Mathcad Community Challenge January 2024 (Prime 8 Express)

(Basic) Calculate the number of ways to shuffle a deck of cards. (Standard 52 card deck, no Jokers).

There are 52 cards that may be placed in position 1, leaving 51 for position 2, leaving 50 for position 3 ... etc.

Hence there are 52x51x50x...x2x1 or 52! ways.

$$52! = 8.066 \cdot 10^{67}$$

(Basic) Calculate the number of ways to shuffle a deck of cards if the suit (heart, diamond, spade, club) doesn't matter.

First let's imagine we only have 4 (distinguishable) cards; one of each suit, say. Then there would be 4! ways of arranging them (using the same reasoning as above).
Now imagine the four cards are replaced by four identical cards (the ace of hearts, say) in every one of those ways. Now all the arrangements are indistinguishable; there is only one distinguishable arrangement. The number of ways has been reduced by 4!

With a complete deck of cards, when the suits don't matter, there are 13 different sets of 4 such indistinguishable cards. Hence the 52! ways must be reduced by (4!)x(4!)x(4!)x.....x(4!) (i.e. thirteen lots of 4!), or:

$$\frac{52!}{(4!)^{13}} = 9.202 \cdot 10^{49}$$

Calculate the number of ways to shuffle a deck of cards consisting of just the face cards (jacks, queens, and kings). If you could shuffle the deck one time a second, how long would it take to shuffle each possible way.

Now we only have 12 cards, so ther number of ways is 12! using our earlier reasoning.

$$12! = 4.79 \cdot 10^{8}$$

At 1 shuffle a second this would take: $T := 12! \cdot s$ or: $T = 15.179\ yr$

How long would it take to shuffle the face cards until there's a 0.5 probability you've shuffled the same way twice?

For the first shuffle there is nothing against which to compare.
For the second shuffle the probability of matching the first is simply:

$$p_1 = \frac{1}{12!}$$

For the third shuffle to match the first or second, we have the probability that we *didn't* match on the second, multiplied by the probability that we matched either the first *or* the second on the third shuffle, or:

$$p_2 = \left(1 - \frac{1}{12!}\right) \cdot \frac{2}{12!}$$

For the fourth shuffle to match the first, second or third, we have the probability that we didn't match the first three, multiplied by the probability that we matched one of the first three on the fourth shuffle, or:

$$p_3 = \left(1 - \frac{1}{12!}\right) \cdot \left(1 - \frac{2}{12!}\right) \cdot \frac{3}{12!}$$

Continuing in this way, we have the probability of matching on the n+1'th shuffle as:

$$p_n = \frac{n}{12!} \cdot \prod_{k=1}^{n-1} \left(1 - \frac{k}{12!}\right)$$

The overall probability of making a match after m+1 shuffles, therefore, is the sum of all these, namely:

$$P(m,a) = \sum_{n=1}^{m} \frac{n}{a} \cdot \prod_{k=1}^{n-1} \left(1 - \frac{k}{a}\right)$$ where a = 12! here.

We want this to be equal to 0.5; i.e. we want the value of m that makes P(m,a) = 0.5. Unfortunately, there is no Solve block in Prime Express, so we must find this value "by hand".

Because the calciulation of P(m,a) can be time-consuming, we'll introduce an approximation to the product terms to speed up the calculations, without sacrificing too much accuracy (I hope!).

Let: $z = \prod_{k=1}^{x} \left(1 - \frac{k}{a}\right)$ and take the log of both sides:

We have: $ln(z) = \sum_{k=1}^{x} ln\left(1 - \frac{k}{a}\right)$ using the summation property of the logarithm of a product.

Now, when δ << 1 then $ln(1-\delta) = -\delta$ approximately.

We will assume k/a << 1 for all values of interest, so:

$$ln(z) = -\left(\sum_{k=1}^{x} \frac{k}{a}\right) \quad \text{hence:} \quad ln(z) = -\frac{x \cdot (x+1)}{2\ a}$$

so that: $z = e^{-\frac{x \cdot (x+1)}{2\ a}}$

This allows us to express P(m,*a*) approximately, as:

$$P(m,a) := \frac{1}{a} \cdot \sum_{n=1}^{m} n \cdot e^{\frac{-(n-1) \cdot n}{2 \cdot a}}$$

We will find it useful to define: $\delta P(m,a) = P(m+1,a) - P(m,a)$

In approximate form, this is: $\delta P(m,a) := \frac{1}{a} \cdot (m+1) \cdot e^{\frac{-m \cdot (m+1)}{2 \cdot a}}$

We'll now use two guessed values for m, and linearly interpolate or extrapolate as necessary to get close to P(m,*a*) = 0.5, using the following:

$$\frac{m_{i+1} - m_i}{0.5 - P(m_i, a)} = \frac{(m_i + 1) - m_i}{\delta P(m_i, a)}$$

so that: $m_{i+1} = \frac{0.5 - P(m_i, a)}{\delta P(m_i, a)} + m_i$

For the initial guess, m0, we'll set all the bracketed terms within the product component of P(m) to be 1, as *a* is likely to be much greater than k for all k's of interest.

$P(m_o, a) = \frac{1}{a} \cdot \sum_{n=1}^{m_0} n$ or: $P(m_0, a) = \frac{m_0 \cdot (m_0 + 1)}{2 \cdot a}$

so: $\frac{m_0 \cdot (m_0 + 1)}{2 \cdot a} = 0.5$ or, approximately: $m_0 = \sqrt{a}$

(We're only after a guessed value, so we've ignored the +1 after m0).

$a := 12!$ $m_0 := \sqrt{a}$

$i := 0..2$ $m_{i+1} := \frac{0.5 - P(m_i, a)}{\delta P(m_i, a)} + m_i$

$P(m_3, a) = 0.5$ $m_3 = 2.577 \cdot 10^4$

So the number of shuffles is: $m := \text{round}(m_3 + 1) = 25770$

The corresponding time taken is: $T := m \cdot s$ or: $T = 7.158\ hr$

If you add the aces, how long would it take for that same 0.5 probability that you've shuffled the same way twice?

Adding the aces gives us 16 instead of 12 cards. The same approach as that above may be used simply by setting *a* to be 16!

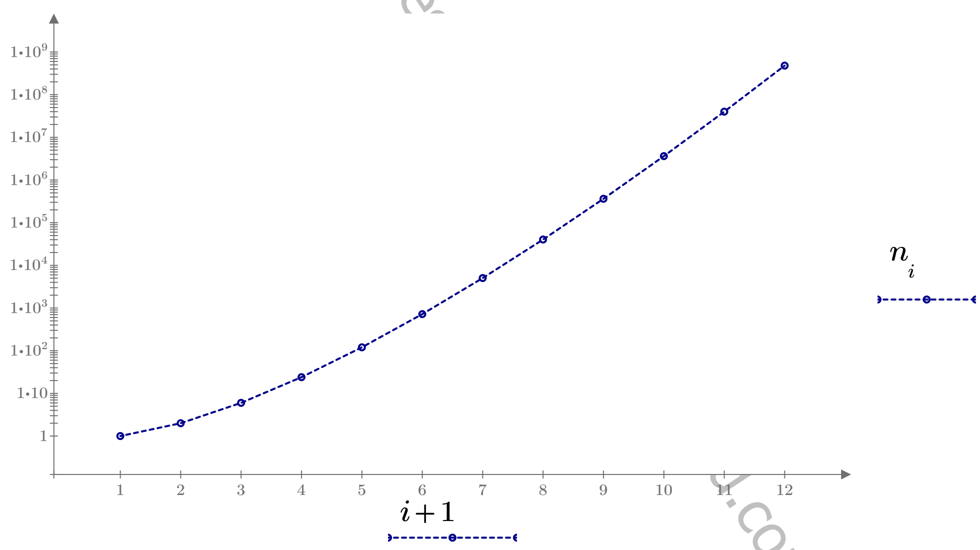
The result of so doing is:

$m := 5385643 \qquad T := m \cdot s \qquad T = 62.334\ day$

Create a Chart Component to depict the number of ways to shuffle a deck from 1 to 12 cards.

Express doesn't allow use of the Chart Component, so the simple XY plot is used here.

$i := 0 .. 11 \qquad n_i := (i+1)!$



(Optional) How many cards would be in a deck in which the number of ways to shuffle would be on an order of magnitude with the number of atoms in the Earth? What about the universe? Assume 10 to the 50th power for number of atoms in the Earth and 10 to the 80th for the universe.

Number of Earth atoms: $N_E := 10^{50}$

We want to find the value of n, such that $n! = N_E$

To do this we'll make use of Stirling's approximation, which should be reasonably accurate (to within an order of magnitude) for large factorials.

One form of Stirling's approximation is:

$$ln(n!) = n \cdot ln(n) - n + \frac{1}{2} \cdot ln(2 \cdot \pi \cdot n) + O\left(\frac{1}{n}\right)$$

We can use this (ignoring the error term). Define the function:

$$f(n, N) := n \cdot \ln(n) - n + \frac{1}{2} \cdot \ln(2 \cdot \pi \cdot n) - \ln(N)$$

We'll use the inbuilt root function to find the value of n that make f(n,N)=0.

Initial guess: $ncard := 52$

$$ncard := \mathbf{root}\left(f\left(ncard, N_E\right), ncard\right) = 41.294$$

$$ncard := \text{round}(ncard) = 41$$

$ncard! = 3.345 \cdot 10^{49}$ Within an order of magnitude of N_E

Doing the same for the Universe, we have: $N_U := 10^{80}$

$$ncard := \mathbf{root}\left(f\left(ncard, N_U\right), ncard\right) = 58.92$$

$$ncard := \text{round}(ncard) = 59$$

$ncard! = 1.387 \cdot 10^{80}$ Within an order of magnitude of N_U