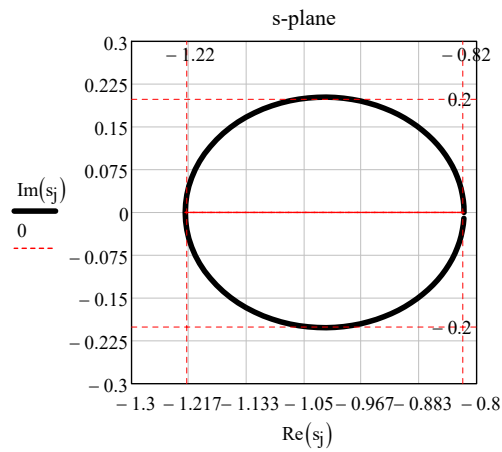


fig.(1.2.5.11)

Z transform calculation of a given Laplace transform (function of s)

$$\begin{array}{l}
 \tau_0 := \tau_0 \quad s := s \quad z := z \quad \alpha := \alpha \\
 \text{substitute, } s = \frac{2}{\tau_0} \cdot \frac{1-z^{-1}}{1+z^{-1}} \\
 \text{collect, } z \rightarrow \\
 \text{collect, } \rho, \alpha \cdot \tau_0^2
 \end{array}
 \quad G1(z) := \frac{-\alpha^2}{(s+\alpha)^2}
 \quad (1.2.5.17)$$

$$\text{Function of } z \quad G1(z) = \frac{-[\tau_0^2 \cdot \alpha^2 \cdot (z+1)^2]}{(\tau_0 \cdot \alpha + 2)^2 \cdot z^2 + 2 \cdot (\tau_0^2 \cdot \alpha^2 - 4) \cdot z + (\tau_0 \cdot \alpha - 2)^2}
 \quad (1.2.5.18)$$

or as a function of z^{-1} :

$$G1(z) = \frac{-(\tau_0^2 \cdot \alpha^2)}{(\tau_0 \cdot \alpha - 2)^2} \cdot \frac{(z^{-1} + 1)^2}{\frac{(\tau_0 \cdot \alpha + 2)^2}{(\tau_0 \cdot \alpha - 2)^2} + 2 \cdot \frac{(\tau_0^2 \cdot \alpha^2 - 4)}{(\tau_0 \cdot \alpha - 2)^2} \cdot z^{-1} + z^{-2}}
 \quad (1.2.5.19)$$

Magnitude and phase plots.

Define the following constants to simplify the calculation:

$$A11 := \frac{-(\tau_0^2 \cdot \alpha^2)}{(\tau_0 \cdot \alpha - 2)^2} \quad B11 := 2 \cdot \frac{(\tau_0^2 \cdot \alpha^2 - 4)}{(\tau_0 \cdot \alpha - 2)^2} \quad C11 := \frac{(\tau_0 \cdot \alpha + 2)^2}{(\tau_0 \cdot \alpha - 2)^2}
 \quad (1.2.5.20)$$

$$A11 = -1.052 \times 10^{-3} \quad B11 = -2.13 \quad C11 = 1.134$$

$$z := z \quad A11 := A11 \quad B11 := B11 \quad C11 := C11$$

$$\text{results: } G1(z) := \frac{A11 \cdot (z^{-1} + 1)^2}{z^{-2} + B11 \cdot z^{-1} + C11}$$

(1.2.5.21)

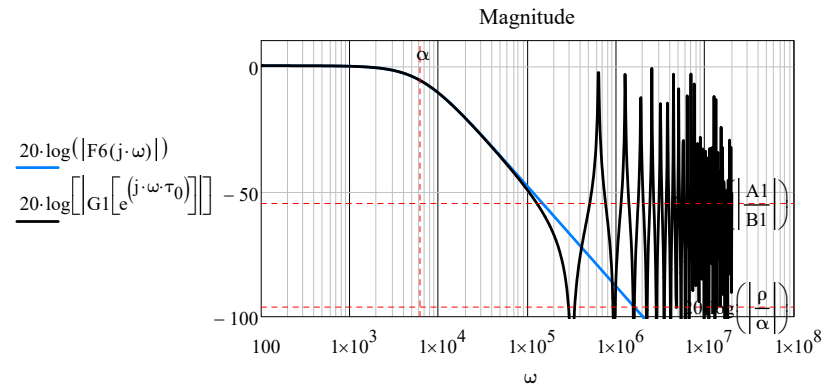


fig.(1.2.5.12)

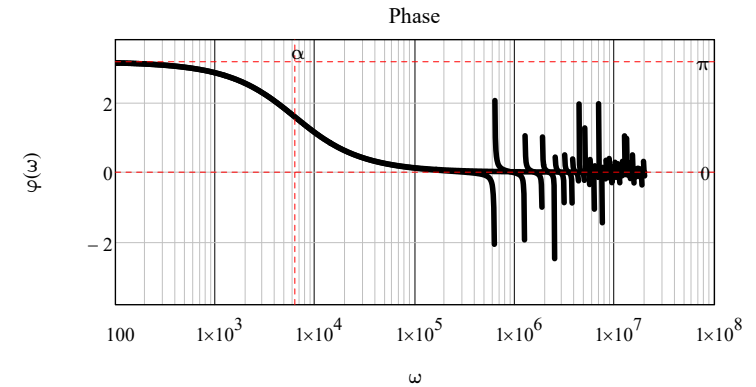


fig.(1.2.5.13)

Plot of G(z) as z moves on a circle

$$z := z \quad A11 := A11 \quad B11 := B11 \quad C11 := C11$$

$$v := \text{denom}(G1(z)) \text{ coeffs}, z \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ B11 \\ C11 \end{pmatrix} \quad (1.2.5.22)$$

search of the poles to represent on the z plane

$$\text{poles1} := \text{polyroots}(v) \quad \text{poles1}^T = (0 \ 0 \ 0.939 \ 0.939)$$

$$p_{0v} := \text{poles1}_0 \quad p_0 = 0 \quad p_{1v} := \text{poles1}_1 \quad p_1 = 0$$

$$z \text{ moves on a circle} \quad r := \text{ceil}(\max(|\text{poles1}|)) \cdot 1.0 \quad r = 2 \quad (1.2.5.23)$$

$$t := t \quad \begin{aligned} \xi(t) &:= r \cdot \cos(t) \\ \psi(t) &:= r \cdot \sin(t) \\ \zeta(t) &:= \xi(t) + j \cdot \psi(t) \end{aligned}$$

$$t := t_0 \cdot t_0 + \frac{t_{\text{fin}} - t_0}{1000} \dots t_{\text{fin}}$$

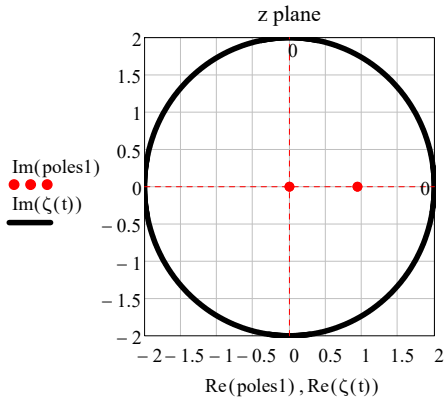


fig.(1.2.5.14)

Sequence calculation of the given z function:

$$z := z \quad A11 := A11 \quad B11 := B11 \quad C11 := C11$$

$$\frac{A11 \cdot (z^{-1} + 1)^2}{z^{-2} + B11 \cdot z^{-1} + C11} \Bigg|_{\substack{\text{invztrans} \\ \text{simplify, max}}} \rightarrow \quad (1.2.5.14)$$

Seek of simplicity define the following constants:

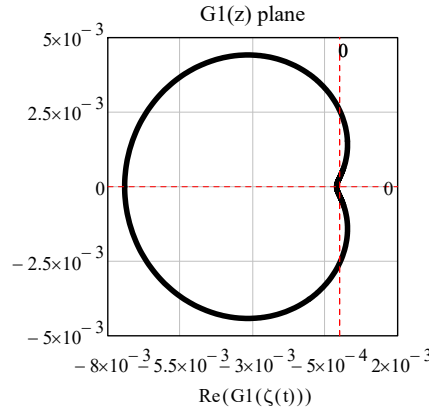


fig.(1.2.5.15)

$$k1 := -\frac{B11 - \sqrt{B11^2 - 4 \cdot C11}}{C11} \quad (1.2.5.15)$$

$$k2 := \frac{(\sqrt{B11^2 - 4 \cdot C11} - B11) \cdot (B11 - 2 \cdot C11) + -2 \cdot C11 \cdot (C11 - 1)}{C11^2 \cdot \sqrt{B11^2 - 4 \cdot C11}}$$

$$k3 := -\frac{B11 + \sqrt{B11^2 - 4 \cdot C11}}{C11}$$

$$k4 := \left[\frac{-(B11 + \sqrt{B11^2 - 4 \cdot C11}) \cdot (B11 - 2 \cdot C11) - 2 \cdot C11 \cdot (C11 - 1)}{C11^2 \cdot \sqrt{B11^2 - 4 \cdot C11}} \right]$$

The resulting sequence is:

$$\text{out1}_n := A11 \cdot \delta(n, 0) + \left(\frac{1}{2}\right)^n \cdot A11 \cdot (-k1^{n-1} \cdot k2 + k3^{n-1} \cdot k4) \quad (1.2.5.16)$$

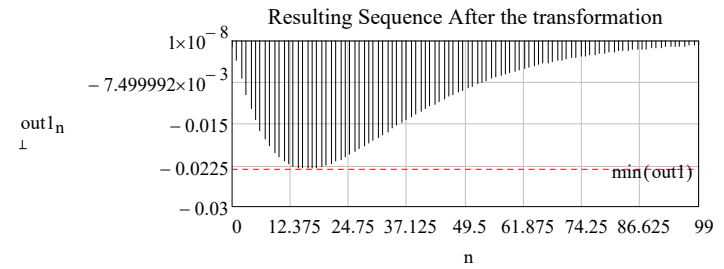


fig.(1.2.5.16)

$$t := 0, 0 + \frac{2 \cdot \beta}{N} \dots 2 \cdot \beta$$

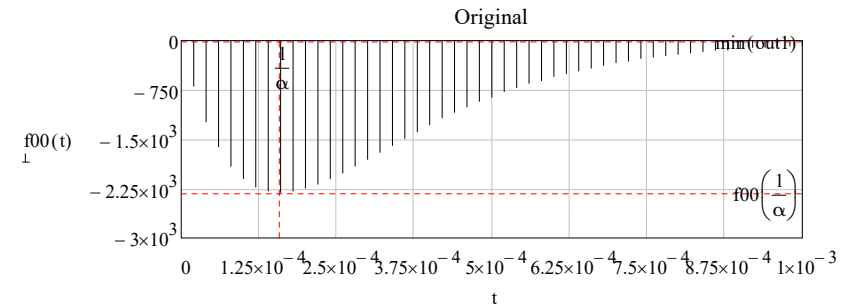


fig.(1.2.5.17)

$$f00\left(\frac{1}{\alpha}\right) = -2311.455$$

$$\frac{\left|f00\left(\frac{1}{\alpha}\right)\right|}{\left|\min(\text{out1})\right|} = 1 \times 10^5$$

$$\min(\text{out1}) = -0.023$$

§1.3) DIFFERENCE EQUATIONS LINEAR OF ORDER N DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS

$$\sum_{k=0}^N (a_k \cdot y_{n-k}) = \sum_{k=0}^N (b_k \cdot x_{n-k})$$

IIR
$$y_n = - \sum_{k=1}^N \left(\frac{a_k}{a_0} \cdot y_{n-k} \right) + \sum_{r=0}^N \left(\frac{b_r}{a_0} \cdot x_{n-r} \right)$$

FIR
$$y_n = \frac{1}{a_0} \cdot \left[\sum_{k=0}^N (b_k \cdot x_{n-k}) \right]$$

$$\frac{1 - a^{-1} \cdot z^{-1}}{1 - a \cdot z^{-1}} \rightarrow \frac{a^{n+2} + \delta(n, 0) - a^n}{a^2}$$

$$\frac{(1 - z^{-1})^2}{1 - \frac{z^{-1}}{2}} \rightarrow \left(\frac{1}{2} \right)^n - 2 \cdot \delta(n - 1, 0)$$

$$\frac{(z - 1)^2}{z - \frac{1}{2}} \rightarrow \frac{1}{2 \cdot 2^n} - 2 \cdot \delta(n, 0) + \text{invztrans}(z, z, n)$$

I consider the following z transform, as a rational function:

$$H(z) = \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}}, \quad (1.3.1)$$

moreover, since H(z) is defined as:

$$H(z) = \frac{Y(z)}{X(z)},$$

(for a causal system, X(z) is the z-transform of the input signal while Y(z) is the corresponding output), I can write:

$$Y(z) \cdot (1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}) = X(z) \cdot (b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}),$$

z inverse transforming I have:

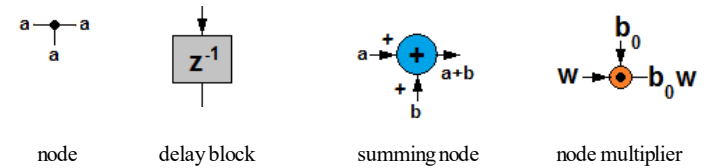
$$y(n) + a_1 \cdot y(n - 1) + a_2 \cdot y(n - 2) = b_0 \cdot x(n) + b_1 \cdot x(n - 1) + b_2 \cdot x(n - 2).$$

The corresponding difference equation is,

$$y(n) = x(n) + b_1 \cdot x(n - 1) + b_2 \cdot x(n - 2) - (a_1 \cdot y(n - 1) + a_2 \cdot y(n - 2)), n > 1. \quad (1.3.2)$$

To these difference equation I must associate the initial conditions y(0) and y(1).

I can match to each difference equation a block diagram that depict the filter structure and the computational procedure for implementing the digital filter. The basic four elements of the realization structure are: branch, delay block, summing node, node multiplier, namely:



fig(1.3.1)

So, for the difference equations defined above, I have the following flow or block diagrams:

DIRECT FORM

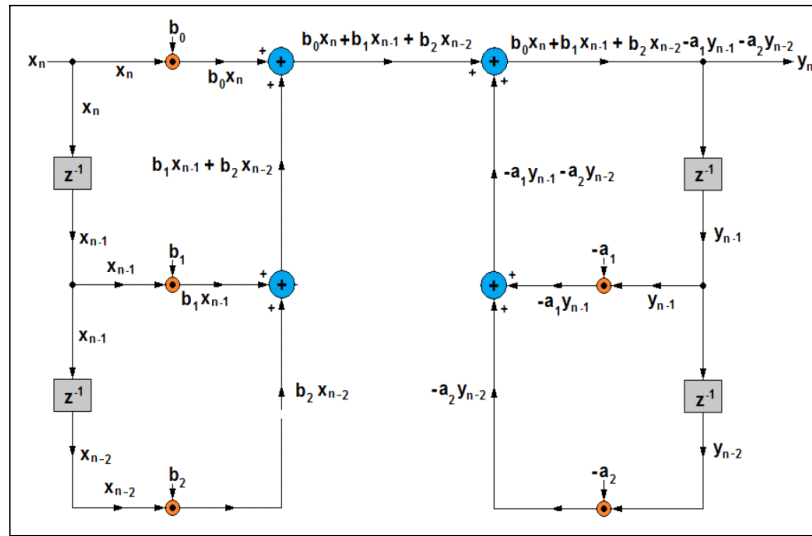


fig.(1.3.2)

Practical second order building block for IIR structure DIRECT FORM

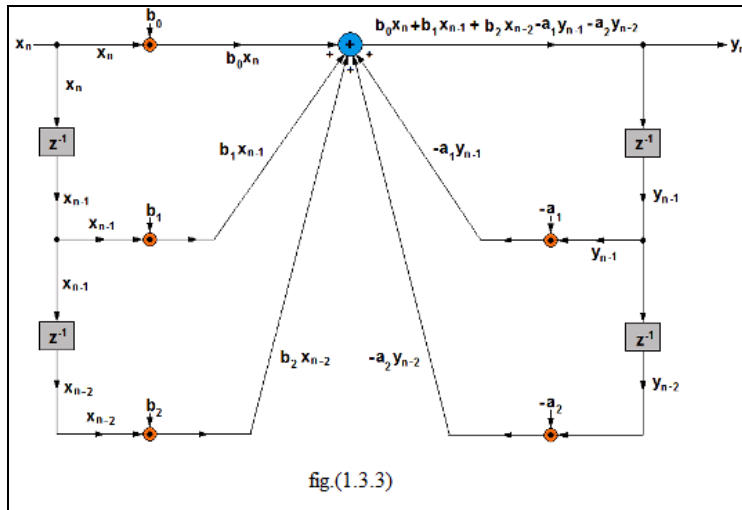


fig.(1.3.3)

Example of input sequence and computing of the corresponding output :

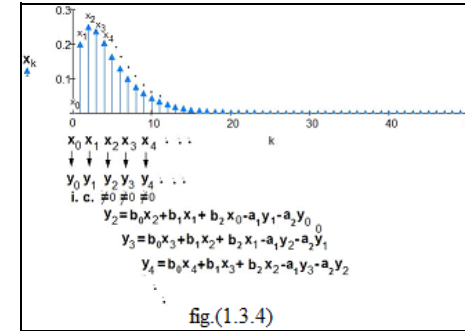


fig.(1.3.4)

DSP registers:

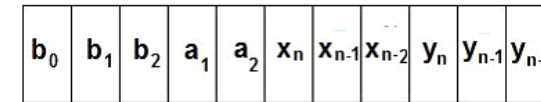


fig.(1.3.5)

A useful stability criterion for linear time invariant systems is that all bounded inputs produce bounded outputs, this is the so called BIBO (bounded input-bounded output) condition. LTI system is said to be stable if and only if it satisfies the

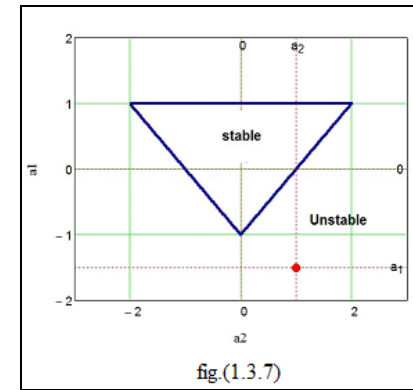
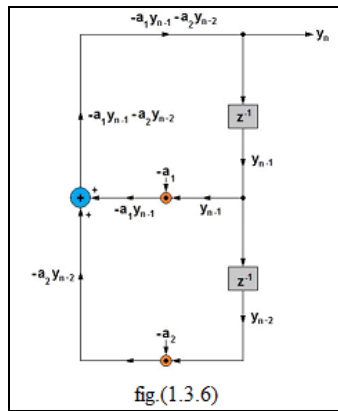
$$\sum_{k=0}^{\infty} |h(k)| < \infty \text{ (IIR),} \quad (1.3.3)$$

where $h(k)$ is the sequence of the impulse response of the system.

Special cases:

$$x(n) = x(n-1) = x(n-2) = 0 \Rightarrow y(n) = -(a_1 \cdot y(n-1) + a_2 \cdot y(n-2)),$$

the resulting block diagram is below depicted:



For system stability the following conditions must be satisfied:

$$0 \leq |a_2| < 1 \quad |a_1| \leq 1 + a_2$$

It is well known that a filter with a transfer function like this:

$$H(z) = \frac{1}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}}, \quad (1.3.4)$$

difference equation: $y(k) = a_1 \cdot y(k-1) + a_2 \cdot y(k-2), \quad (1.3.5)$

is stable if and only if the parameters (a_1, a_2) lie inside the triangle drawn below.

Under some conditions this can be a digital oscillator:

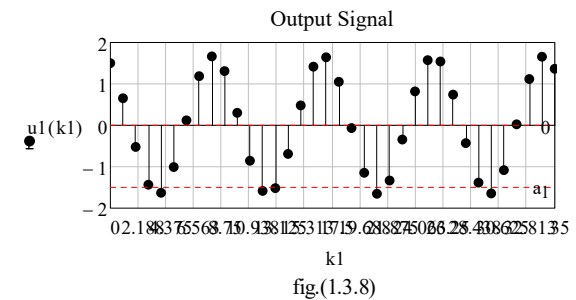
Example: if I place:

$$a_1 := -1.5 \quad a_2 := 1.0$$

$$n := 2..M1 \quad k1 := 0..35$$

and define the following recurrence relation:

$$u1(k1) := \begin{cases} -(a_1 u1(k1-1) + a_2 u1(k1-2)) & \text{if } k1 > 1 \\ 1.5 & \text{if } k1 = 0 \\ 0.65 & \text{if } k1 = 1 \end{cases} \quad (1.3.6)$$



Instead of using a recurrence relation, I can define an array, as follows, which is faster than the previous one, but with large expense of memory:

$$v_0 := 1.5 \quad v_1 := 0.65$$

$$v_n := -(a_1 \cdot v_{n-1} + a_2 \cdot v_{n-2}) \quad (1.3.7)$$

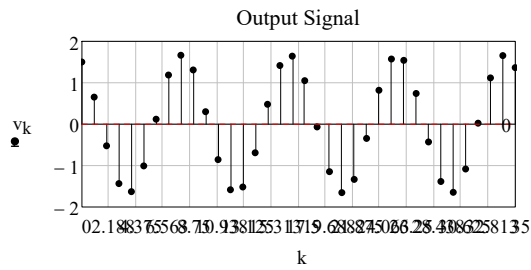


fig.(1.3.9)

Stability "Triangle"

for the previous example I have: Check_stability(a) = "Unstable"
 while if I use some other coefficients the system become stable as I can see changing them.

$a_1 = -1.5$ $a_2 = 1$ $x := -3, -3 + 0.01 \dots 3$

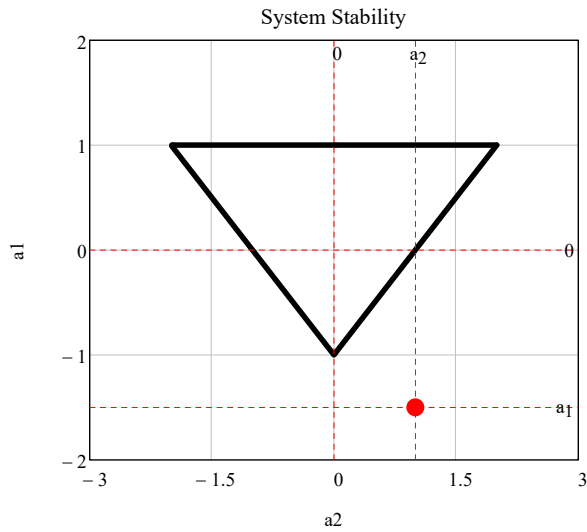


fig.(1.3.10)

CANONICAL SECOND ORDER SECTION

On the other hand, for a second order filter, having placed:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}, \quad (1.3.8)$$

namely:

$$\frac{Y(z)}{W(z)} = b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}, \quad (1.3.9)$$

$$\frac{W(z)}{X(z)} = \frac{1}{\left(1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}\right)}, \quad (1.3.10)$$

or

$$W(z) = \frac{X(z)}{1 + \sum_{p=1}^2 \left(a_p \cdot z^{-p}\right)} \quad (1.3.11)$$

and

$$Y(z) = b_0 \cdot W(z) + b_1 \cdot z^{-1} \cdot W(z) + b_2 \cdot z^{-2} \cdot W(z), \quad (1.3.12)$$

$$X(z) = W(z) + a_1 \cdot z^{-1} \cdot W(z) + a_2 \cdot z^{-2} \cdot W(z), \quad (1.3.13)$$

The corresponding set of difference equations is:

$$\text{state at time } n \quad w(n) = x(n) - a_1 \cdot w(n-1) - a_2 \cdot w(n-2), \quad (1.3.14)$$

$$\text{output signal} \quad y(n) = b_0 \cdot w(n) + b_1 \cdot w(n-1) + b_2 \cdot w(n-2), \quad (1.3.15)$$

whose block diagram is given in the next drawing:

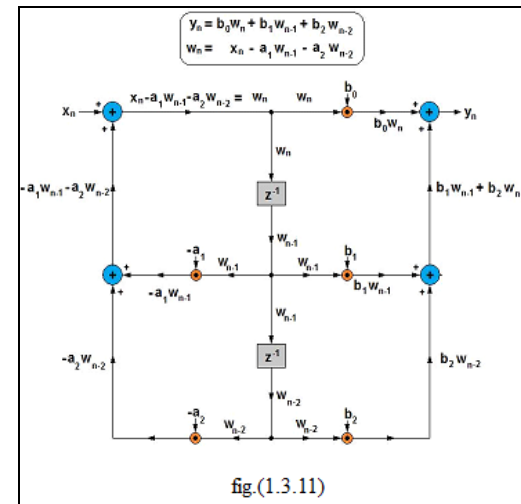


fig.(1.3.11)

DSP registers:

b_0
b_1
b_2
a_1
a_2
w_n
w_{n-1}
w_{n-2}
y_n
x_n

fig.(1.3.12)

Practical second order building block for IIR structure. CANONICAL SECOND ORDER SECTION

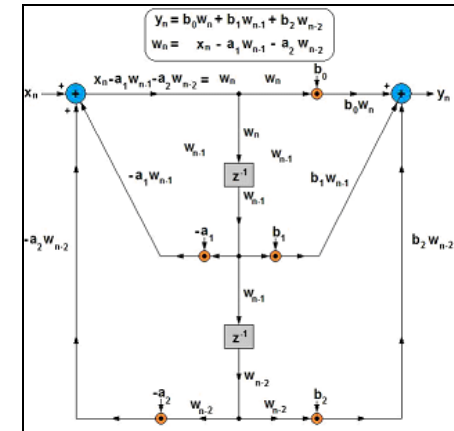


fig.(1.3.13)

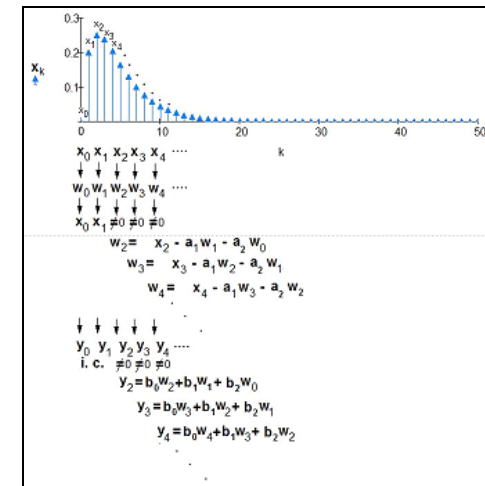


fig.(1.3.14)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

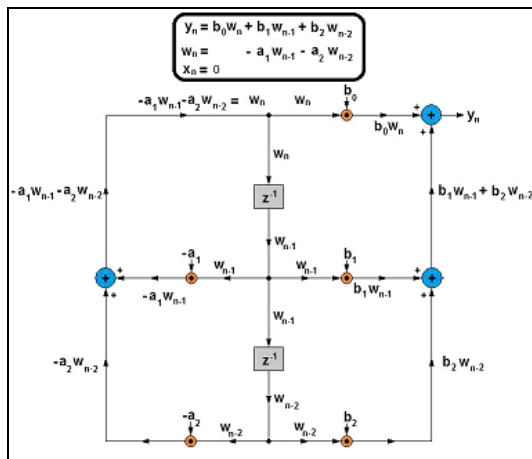


fig.(1.3.15)

Example $a_1 := 0.5$ $a_2 := 0.9$ $b_0 := 15$ $b_1 := -2.25$ $b_2 := 0$

$$H10(z) := \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} \quad (1.3.16)$$

$n := 2..M1$ $k1 := 0..M1$ $x_n := 0$
 $x_{k1} := u(k1)$ (1.3.17)

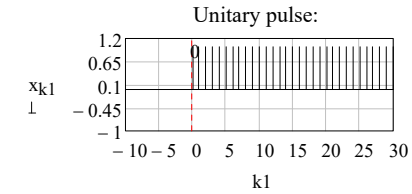


fig.(1.3.16)

$w_0 := x_0$

$w_1 := x_1$

$$w_n := x_n - a_1 \cdot w_{n-1} - a_2 \cdot w_{n-2} \quad (1.3.18)$$

$$y1_n := b_0 \cdot w_n + b_1 \cdot w_{n-1} + b_2 \cdot w_{n-2} \quad (1.3.19)$$

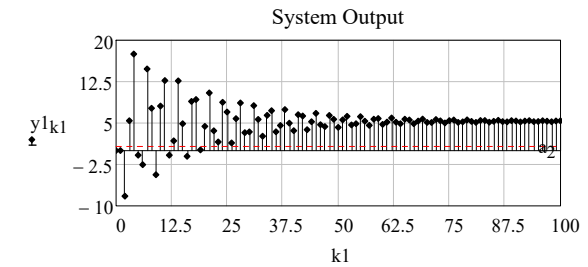


fig.(1.3.17)

Searching the poles of H(z):

$$v := \text{denom}(H10(z)) \text{ coeffs}, z \rightarrow \begin{pmatrix} 0 \\ 36.0 \\ 20.0 \\ 40.0 \end{pmatrix} \quad (1.3.20)$$

$p := \text{polyroots}(v)$ $\text{rows}(p) = 3$

$$p^T = (-0.25 + 0.915j \quad -0.25 - 0.915j \quad 0) \quad (1.3.21)$$



for the previous example I have:

Check_stability(a) = "Stable"

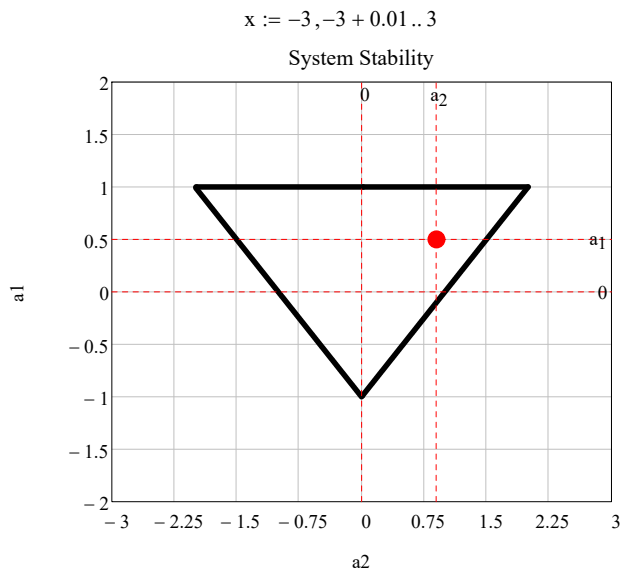


fig.(1.3.18)

$a_1 = 0.5 \quad a_2 = 0.9$

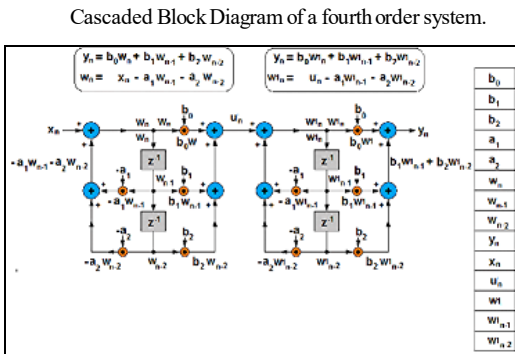


fig.(1.3.19)

Cascaded Block Diagram of a eighth order system.

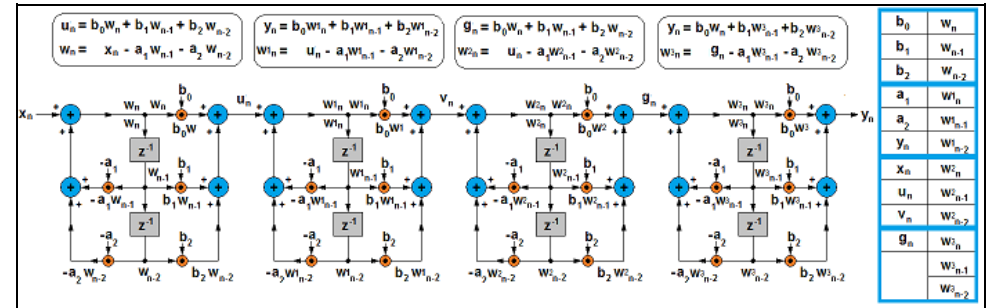


fig.(1.3.20)

Generally I can implement a digital filter of N^o order using the cascade structure of second order subsystems. For this worth the following expression.

(for example: $N_{\text{casc}} := 7$),
$$H(z) = B \cdot \prod_{k=1}^L \frac{1 + \beta_1 \cdot k \cdot z^{-1} + \beta_2 \cdot k \cdot z^{-2}}{1 - \alpha_1 \cdot k \cdot z^{-1} - \alpha_2 \cdot k \cdot z^{-2}} \quad (1.3.22)$$

where:
$$L := \text{floor}\left(\frac{N+1}{2}\right), \quad L = 4 \text{ is the number of cascaded blocks.}$$

(example: $l := 0..2, m := 0..L$, create the arrays $\beta_{l,m} := 0.0, \alpha_{l,m} := 0. B = \text{cost}$)

If N is odd the latter equation contains a first order term. The output signal is a function of the states $w_i(n-1), w_i(n-2), i=1, \dots, L$ and of the input signal $x(n)$, as follows:

$$y_n = w_{L,n} + \beta_{1,L} \cdot w_{L,n-1} + \beta_{2,L} \cdot w_{L,n-2}, \quad (1.3.23)$$

$$w_{L,n} = w_{L-1,n} + \beta_{1,L-1} \cdot w_{L-1,n-1} + \beta_{2,L-1} \cdot w_{L-1,n-2} + \alpha_{1,L} \cdot w_{L,n-1} + \alpha_{2,L} \cdot w_{L,n-2},$$

$$\vdots$$

$$w_{2,n} = w_{1,n} + \beta_{1,1} \cdot w_{1,n-1} + \beta_{2,1} \cdot w_{1,n-2} + \alpha_{1,2} \cdot w_{2,n-1} + \alpha_{2,2} \cdot w_{2,n-2},$$

$$w_{1,n} = B \cdot x_n + \alpha_{1,1} \cdot w_{1,n-1} + \alpha_{2,1} \cdot w_{1,n-2}.$$

Special cases.

Any sampled monochromatic signal switched on at $n=0$, of amplitude A , phase ϕ_1 , angular frequency ω_1 and sampling frequency f_s , at sampled time $n=0$ is given by:

$$x_n = A \cdot e^{j(\varphi_0 \cdot n + \phi_1)}, \text{ for } n \geq 0, \quad (1.3.24)$$

$$x_n = 0 \quad \text{for } n < 0.$$

normalized angular frequency: $\varphi_0 = \frac{\omega_1}{f_s}, \quad (1.3.25)$

It follows that for two complex samples, namely:

$$x_0 = A \cdot e^{j \cdot \phi_1},$$

$$x_1 = A \cdot e^{j \cdot (\varphi_0 + \phi_1)},$$

I can obtain:

normalized angular frequency $\varphi_0 = -j \cdot \ln\left(\frac{x_1}{x_0}\right),$

signal amplitude $A_x = |x_0| = |x_1|,$

phase at sampled time $n=0$ $\phi_1 = -j \cdot \ln\left(\frac{x_0}{A_x}\right).$

Example: $f := 1 \cdot \text{kHz}, T := \frac{1}{f}, \omega_1 := 2 \cdot \pi \cdot f, \phi_1 := \frac{\pi}{5}, A_x := 10, f_s := 12 \cdot f, T_{\text{eff}} := \frac{1}{f_s}, \varphi_0 := \frac{\omega_1}{f_s},$

$$x(k) := A_x \cdot e^{j(\varphi_0 \cdot k + \phi_1)}, \quad (1.3.26)$$

$$\varphi_0 := -j \cdot \ln\left(\frac{x(1)}{x(0)}\right), \quad \varphi_0 = 0.524,$$

$$\phi_1 := -j \cdot \ln\left(\frac{x(0)}{A_x}\right), \quad \phi_1 = 0.628,$$

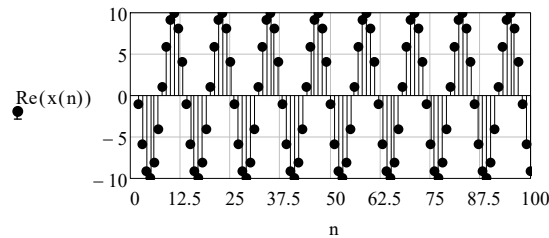


fig.(1.3.21)

$$z = e^{j \cdot \left(\omega_1 \cdot \frac{1}{f_s}\right)}$$

$$H(z) = \frac{X(z)}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} = \frac{X(z)}{1 + \sum_{p=1}^2 (a_p \cdot z^{-p})} \quad (1.3.27)$$

$$H(z) = \frac{X(z)}{1 + \sum_{p=1}^2 \left[a_p \cdot e^{-j \cdot \left(\omega_1 \cdot \frac{p}{f_s}\right)} \right]} \quad (1.3.28)$$

$$h(k) := \frac{1}{\left[1 + \sum_{p=1}^2 \left[a_p \cdot e^{-j \cdot \left(\omega_1 \cdot \frac{p}{f_s}\right)} \right] \right]} \left[A_x \cdot e^{j \cdot \left(\omega_1 \cdot \frac{k}{f_s} + \phi_1\right)} \right] \quad (1.3.29)$$

$$x(0) = 8.09 + 5.878j \quad A_x \cdot e^{j \cdot \phi_1} = 8.09 + 5.878j \quad (1.3.30)$$

$$x(1) = 4.067 + 9.135j \quad x(0) \cdot e^{j \cdot \frac{\omega_1}{f_s}} = 4.067 + 9.135j \quad (1.3.31)$$

$$h(k) := \frac{1}{\left[1 + \sum_{p=1}^2 \left[a_p \cdot \left(\frac{x(1)}{x(0)}\right)^{-p} \right] \right]} \cdot x(0) \cdot \left(\frac{x(1)}{x(0)}\right)^k \quad (1.3.32)$$

Thus, known the first two sample of the monochromatic input, I can determine the memory of the filter so that the filter is in its settled state.

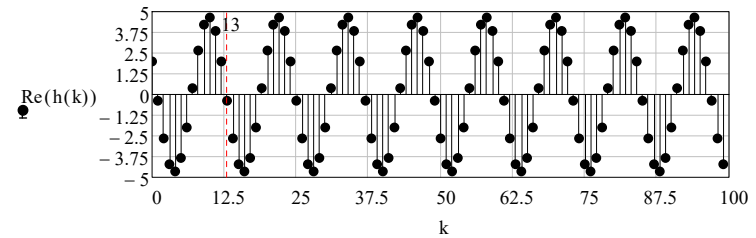


fig.(1.3.22)

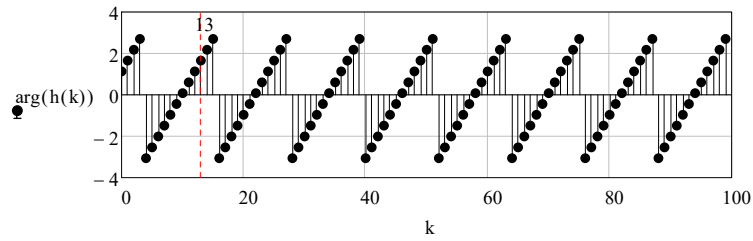


fig.(1.3.23)

For a monochromatic input signal and a cascaded structure worth the relation for state of the λ stage

$$\lambda := 2..L, \quad \nu := 0..-2,$$

$$H_{\nu, \lambda} := \frac{1 + \beta_{1, (\lambda-1)} \cdot \left(\frac{x(1)}{x(0)}\right)^{-1} + \beta_{2, (\lambda-1)} \cdot \left(\frac{x(1)}{x(0)}\right)^{-2}}{1 - \alpha_{1, \lambda} \cdot \left(\frac{x(1)}{x(0)}\right)^{-1} - \alpha_{2, \lambda} \cdot \left(\frac{x(1)}{x(0)}\right)^{-2}} \cdot W_{0, \lambda-1} \cdot \left(\frac{x(1)}{x(0)}\right)^{\nu}, \quad (1.3.33)$$

$$H_{n, 1} := \frac{B}{1 - \alpha_{1, \lambda} \cdot \left(\frac{x(1)}{x(0)}\right)^{-1} - \alpha_{2, \lambda} \cdot \left(\frac{x(1)}{x(0)}\right)^{-2}} \cdot x(0) \cdot \left(\frac{x(1)}{x(0)}\right)^n \quad (1.3.34)$$

Knowing H_1 I can pre initialize the memory of an IIR filter in the cascade structure to shorten the filter settling time.

§1.4) Consequences due the quantization error in IIR digital filters

In a DSP a finite number of bits is used to represent constants and variables and to perform arithmetic operations. Typical word lengths in modern DSP processors are 16 or 32 bit. The use of finite word lengths produces errors which influences the performances of the DSP.

The causes of errors in DSP are:

- 1) ADC quantization errors,
- 2) Difference equation coefficients quantization errors,
- 3) Overflow errors due to operation results which are greater than word length,
- 4) Roundoff errors.

§1.5) PERIODIC SEQUENCES. FOURIER SERIES.

A periodic sequence with period N , is so defined:

$$x_p(n) = x_p(n + k \cdot N), \quad k=0,1,2,\dots,N. \quad (1.5.1)$$

Its Fourier series is:

$$x_p(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} \left(X_p(k) \cdot e^{j \cdot \frac{2 \cdot \pi}{N} \cdot n \cdot k} \right). \quad (1.5.2)$$

$$X_p(k) = \sum_{n=0}^{N-1} \left(x_p(n) \cdot e^{-j \cdot \frac{2 \cdot \pi}{N} \cdot n \cdot k} \right). \quad (1.5.3)$$

placing:
$$W_N = e^{-j \cdot \frac{2 \cdot \pi}{N}}, \quad (1.5.4)$$

I can write
$$X_p(k) = \sum_{n=0}^{N-1} \left(x_p(n) \cdot W_N^{n \cdot k} \right), \quad (1.5.5)$$

$$x_p(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} \left(X_p(k) \cdot W_N^{-n \cdot k} \right). \quad (1.5.6)$$

Indicating in a period:

$$x(n) = x_p(n) \quad \text{for } 0 \leq n \leq N - 1$$

$$x(n) = 0 \quad \text{otherwise,}$$

I can deduce its z transform:

$$X(z) = \sum_{n=0}^{N-1} \left(x(n) \cdot z^{-n} \right), \quad (1.5.7)$$

$$X_p(k) = X(W_N^{-k}). \quad (1.5.8)$$

Example:

$$\nu := M1 - 1 \quad \text{period: } \underline{N} := 10 \quad \text{Amplitude: } V_m := 1$$

$$x1(n) := \frac{V_m}{2} \cdot \left[\sum_{k=0}^{\nu} \left[u(n - k \cdot N) - 2 \cdot u\left[n - \left(\frac{2 \cdot k + 1}{2} \cdot N\right)\right] + u[n - (k + 1) \cdot N] \right] + u(n) \right] \quad (1.5.9)$$

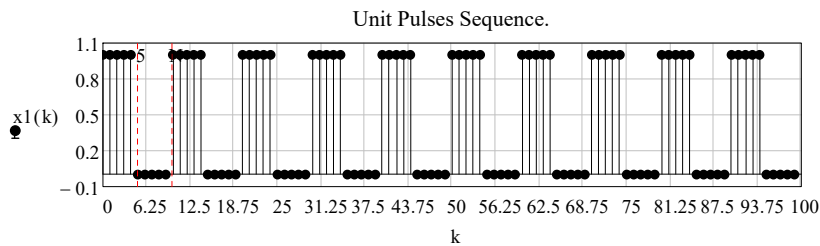


fig.(1.5.1)

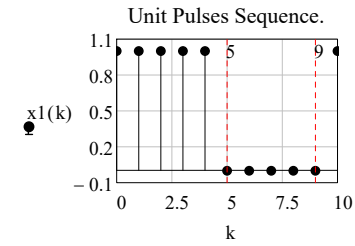


fig.(1.5.2)

$$X_p(k) = \sum_{n=0}^4 W_{10}^{n \cdot k} = \sum_{n=0}^4 e^{-j \cdot \frac{2 \cdot \pi}{10} \cdot n \cdot k} = e^{-j \cdot \frac{4 \cdot \pi}{10} \cdot k} \cdot \frac{\sin\left(k \cdot \frac{\pi}{2}\right)}{\sin\left(k \cdot \frac{\pi}{10}\right)} \quad (1.5.10)$$

$$\lim_{k \rightarrow 30} \frac{\sin\left(k \cdot \frac{\pi}{2}\right)}{\sin\left(k \cdot \frac{\pi}{10}\right)} \rightarrow 5$$

Function $X_p(k)$

$$|X_p(30)| = 5$$

$$|X_p(70)| = 5$$

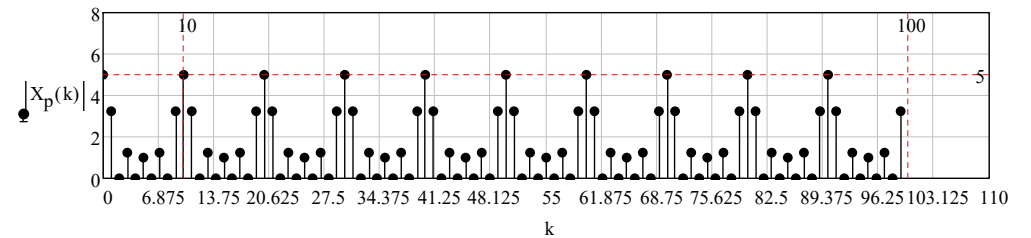


fig.(1.5.3)

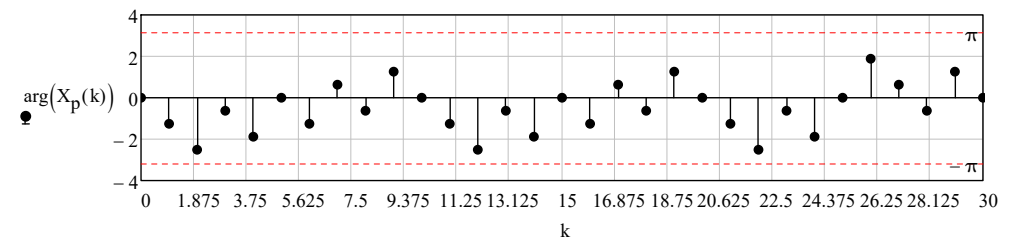


fig.(1.5.4)

$$X(z) = \sum_{n=0}^{N-1} (x_p(n) \cdot z^{-n}) \quad (1.5.11)$$

$$X(\omega) := e^{-j \cdot 2 \cdot \omega} \cdot \frac{\sin\left(\frac{5}{2} \cdot \omega\right)}{\sin\left(\frac{\omega}{2}\right)} \quad \text{for } z = e^{j \cdot \omega} \quad (1.5.12)$$

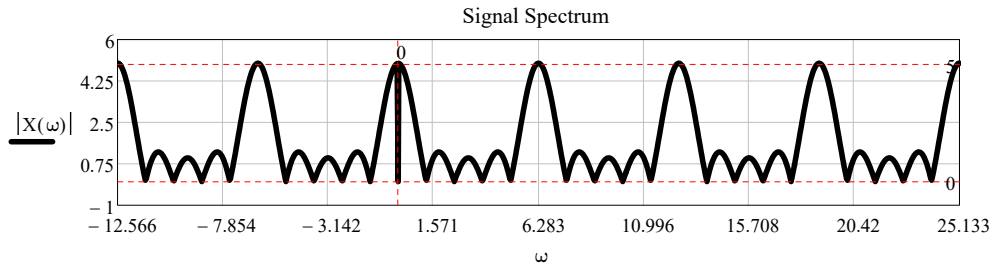


fig.(1.5.5)

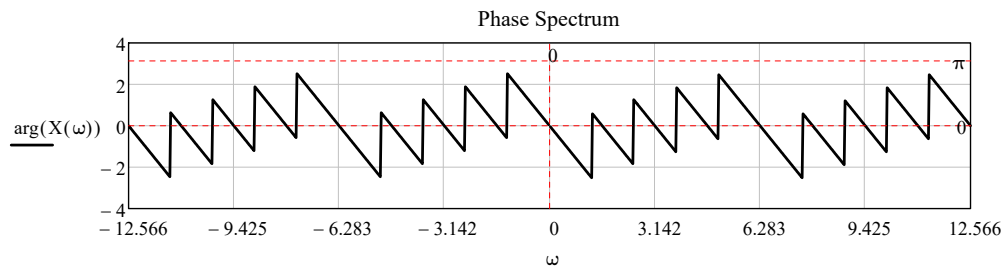


fig.(1.5.6)

FFT application: Frequency response estimation.

The frequency response of a discrete-time system is simply the Fourier transform of its impulse response.

Given the two vectors of the coefficients of the numerator (N) and denominator (M) a and b, I create the FFT of a and b namely A(k) and B(k). The ratio of the two FFTs gives the frequency response.

$$H\left(e^{j \cdot \omega_k \cdot T}\right) = \frac{A(k)}{B(k)}, \quad k = 0 \dots \frac{N}{2} \quad (1.5.13)$$

Example:

$$H(z) = \frac{z+1}{z-0.7071} = \frac{1+z^{-1}}{1-0.7071 \cdot z^{-1}} \quad (1.5.14)$$

k := 0..2·M1

a_k := 0 b_k := 0

a₀ := 1 a₁ := -0.7071 a₂ := 0. b₀ := 1.0 b₁ := 1 b₂ := 0

$$H(z) := \frac{b_0 + b_1 \cdot z^{-1}}{1 + a_1 \cdot z^{-1}} \quad (1.5.15)$$

$$a^T = [1 \quad -0.707 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \dots]$$

$$b^T = [1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \dots]$$

$$A_k := 0 \quad B_k := 0$$

$$A := \text{CFFT}(a)$$

(1.5.16)

$$B := \text{CFFT}(b)$$

$$h_k := \frac{B_k}{A_k}$$

$$h^T = [6.828 \quad 6.772-0.617j \quad 6.609-1.204j \quad 6.353-1.737j \quad 6.026-2.199j \quad 5.651-2.579j \quad \dots]$$

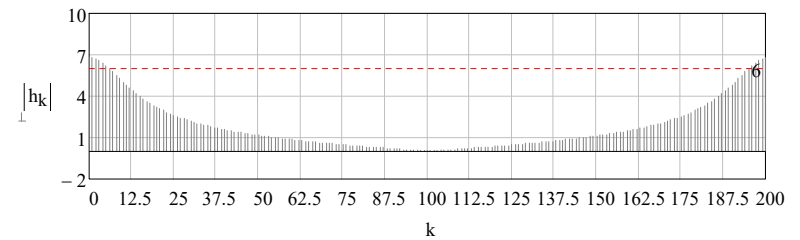


fig.(1.5.7)

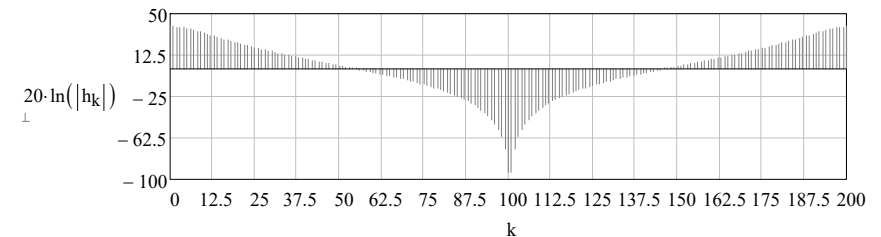


fig.(1.5.8)

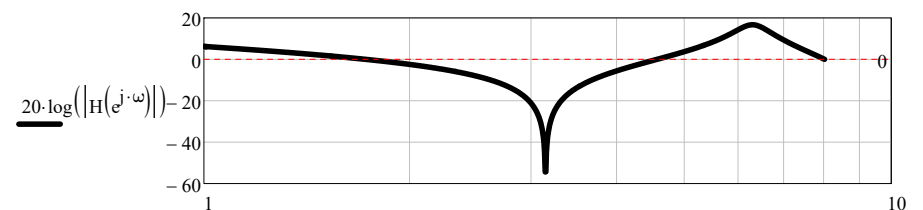


fig.(1.5.9)

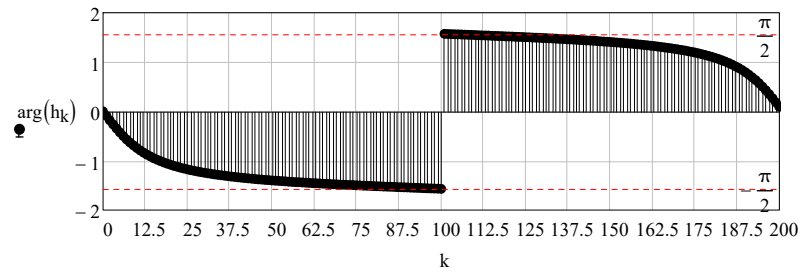


fig.(1.5.10)

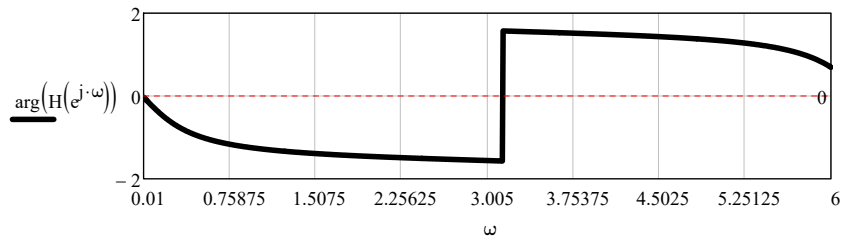


fig.(1.5.11)

□